Two-loop analysis of axial vector current propagators in chiral perturbation theory

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We perform a calculation of the isospin and hypercharge axial vector current propagators [$\Delta^{\mu\nu}_{A3}(q)$ and $\Delta^{\mu\nu}_{A8}(q)$] to two loops in $SU(3)\times SU(3)$ chiral perturbation theory. A large number of $\mathcal{O}(p^6)$ divergent counterterms are fixed, and complete two-loop renormalized expressions for the pion and eta masses and decay constants are obtained. The calculated isospin and hypercharge axial vector polarization functions are used as input in new chiral sum rules, valid to second order in the light quark masses. Some phenomenological implications of these sum rules are considered. [S0556-2821(98)06015-9]

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I. INTRODUCTION

Although low energy quantum chromodynamics remains analytically intractable, the calculational scheme of chiral perturbation theory [1] (ChPT) has led to many valuable contributions. Following the seminal papers of Gasser and Leutwyler [2,3], numerous studies conducted in the following decade convincingly demonstrated the power of ChPT. The state of the art up to 1994 is summarized in several reviews e.g. [4,5] (see also [6,7]). The exploration of ChPT continues to this day, and two-loop studies represent an active frontier area of research. These include processes which have leading contributions in the chiral expansion at order p^4 [8–13] or even p^6 [14], as well as systems for which precision tests will soon be available, e.g. the low-energy behavior of $\pi\pi$ scattering [15–17]. While the case of $SU(2)\times SU(2)$ ChPT to two-loop order has been relatively well explored (in particular see [17]), works in $SU(3)\times SU(3)$ ChPT are still few in number [9,11,13,18,19].

Recently, we performed a calculation of the isospin and hypercharge vector current propagators [$\Delta^{\mu\nu}_{V3}(q)$ and $\Delta^{\mu\nu}_{V8}(q)$] to two-loop order in $SU(3)\times SU(3)$ chiral perturbation theory [9]. A partial motivation for working in the three-flavor sector stems from its inherently richer phenomenology. In particular, it becomes possible to derive new chiral sum rules which explicitly probe the $SU(3)$-breaking sector. With the aid of improved experimental information on spectral functions with strangeness content, it should become possible to evaluate and test these sum rules.

We have completed this program of calculation by determining the corresponding isospin and hypercharge axial vector current propagators [$\Delta^{\mu\nu}_{A3}(q)$ and $\Delta^{\mu\nu}_{A8}(q)$]. Determination of axial vector propagators is much more technically demanding than for the vector propagators, but at the same time yields an extended set of results, among which are (1) a large number of constraints on the set of $\mathcal{O}(p^6)$ counterterms, (2) predictions for the threshold behavior of the 3π, $\bar{K}K\pi$, $\bar{K}K\pi$, $\eta\pi\pi$, etc. axial vector spectral functions, (3) an extensive analysis of the so-called ''sunset'' diagrams, (4) new axial vector spectral function sum rules, (5) a new contribution [20] to the Das-Mathur-Okubo sum rule [21], and (6) a complete two-loop renormalization of the masses and decay constants of the pion and eta mesons.

The presentation begins in Sec. II with a summary of results for the tree-level and one-loop sectors as well as an overview of the corresponding procedure in the two-loop sector, including a discussion of the $\mathcal{O}(p^6)$ counterterms. The construction of a proper renormalization procedure forms the subject of Sec. III. It provides the framework for the removal of divergences, described in Sec. IV, and leads to finite renormalized expressions for the isospin polarization functions given in Sec. V. Section VI deals with the determination of spectral functions, and the subject of chiral sum rules is discussed in Sec. VII. Our conclusions are presented in Sec. VIII. Technical details regarding sunset integrals are presented in the Appendix. At several points in the paper we compare results as expressed in the ''$\bar{\lambda}$-subtraction'' renormalization used here with a variant of the modified minimal subtraction ($\overline{\rm MS}$) scheme. Some of the results of the axial vector study described here (e.g. two-loop renormalized masses and decay constants) occur in exceedingly cumbersome form. They have been collected in Ref. [22].

II. TREE-LEVEL AND ONE-LOOP ANALYSES

The one-loop chiral analysis of the isospin axial vector current propagator was first carried out by Gasser and Leutwyler who used the background-field formalism and worked in an $SU(2)$ basis of fields [2]. We shall describe a re-calculation of the isospin axial vector propagator through one-loop order, but now done within the context of a Feynman diagram calculation and using an $SU(3)$ basis of fields. Then we present an overview of the two-loop sector.

A. Basic definitions and calculational procedure

In this paper, we shall deal with the axial vector current propagators

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$$\Delta^{\mu\nu}_{\mathrm{A}a}(q) \equiv i \int d^4x e^{iq\cdot x} \langle 0 | T(A^\mu_a(x) A^\nu_a(0)) | 0 \rangle \quad (a=3,8 \ \text{not summed}), \quad (1)$$

where our normalization for the $SU(3)$ octet of axial vector currents is standard,

$$A^\mu_a = \bar{q} \frac{\lambda_a}{2} \gamma^\mu \gamma_5 q \quad (a=1,8). \quad (2)$$

The propagators of Eq. (1) have the spectral content

$$\frac{1}{\pi} \operatorname{Im} \Delta^{\mu\nu}_{\mathrm{A}a}(q) = (q^\mu q^\nu - q^2 g^{\mu\nu}) \rho^{(1)}_{\mathrm{A}a}(q^2) + q^\mu q^\nu \rho^{(0)}_{\mathrm{A}a}(q^2), \quad (3)$$

as expressed in terms of the spin-one and spin-zero spectral functions $\rho^{(1)}_{\mathrm{A}a}$ and $\rho^{(0)}_{\mathrm{A}a}$. This motivates the following tensorial decomposition usually adopted in the literature:

$$\Delta^{\mu\nu}_{\mathrm{A}a}(q) = (q^\mu q^\nu - q^2 g^{\mu\nu}) \Pi^{(1)}_{\mathrm{A}a}(q^2) + q^\mu q^\nu \Pi^{(0)}_{\mathrm{A}a}(q^2), \quad (4)$$

where $\Pi^{(1)}_{\mathrm{A}a}$ and $\Pi^{(0)}_{\mathrm{A}a}$ are the spin-one and spin-zero polarization functions. The low-energy behavior of the spin-zero spectral function $\rho^{(0)}_{\mathrm{A}a}$ is dominated by the pole contribution associated with propagation of a Goldstone mode:

$$\rho^{(0)}_{\mathrm{A}a}(s) \equiv F_a^2 \delta(s - M_a^2) + \overline{\rho}^{(0)}_{\mathrm{A}a}(s). \quad (5)$$

In the chiral limit, one has $M_a \to 0$ and $\overline{\rho}^{(0)}_{\mathrm{A}a}(s) \to 0$ as well.

Adopting the approach carried out in Ref. [9], we make use of external axial vector sources [23] to determine the axial vector propagators. The procedure is simply to compute the $\mathcal{S}$-matrix element connecting initial and final states of an axial vector source. Analogous to the analysis of Ref. [9], the invariant amplitude is then guaranteed to be the axial vector current propagator, say of flavor a,

$$\langle a_a(q',\lambda') | \mathcal{S} - 1 | a_a(q,\lambda) \rangle = i(2\pi)^4 \delta^{(4)}(q'-q) \epsilon^\dagger_\mu(q',\lambda') \Delta^{\mu\nu}_{\mathrm{A}a}(q) \epsilon_\nu(q,\lambda). \quad (6)$$

We shall typically use the invariant amplitude symbol $\mathcal{M}_{\mu\nu a}$ to denote various individual contributions [tree, tadpole, counterterm, one-particle irreducible (1PI), etc.] to the full propagator.

B. Tree-level analysis

For definiteness, the analysis in the remainder of this section will refer to the isospin flavor. Given the $\mathcal{O}(p^2)$ chiral Lagrangian, it is straightforward to determine the lowest-order propagator contributions:

$$\mathcal{M}^{(\mathrm{tree})}_{\mu\nu 3} = F_0^2 g_{\mu\nu} - \frac{F_0^2}{q^2 - m_\pi^2 + i\epsilon} q_\mu q_\nu. \quad (7)$$

The two terms represent respectively contributions from a contact interaction [Fig. 1(a)] and a pion-pole term [Fig. 1(b)]. Although $\mathcal{M}^{(\mathrm{tree})}_{\mu\nu 3}$ has an exceedingly simple form, it is

FIG. 1. Lowest-order graphs for the axial vector propagator.

nonetheless worthwhile to briefly point out two of its features. First, as follows from unitarity there is an imaginary part corresponding to the pion single-particle intermediate state:

$$\operatorname{Im} \mathcal{M}^{(\mathrm{tree})}_{\mu\nu 3} = \pi F_0^2 \delta(q^2 - m_\pi^2) q_\mu q_\nu. \quad (8)$$

However, there is also a non-pole contribution to $\mathcal{M}^{(\mathrm{tree})}_{\mu\nu 3}$. Its presence is needed to ensure the proper behavior in the chiral limit ($\partial_\mu A^\mu_3 = 0$), where $\mathcal{R}e\mathcal{M}^{(\mathrm{tree})}_{\mu\nu 3}$ is required to obtain a purely spin-one (or "transverse") form. This is indeed the case, as we find by taking $m_\pi^2 \to 0$ in Eq. (7):

$$\mathcal{M}^{(\mathrm{tree})}_{\mu\nu 3}|_{m_\pi = 0} = -\frac{F_0^2}{q^2}(q_\mu q_\nu - q^2 g_{\mu\nu}). \quad (9)$$

C. Results through one-loop order

At the one-loop level, the axial vector current propagators are also determined from the $\mathcal{O}(p^2)$ chiral Lagrangian and correspond to the Feynman diagrams appearing in Fig. 2. The loop correction to the lowest-order contact amplitude appears in Fig. 2(a) and Figs. 2(b), 2(c) depict corrections to the pion-pole amplitude. There are also several $\mathcal{O}(p^4)$ counterterm diagrams, a contact term [Fig. 2(d)] and contributions to the pion pole term [Figs. 2(e), 2(f)]. The complete expression for the isospin axial vector current propagator through one-loop order is thus given by

$$\Delta_{\mathrm{A}\mu\nu 3} = \mathcal{M}^{(\mathrm{tree})}_{\mu\nu 3} + \mathcal{M}^{(\mathrm{tadpole})}_{\mu\nu 3} + \mathcal{M}^{(\mathrm{CT})}_{\mu\nu 3}. \quad (10)$$

Inclusion of $\mathcal{O}(p^4)$ counterterm contributions is necessary to absorb the singular dependence in $\mathcal{M}^{(\mathrm{tadpole})}_{\mu\nu 3}$ associated with the loop integral

$$A(m^2) \equiv \int d\tilde{k} \frac{1}{k^2 - m^2} = \mu^{d-4}\left[-2im^2\overline{\lambda} - \frac{im^2}{16\pi^2} \log\left(\frac{m^2}{\mu^2}\right) + \dots \right], \quad (11)$$

where $d\tilde{k} \equiv d^dk/(2\pi)^d$, μ is the mass scale associated with the use of dimensional regularization and $\overline{\lambda}$ is the singular quantity:

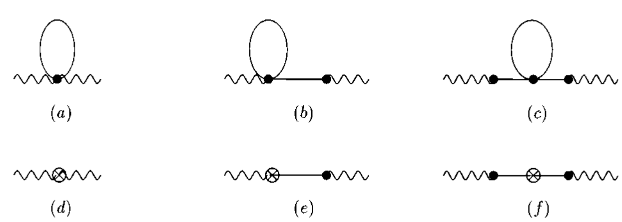

FIG. 2. One-loop graphs.

$$\overline{\lambda}=\frac{1}{16\pi^2}\left[\frac{1}{d-4}-\frac{1}{2}\left(\log 4\pi-\gamma+1\right)\right]. \tag{12}$$

The $\mathcal{O}(p^4)$ counterterms are the $\{L_i\}$ $(i=1,\ldots,10)$ and $\{H_j\}$ $(j=1,2)$ appearing in the $\mathcal{O}(p^4)$ chiral Lagrangian of Gasser and Leutwyler [3].[1] Each is expressible as an expansion in $\overline{\lambda}$, e.g.

$$L_l=\mu^{(d-4)}\sum_{n=1}^{-\infty} L_l^{(n)}(\mu)\overline{\lambda}^n$$
$$=\mu^{(d-4)}[L_l^{(1)}(\mu)\overline{\lambda}+L_l^{(0)}(\mu)+L_l^{(-1)}(\mu)\overline{\lambda}^{-1}+\ldots]. \tag{13}$$

This leads to renormalizations of the pion's mass and decay constant. A detailed account of the renormalization procedure is deferred to Sec. III. However, we note here that the renormalized masses and decay constants have the expansions

$$F^2=F^{2(0)}+F^{2(2)}+F^{2(4)}+\ldots,$$
$$M^2=M^{2(2)}+M^{2(4)}+M^{2(6)}+\ldots, \tag{14}$$

where the flavor notation is temporarily suppressed and the superscript indices $\{(i)\}$ denote quantities evaluated at chiral order $\{p^i\}$. To one loop [3], the explicit expressions are given by[2]

$$F_\pi^2=F_0^2+8[(m_\pi^2+2m_K^2)L_4^{(0)}+m_\pi^2L_5^{(0)}]-\frac{2m_\pi^2}{16\pi^2}\ln\frac{m_\pi^2}{\mu^2}-\frac{m_K^2}{16\pi^2}\ln\frac{m_K^2}{\mu^2},$$

$$M_\pi^2=m_\pi^2+\frac{1}{F_0^2}\left[\frac{m_\pi^4}{32\pi^2}\ln\frac{m_\pi^2}{\mu^2}-\frac{m_\pi^2m_\eta^2}{96\pi^2}\ln\frac{m_\eta^2}{\mu^2}-8m_\pi^2[(m_\pi^2+2m_K^2)(L_4^{(0)}-2L_6^{(0)})+m_\pi^2(L_5^{(0)}-2L_8^{(0)})]\right]. \tag{15}$$

Upon combining the information gathered in Eqs. (10), (15) as well as the divergent part of the one-loop functional given in Ref. [3], one obtains, for the renormalized isospin propagator through one-loop order,

$$\Delta_{A3}^{\mu\nu}=F_\pi^2g^{\mu\nu}+2(L_{10}^{(0)}-2H_1^{(0)})(q^\mu q^\nu-q^2g^{\mu\nu})-\frac{F_\pi^2}{q^2-M_\pi^2}q^\mu q^\nu. \tag{16}$$

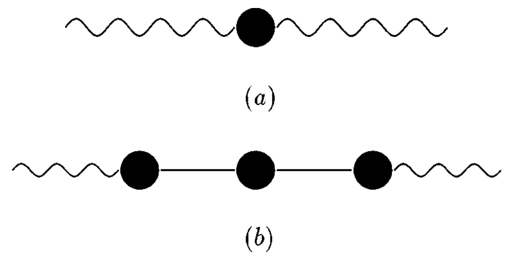

FIG. 3. Generic corrections to the axial vector propagator.

The hypercharge propagator $\Delta_{A8}^{\mu\nu}$ is found analogously. Both the isospin and hypercharge amplitudes contain the regularization dependent constant $H_1^{(0)}$, and are therefore unphysical. A physically observable quantity is obtained from the difference

$$\Delta_{A3}^{\mu\nu}-\Delta_{A8}^{\mu\nu}=(F_\pi^2-F_\eta^2)g^{\mu\nu}-q^\mu q^\nu\left[\frac{F_\pi^2}{q^2-M_\pi^2}-\frac{F_\eta^2}{q^2-M_\eta^2}\right]. \tag{17}$$

D. Overview of two-loop amplitudes

The general structure of propagator corrections to arbitrary order is displayed in Fig. 3, the one-particle irreducible (1PI) diagrams of Fig. 3(a) and the one-particle reducible (1PR) diagrams of Fig. 3(b). The latter consists of both vertex and self-energy corrections.

The 1PI diagrams are displayed in Fig. 4 and Fig. 5. The processes in Fig. 4 are analogous to two-loop contributions occurring for the vector current propagators [9], but the graph of Fig. 5 (the so-called ''sunset graph'') has no counterpart in the vector system. For convenience, we compile definitions and explicit representations for sunset-related functions in the Appendix. The 1PR two-loop vertex $\Gamma_{\mu a}^{(4)}$ and the two-loop self-energy $\Sigma_a^{(6)}$ are the sum of amplitudes depicted respectively in Figs. 6(a)–6(e) and in Figs. 7(a)–7(e). The vertex amplitude $\Gamma_{\mu a}^{(4)}$ is the product of the four-momentum q^μ and an invariant function $\Gamma_a(q^2)$:

$$\Gamma_a^\mu(q)\equiv iq^\mu\Gamma_a(q^2). \tag{18}$$

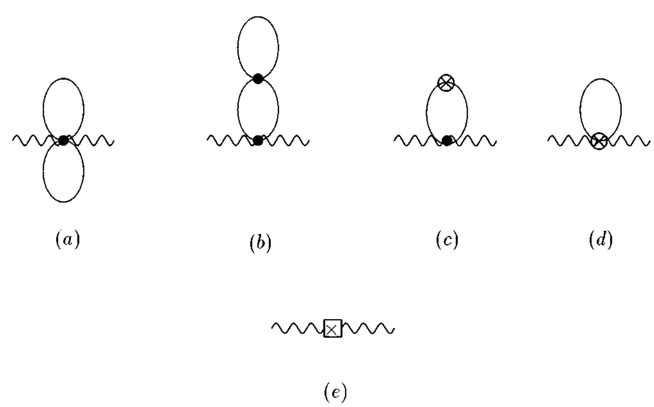

FIG. 4. Two-loop 1PI non-sunset graphs.

[1]See also the discussion surrounding Eq. (26) of Ref. [9].

[2]At one-loop order, our counterterms $\{L_i^{(0)}\}$ and $\{H_i^{(0)}\}$ are equivalent to the $\{L_i^r\}$ and $\{H_i^r\}$ of Ref. [3].

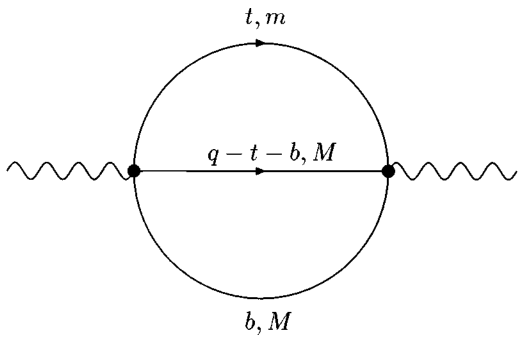

FIG. 5. The two-loop 1PI sunset graph.

Sunset contributions for the vertex and self-energy functions correspond to Figs. 6(e), 7(e).

Rather than giving the exceedingly cumbersome expressions for these quantities here, it suffices to note the following features:

(1) The leading singularities in both the 1PI and 1PR sectors go as $\overline{\lambda}^2$.
(2) All the non-sunset vertex quantities Γ_a are *independent* of the propagator momentum q^2.
(3) All the non-sunset self-energies are at most *linear* in the propagator momentum q^2.

To construct the counterterm amplitudes needed to subtract off divergences and scale dependence contained in the 1PI and 1PR two-loop graphs, we refer to collection $\{B_l\}$ of $\mathcal{O}(p^6)$ counterterms due to Fearing and Scherer [24]. The $\{B_l\}$ are expressed in dimensional regularization as[3]

$$B_l = \mu^{2(d-4)} \sum_{n=2}^{-\infty} B_l^{(n)}(\mu) \overline{\lambda}^n$$
$$= \mu^{2(d-4)} [B_l^{(2)}(\mu)\overline{\lambda}^2 + B_l^{(1)}(\mu)\overline{\lambda} + B_l^{(0)}(\mu) + \dots]. \tag{19}$$

This representation, as with Eq. (13) for the $\mathcal{O}(p^4)$ counterterms $\{L_l\}$, is an expansion in the singular quantity $\overline{\lambda}$. Our two-loop analysis reveals that 23 Fearing-Scherer counterterms ultimately contribute to the axial vector propagators. Of these, only five are explicitly present in the isospin and hypercharge polarization functions; the rest occur in the renormalization of masses and decay constants at two-loop order.

III. RENORMALIZATION PROCEDURE

The result of calculating all relevant Feynman diagrams through two-loop order is to obtain an expression for the axial vector propagator $\Delta_{\mathrm{A}a}^{\mu\nu}$ having contributions from both the 1PI part $\mathcal{M}_a^{\mu\nu}$ and the 1PR pole term:

[3]The dependence of $L_l^{(n)}(\mu)$ and $B_l^{(n)}(\mu)$ upon the scale μ is determined from the renormalization group equations and has been explicitly given in Ref. [9].

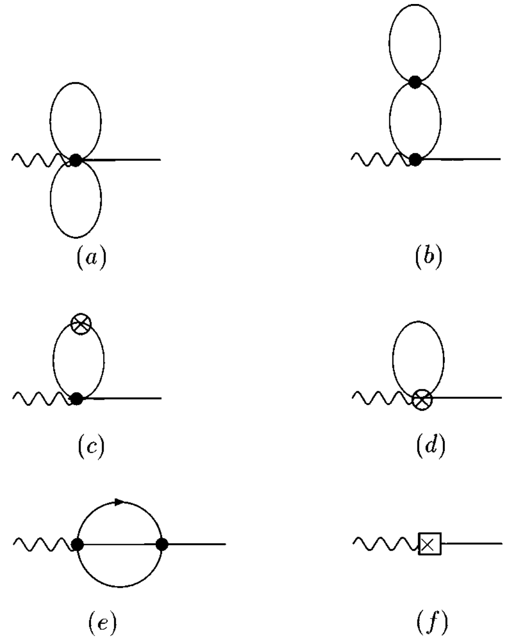

FIG. 6. Two-loop 1PI vertex graphs.

$$\Delta_{\mathrm{A}a}^{\mu\nu}(q) = \mathcal{M}_a^{\mu\nu}(q) - q^\mu q^\nu \frac{\Gamma_a^2(q^2)}{q^2 - m_a^2 + \Sigma_a(q^2)} \quad (a=3,8). \tag{20}$$

In the above, the $\mathcal{O}(p^6)$ counterterms are understood to be already included in $\mathcal{M}_a^{\mu\nu}$, in the self-energy Σ_a and also in $\Gamma_a(q^2)$. The renormalized mass M_a and decay constant F_a are defined as parameters occurring in the meson pole term

$$q^\mu q^\nu \frac{\Gamma_a^2(q^2)}{q^2 - m_a^2 + \Sigma_a(q^2)} \equiv q^\mu q^\nu \left(\frac{F_a^2}{q^2 - M_a^2} + R_a(q^2) \right), \tag{21}$$

where $R_a(q^2)$ is a remainder term having no poles.

A. Identification of the meson mass

From Eqs. (20),(21), it follows that M_a^2 is a solution of the implicit relation

$$M_a^2 = m_a^2 - \Sigma_a(M_a^2). \tag{22}$$

Since we have already calculated $\Sigma(m^2)$ (in the following, we temporarily omit flavor indices), it makes sense to expand the self-energy $\Sigma(M^2)$ as

$$\Sigma(M^2) = \Sigma(m^2) + \Sigma'(m^2)(M^2 - m^2) + \dots . \tag{23}$$

Then, expressing the squared-mass perturbatively,

$$M^2 = M^{(2)2} + M^{(4)2} + M^{(6)2} + \dots, \tag{24}$$

and similarly for the self-energy, we obtain the perturbative chain

$$M^{(2)2} = m^2, \tag{25}$$

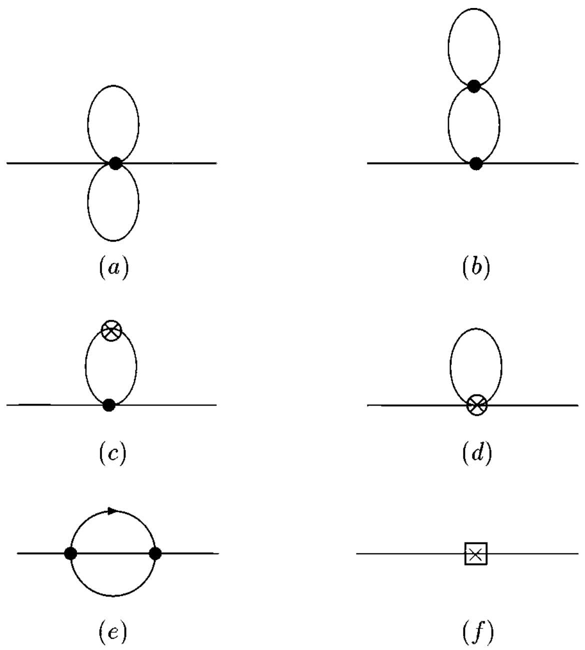

FIG. 7. Two-loop 1PI self-energy graphs.

$$M^{(4)2} = -\Sigma^{(4)}(m^2), \tag{26}$$

$$M^{(6)2} = -\Sigma^{(6)}(m^2) + \Sigma^{(4)\prime}(m^2)\Sigma^{(4)}(m^2), \tag{27}$$

where we have noted that

$$M^2 - m^2 = \mathcal{O}(q^4) = -\Sigma^{(4)}(m^2) + \ \ldots\ . \tag{28}$$

We exhibit this procedure at the one-loop level for the pion mass, beginning with the fourth-order self-energy

$$\Sigma_3^{(4)}(q^2) = \frac{i}{F_0^2}\left[\frac{m_\pi^2 - 4q^2}{6}A(\pi) + \frac{m_\pi^2 - q^2}{3}A(K) + \frac{m_\pi^2}{6}A(\eta)\right] + \frac{8}{F_0^2}[(m_\pi^2 + 2m_K^2)(q^2L_4 - 2m_\pi^2 L_6) + m_\pi^2(q^2L_5 - 2m_\pi^2 L_8)], \tag{29}$$

where A is defined in Eq. (11). We then obtain, from Eq. (26),

$$M_\pi^{(4)2} = \frac{i}{F_0^2}\left[\frac{3A(\pi) - A(\eta)}{6}\right] - \frac{8m_\pi^2}{F_0^2}[(m_\pi^2 + 2m_K^2)(L_4 - 2L_6) + m_\pi^2(L_5 - 2L_8)]. \tag{30}$$

Expanding the A-integrals in a Laurent series around $d=4$ and rewriting the $\{L_i\}$ in terms of the renormalized quantities $\{L_i^{(0)}\}$, we readily obtain the result cited earlier in Eq. (15). The analysis at two-loop level proceeds analogously.

B. Identification of the meson decay constant

With the aid of Eq. (22), it is straightforward to write the meson pole term in the form (again temporarily suspending flavor indices)

$$\frac{\Gamma^2(q^2)}{q^2 - m^2 - \Sigma(q^2)} = \frac{\Gamma^2(q^2)}{q^2 - M^2}\,\frac{1}{1 + \tilde{\Sigma}(q^2)} = \frac{\Gamma^2(q^2)[1 - \tilde{\Sigma}(q^2) + \tilde{\Sigma}^2(q^2) + \ \ldots\ ]}{q^2 - M^2}, \tag{31}$$

where $\tilde{\Sigma}(q^2)$ is the divided difference:

$$\tilde{\Sigma}(q^2) \equiv \frac{\Sigma(q^2) - \Sigma(M^2)}{q^2 - M^2}. \tag{32}$$

From the definition in Eq. (21) of the squared decay constant F^2, we have

$$F^2 = \lim_{q^2 = M^2}[\Gamma^2(q^2)\{1 - \tilde{\Sigma}(q^2) + [\tilde{\Sigma}(q^2)]^2 + \ldots\}]. \tag{33}$$

Let us analyze this relation perturbatively.

We begin by writing the vertex quantity $\Gamma(q^2)$ (evaluated at $q^2 = M^2$) as

$$\Gamma(M^2) = \Gamma^{(0)} + \Gamma^{(2)} + \Gamma^{(4)}(M^2) + \ \ldots, \tag{34}$$

where both $\Gamma^{(0)}$ and $\Gamma^{(2)}$ are independent of q^2. Expanding $\Gamma^{(4)}(M^2)$ as

$$\Gamma^{(4)}(M^2) = \Gamma^{(4)}(m^2) + \Gamma^{(4)\prime}(m^2)(M^2 - m^2) + \ \ldots, \tag{35}$$

we see from chiral counting that to the order at which we are working, one is justified in replacing $\Gamma^{(4)}(M^2)$ by $\Gamma^{(4)}(m^2)$. As for the self-energy dependence in Eq. (33), we first observe that

$$\tilde{\Sigma}(M^2) = \lim_{q^2 = M^2}\tilde{\Sigma}(q^2) = \Sigma'(M^2). \tag{36}$$

Recalling that $\Sigma^{(4)}(q^2)$ is linear in q^2, we have the perturbative expression

$$\Sigma'(M^2) = \Sigma^{(4)\prime} + \Sigma^{(6)\prime}(M^2) + \ldots = \Sigma^{(4)\prime} + \Sigma^{(6)\prime}(m^2) + \ \ldots, \tag{37}$$

where the error in replacing $\Sigma^{(6)\prime}(M^2)$ by $\Sigma^{(6)\prime}(m^2)$ appears in higher order. Thus, the perturbative content of Eq. (33) reduces to

$$F^2 = [\Gamma^{(0)} + \Gamma^{(2)} + \Gamma^{(4)}(m^2)]^2 \times [1 - \Sigma^{(4)\prime} - \Sigma^{(6)\prime}(m^2) + (\Sigma^{(4)\prime})^2] + \ \ldots\ . \tag{38}$$

Upon organizing terms in ascending chiral powers, we obtain the following expression, valid through two-loop order, for the decay constant:

$$F=\Gamma^{(0)}+\left[\Gamma^{(2)}-\frac{1}{2}\Gamma^{(0)}\Sigma^{(4)\prime}\right]+\left[\Gamma^{(4)}(m^2)-\frac{1}{2}\Gamma^{(2)}\Sigma^{(4)\prime}+\Gamma^{(0)}\left(-\frac{1}{2}\Sigma^{(6)\prime}(m^2)+\frac{3}{8}(\Sigma^{(4)\prime})^2\right)\right]+\dots, \tag{39}$$

where terms of a given order have been collected together.

As an example, Eq. (39) readily provides a determination of the isospin decay constant through one-loop order. The corresponding vertex quantity is

$$\Gamma_\pi=F_0-\frac{2i}{3F_0}[2A(\pi)+A(K)]+\frac{8}{F_0}[(m_\pi^2+2m_K^2)L_4+m_\pi^2L_5]+\dots. \tag{40}$$

Upon inferring $\Sigma_3^{(4)\prime}$ from Eq. (29), we obtain

$$F_\pi^{(0)}+F_\pi^{(2)}=F_0-\frac{i}{2F_0}[2A(\pi)+A(K)]+\frac{4}{F_0}[(m_\pi^2+2m_K^2)L_4+m_\pi^2L_5]. \tag{41}$$

The one-loop expression for the pion decay constant [appearing in Eq. (15)] follows from the above relation. The two-loop isospin corrections are found analogously, as are the corrections to the eta decay constant.

C. Remainder term

The preceding work allows the extraction of the remainder term $R(q^2)$ defined earlier in Eq. (21). Making use of Eqs. (24)–(27) as well as Eq. (39), a straightforward calculation yields the expression

$$R(q^2)=2F_0\frac{\Gamma^{(4)}(q^2)-\Gamma^{(4)}(m^2)}{q^2-m^2}-F_0^2\frac{\Sigma^{(6)}(q^2)-\Sigma^{(6)}(m^2)-(q^2-m^2)\Sigma^{(6)\prime}(m^2)}{(q^2-m^2)^2}. \tag{42}$$

It is manifest that $R(q^2)$ has no pole at $q^2=m^2$. Moreover, since the non-sunset vertex functions $\Gamma^{(4)}$, (a)–(d), of Fig. 6 are constant in q^2, they do not contribute to $R(q^2)$. Nor do the non-sunset self-energies $\Sigma^{(6)}$, (a)–(d), since they are at most linear in q^2. Thus, the only contributors to $R(q^2)$ are sunset amplitudes, and it is straightforward to obtain the isospin and hypercharge remainder functions directly from Eq. (42).

D. Polarization amplitudes $\hat{\Pi}^{(1)}$ and $\hat{\Pi}^{(0)}$

From the tree-level and one-loop results, we anticipate that the two-loop 1PI amplitude $\mathcal{M}_{\mu\nu}^{(4)}$ will contain an additive term involving the meson decay constant, and can thus be written

$$\mathcal{M}_{\mu\nu}^{(4)}\equiv g_{\mu\nu}(F^2)^{(4)}+\hat{\mathcal{M}}_{\mu\nu}, \tag{43}$$

where $\hat{\mathcal{M}}_{\mu\nu}$ denotes the residual part of the 1PI amplitude. It turns out that much of the rather complicated content in the 1PI amplitudes is attributable to the two-loop squared decay constant $(F^2)^{(4)}$, as can be verified from decay constant results derived earlier in this section. First, we re-express the expansion of Eq. (38) as

$$F^2=(F^2)^{(0)}+(F^2)^{(2)}+(F^2)^{(4)}+\dots, \tag{44}$$

with $(F^2)^{(0)}=F_0^2$, $(F^2)^{(2)}=2F_0F^{(2)}$ and

$$(F^2)^{(4)}=(F^{(2)})^2+2F_0F^{(4)}. \tag{45}$$

One then compares the $g_{\mu\nu}$ part of $\mathcal{M}_{\mu\nu}^{(4)}$ with $(F^2)^{(4)}$. The residual amplitude $\hat{\mathcal{M}}_{\mu\nu}$ is simply the difference of these.

It is convenient to replace the residual amplitude $\hat{\mathcal{M}}^{\mu\nu}$ and the remainder term $-q^\mu q^\nu R(q^2)$ (which arises from the meson pole term but itself contains no poles) by the equivalent quantities $\hat{\Pi}^{(1)}$ and $\hat{\Pi}^{(0)}$:

$$\hat{\mathcal{M}}^{\mu\nu}(q)-q^\mu q^\nu R(q^2)\equiv(q^\mu q^\nu-q^2g^{\mu\nu})\hat{\Pi}^{(1)}(q^2)+g^{\mu\nu}\hat{\Pi}^{(0)}(q^2). \tag{46}$$

We shall employ $\hat{\Pi}^{(1)}$ and $\hat{\Pi}^{(0)}$ throughout the rest of the paper.

IV. REMOVAL OF DIVERGENCES

Thus far, we have derived lengthy expressions for the various two-loop components of the axial vector propagators and then determined the renormalization structure of the masses and decay constants. At this stage of the calculation, there are many terms which diverge as $d\to4$ and which must therefore be removed from the description. Below, we carry out the subtraction procedure by expanding the relevant quantities in powers of the parameter $\bar{\lambda}$ and then using $\mathcal{O}(p^6)$ and $\mathcal{O}(p^4)$ counterterms to cancel the singular contributions. In particular, this process will determine a subset of the so-called β-functions of the complete $\mathcal{O}(p^6)$ Lagrangian. This is of special interest because the divergence structure of the generating functional to two-loop level can be obtained in closed form [25], and our results derived in the following can be used as checks of such future calculations.

A. Removal of $\bar{\lambda}^2$ dependence

It will simplify the following discussion to define the counterterm combinations

$$A\equiv2B_{14}-B_{17},\quad B\equiv B_{16}+B_{18},$$

$$C\equiv B_{15}-B_{20}, \quad D=B_{19}+B_{21},$$

$$E\equiv B_{19}-B_{21}, \quad F\equiv 3B_1+2B_2. \tag{47}$$

Consider first the decay constant sector. Upon demanding that all $\overline{\lambda}^2$ dependence vanish, we obtain six equations containing five variables. There are *six* equations because the pion and eta constants each have an explicit dependence on m_π^4, $m_\pi^2 m_K^2$ and m_K^4 factors and the singular behavior must be subtracted for each of these. We find the equation set to be degenerate, and one obtains just the following four conditions:

$$A^{(2)}-3E^{(2)}=-\frac{31}{24F_0^2}, \quad B^{(2)}-2E^{(2)}=-\frac{53}{72F_0^2},$$

$$C^{(2)}+E^{(2)}=\frac{13}{18F_0^2}, \quad D^{(2)}=\frac{73}{144F_0^2}. \tag{48}$$

The information contained in the above set is unique and any other way of expressing the solution must be equivalent.

In the mass sector, we find subtraction at the $\overline{\lambda}^2$ level to yield *seven* equations in 11 variables. The number of equations follows from the dependence of each mass on m_π^6, $m_\pi^4 m_K^2$ and $m_\pi^2 m_K^4$ factors (this implies six equations), along with the fact that m_K^6 dependence is absent from $M_\pi^{(6)2}$. This latter fact arises because there is no m_K^6 counterterm contribution in $M_\pi^{(6)2}$, and thus there must be a cancellation between sunset and nonsunset numerical terms. Such a cancellation occurs and constitutes an important check on our determination of the sunset contribution. The equation set for the $M^{(6)2}$ masses is found to be degenerate and just five constraints can be obtained, e.g.

$$B_3^{(2)}=\frac{4}{27F_0^2}-\frac{1}{6}F^{(2)}-B_4^{(2)}-B_5^{(2)}-3B_7^{(2)},$$

$$B_6^{(2)}=-\frac{16}{81F_0^2}+\frac{1}{18}F^{(2)}+\frac{1}{3}B_4^{(2)}+\frac{1}{3}B_5^{(2)}+B_7^{(2)},$$

$$B_{14}^{(2)}=\frac{1}{48}\left[-\frac{37}{F_0^2}+72B_4^{(2)}+72B_5^{(2)}+216B_7^{(2)}\right],$$

$$B_{15}^{(2)}=\frac{307}{216F_0^2}-\frac{1}{3}F^{(2)}-B_4^{(2)}-2B_5^{(2)}-3B_7^{(2)},$$

$$B_{16}^{(2)}=-\frac{91}{72F_0^2}+\frac{1}{6}F^{(2)}+2B_4^{(2)}+B_5^{(2)}+3B_7^{(2)}. \tag{49}$$

Finally, removal of the $\overline{\lambda}^2$-dependence for the polarization functions $\hat{\Pi}^{(0)}$ and $\hat{\Pi}^{(1)}$ yields a final set of constraints at this order:

$$B_{11}^{(2)}=B_{13}^{(2)}=0, \quad B_{33}^{(2)}=2B_{32}^{(2)},$$

$$B_{28}^{(2)}-B_{46}^{(2)}=-\frac{3}{16F_0^2}, \quad B_{29}^{(2)}-2B_{49}^{(2)}=-\frac{1}{8F_0^2}. \tag{50}$$

B. Removal of the $\overline{\lambda}$ dependence

In a similar manner, we obtain constraints for the $\{B^{(1)}\}$ counterterms. We list these below without further comment, beginning with those following from decay constants,

$$A^{(1)}-3E^{(1)}=\frac{1}{F_0^2}\left[-\frac{175}{9216\pi^2}-\frac{28}{3}L_1^{(0)}-\frac{34}{3}L_2^{(0)}-\frac{25}{3}L_3^{(0)}+\frac{26}{3}L_4^{(0)}-\frac{8}{3}L_5^{(0)}-12L_6^{(0)}+12L_8^{(0)}\right],$$

$$B^{(1)}-2E^{(1)}=\frac{1}{F_0^2}\left[\frac{19}{1536\pi^2}-\frac{32}{9}L_1^{(0)}-\frac{8}{9}L_2^{(0)}-\frac{8}{9}L_3^{(0)}+\frac{106}{9}L_4^{(0)}-\frac{22}{9}L_5^{(0)}-20L_6^{(0)}\right],$$

$$C^{(1)}+E^{(1)}=\frac{1}{F_0^2}\left[\frac{691}{82944\pi^2}+\frac{28}{9}L_1^{(0)}+\frac{34}{9}L_2^{(0)}+\frac{59}{18}L_3^{(0)}-\frac{26}{9}L_4^{(0)}+3L_5^{(0)}+4L_6^{(0)}-6L_8^{(0)}\right],$$

$$D^{(1)}=\frac{1}{F_0^2}\left[\frac{43}{3072\pi^2}+\frac{104}{9}L_1^{(0)}+\frac{26}{9}L_2^{(0)}+\frac{61}{18}L_3^{(0)}-\frac{34}{9}L_4^{(0)}+L_5^{(0)}-4L_6^{(0)}-2L_8^{(0)}\right], \tag{51}$$

then masses,

$$B_3^{(1)}=\frac{1}{F_0^2}\left[\frac{5}{216\pi^2}-\frac{20}{3}L_4^{(0)}-\frac{23}{3}L_5^{(0)}+\frac{40}{3}L_6^{(0)}+40L_7^{(0)}+\frac{86}{3}L_8^{(0)}\right]-\frac{1}{6}F^{(1)}-B_4^{(1)}-B_5^{(1)}-3B_7^{(1)},$$

$$B_6^{(1)}=\frac{1}{648}\left[\frac{1}{F_0^2}\left(-\frac{5}{\pi^2}-3168L_4^{(0)}-24L_5^{(0)}+6336L_6^{(0)}-9792L_7^{(0)}-3216L_8^{(0)}\right)+36F^{(1)}+216B_4^{(1)}+216B_5^{(1)}+648B_7^{(1)}\right],$$

$$B_{14}^{(1)}=\frac{1}{F_0^2}\left[-\frac{167}{4608\pi^2}+8L_6^{(0)}-64L_7^{(0)}-\frac{62}{3}L_8^{(0)}\right]+\frac{3}{2}B_4^{(1)}+\frac{3}{2}B_5^{(1)}+\frac{9}{2}B_7^{(1)},$$

$$B_{15}^{(1)}=\frac{1}{F_0^2}\left[\frac{371}{20736\pi^2}+2L_5^{(0)}-\frac{16}{3}L_6^{(0)}+72L_7^{(0)}+24L_8^{(0)}\right]$$
$$-\frac{1}{3}F^{(1)}-B_4^{(1)}-2B_5^{(1)}-3B_7^{(1)},$$
$$B_{16}^{(1)}=\frac{1}{F_0^2}\left[-\frac{9}{512\pi^2}-L_5^{(0)}-\frac{152}{3}L_7^{(0)}-\frac{62}{3}L_8^{(0)}\right]$$
$$+\frac{1}{6}F^{(1)}+2B_4^{(1)}+B_5^{(1)}+3B_7^{(1)}, \quad (52)$$

and finally from the polarizations $\hat{\Pi}^{(0)}$ and $\hat{\Pi}^{(1)}$,

$$B_{11}^{(1)}=-\frac{49}{576}\frac{1}{16\pi^2F_0^2},\quad B_{13}^{(1)}=\frac{173}{5184}\frac{1}{16\pi^2F_0^2},$$
$$2B_{32}^{(1)}-B_{33}^{(1)}=\frac{3}{64}\frac{1}{16\pi^2F_0^2},$$
$$B_{28}^{(1)}-B_{46}^{(1)}=\frac{1}{F_0^2}\left[-\frac{5}{64}\frac{1}{16\pi^2}+\frac{3}{2}L_{10}^{(0)}\right],$$
$$B_{29}^{(1)}-2B_{49}^{(1)}=\frac{1}{F_0^2}\left[-\frac{17}{96}\frac{1}{16\pi^2}+L_{10}^{(0)}\right]. \quad (53)$$

This completes the subtraction part of the calculation.

C. $\bar{\lambda}$-subtraction and $\overline{\mathrm{MS}}$ renormalization schemes

The renormalization procedure employed originally in Ref. [9] and adopted in this paper amounts to $\bar{\lambda}$-subtraction; cf. Eqs. (13),(19). Alternatively, one could employ minimal subtraction (MS):

$$L_l(d)=\frac{\mu^{2\omega}}{(4\pi)^2}\left[\frac{\Gamma_l}{2\omega}+L_{l,r}^{\mathrm{MS}}(\mu,\omega)+\ldots\right],$$
$$B_l(d)=\frac{\mu^{4\omega}}{(4\pi)^4}\left[\frac{B_l^{(2)\mathrm{MS}}}{(2\omega)^2}+\frac{B_l^{(1)\mathrm{MS}}}{2\omega}+B_{l,r}^{(0)\mathrm{MS}}(\mu,\omega)+\ldots\right], \quad (54)$$

where $\omega\equiv d/2-2$. Yet another approach is modified minimal subtraction ($\overline{\mathrm{MS}}$):

$$L_l(d)=\frac{(\mu c)^{2\omega}}{(4\pi)^2}\left[\frac{\Gamma_l}{2\omega}+L_{l,r}^{\overline{\mathrm{MS}}}(\mu,\omega)+\ldots\right]$$
$$B_l(d)=\frac{(\mu c)^{4\omega}}{(4\pi)^4}\left[\frac{B_l^{(2)\mathrm{MS}}}{(2\omega)^2}+\frac{B_l^{(1)\mathrm{MS}}}{2\omega}\right.$$
$$\left.+(4\pi)^4B_{l,r}^{(0)\overline{\mathrm{MS}}}(\mu)+\ldots\right], \quad (55)$$

with the standard ChPT choice

$$\ln c=-\frac{1}{2}[1-\gamma_{\mathrm{E}}+\ln(4\pi)]\equiv -C. \quad (56)$$

Of course, there must be only finite differences between these three procedures, amounting to additional finite renormalizations.

As an illustration, we relate the $\{B_l^{(n)}\}$ and $\{L_l^{(n)}\}$ renormalization constants employed here to those defined in the $\overline{\mathrm{MS}}$ approach of Ref. [17]. In the latter scheme, one writes further that

$$L_{l,r}^{\overline{\mathrm{MS}}}(\mu,\omega)=L_{l,r}^{\overline{\mathrm{MS}}}(\mu,0)+L_{l,r}^{\overline{\mathrm{MS}}\prime}(\mu,0)\omega+\ldots \quad (57)$$

and sets

$$L_{l,r}^{\overline{\mathrm{MS}}\prime}(\mu,0)=0. \quad (58)$$

The ''convention'' established by Eq. (58) is allowed because it can be shown [17] that the effect of the quantity $L_{l,r}^{\overline{\mathrm{MS}}\prime}(\mu,0)$ is to add a local contribution at order p^6, which can always be absorbed into the couplings of the $\mathcal{O}(p^6)$ Lagrangian. Comparison of the two approaches yields

$$L_l^{(1)}=\Gamma_l,$$
$$L_l^{(0)}=\frac{1}{(4\pi)^2}L_{l,r}^{\overline{\mathrm{MS}}}(\mu,0)\equiv L_l^r(\mu),$$
$$L_l^{(-1)}=\frac{C}{(4\pi)^4}\left[-L_{l,r}^{\overline{\mathrm{MS}}}(\mu,0)+\frac{C\Gamma_l}{2}\right] \quad (59)$$

and

$$B_l^{(2)}=B_l^{(2)\mathrm{MS}},$$
$$B_l^{(1)}=\frac{1}{(4\pi)^2}B_l^{(1)\mathrm{MS}},$$
$$B_l^{(0)}(\mu)=B_{l,r}^{\overline{\mathrm{MS}}}(\mu)-\frac{C}{(4\pi)^4}B_l^{(1)\mathrm{MS}}+\frac{C^2}{(4\pi)^4}B_l^{(2)\mathrm{MS}}. \quad (60)$$

We stress that the content of Eqs. (59),(60) is partly a reflection of the convention of Eq. (58). Finally, one can combine the relations of Eq. (60) to write

$$B_{l,r}^{\overline{\mathrm{MS}}}(\mu)=B_l^{(0)}(\mu)+\frac{C}{(4\pi)^2}B_l^{(1)}-\frac{C^2}{(4\pi)^4}B_l^{(2)}. \quad (61)$$

We shall return to the comparison between the $\bar{\lambda}$-subtraction and $\overline{\mathrm{MS}}$ renormalizations at the end of Sec. V.

V. RENORMALIZED ISOSPIN POLARIZATION FUNCTIONS

Removal of the $\bar{\lambda}^2$ and $\bar{\lambda}^1$ singular dependence from the theory leaves the $\bar{\lambda}^0$ sector, in which all quantities are finite. Such contributions can be expressed entirely in terms of physical quantities upon replacing the tree-level parameters $m_\pi^2,m_K^2,m_\eta^2,F_0$ with $M_\pi^2,M_K^2,M_\eta^2,F_\pi$. Any error thereby induced appears in higher orders.

For any observable $\mathcal{O}$ (e.g. $\mathcal{O}=\hat{\Pi}_3^{(1)}$, M_π, F_π, etc.)

evaluated at the two-loop level, three classes of finite contributions are encountered:

$$\mathcal{O}=\mathcal{O}_{\rm REM}+\mathcal{O}_{\rm CT}+\mathcal{O}_{\rm SUN}. \tag{62}$$

$\mathcal{O}_{\rm REM}$ refers to the finite $\overline{\lambda}^0$ ''remnants'' in $\{A(k)A(l)\}$ or $\{A(k)L_l\}$ contributions and arises from either the product of two $\overline{\lambda}^0$ factors or the product of $\overline{\lambda}^1$ and $\overline{\lambda}^{-1}$ factors. $\mathcal{O}_{\rm CT}$ denotes any term containing the $\{B_l^{(0)}\}$ p^6-counterterms (CTs), whereas $\mathcal{O}_{\rm SUN}$ represents contributions from finite integrals [cf. Eqs. (A21),(A22) of the Appendix] which occur solely in sunset amplitudes. Since the finite results are quite lengthy, it will suffice to consider just the isospin polarization functions among the full set of observables occurring in this calculation.

Before proceeding, we address a technical issue related to the presence of $L_l^{(-1)}$ terms appearing in the ''remnant'' amplitudes. Such terms are always multiplied by polynomials in the quark masses and hence can be absorbed by the $\mathcal{O}(p^6)$ counterterms, as expected from general renormalization theorems [17]. In the vector current analysis of Ref. [9], we defined the dimensionless quantities (now expressed in terms of the $\{B_l\}$)

$$P\equiv 4F_\pi^2(-2B_{30}^{(0)}+B_{31}^{(0)})+4L_9^{(-1)},$$
$$Q\equiv 2F_\pi^2B_{47}^{(0)}-3(L_9^{(-1)}+L_{10}^{(-1)}),$$
$$R\equiv 2F_\pi^2B_{50}^{(0)}-(L_9^{(-1)}+L_{10}^{(-1)}), \tag{63}$$

and thus removed all explicit $L_l^{(-1)}$ dependence. We repeat that procedure here by defining the axial vector quantities

$$P_{\rm A}\equiv 4F_\pi^2(-2B_{32}^{(0)}+B_{33}^{(0)}),$$
$$Q_{\rm A}\equiv 2F_\pi^2(-B_{28}^{(0)}+B_{46}^{(0)})+3L_{10}^{(-1)},$$
$$R_{\rm A}\equiv F_\pi^2(-B_{29}^{(0)}+2B_{49}^{(0)})+L_{10}^{(-1)}, \tag{64}$$

such that the counterterm dependence of $\Pi^{(1)}_{\rm A3,8}$ is identical to that established originally for $\Pi_{\rm V3,8}$ [9].

A. Results

Consider first the spin-one isospin polarization amplitude. The remnant part $F_\pi^2\hat{\Pi}^{(1)}_{3,\rm REM}$ consists of terms proportional to M_π^2, M_K^2 and q^2, each multiplied by numerical quantities and chiral logarithms:

$$\begin{aligned}F_\pi^2\hat{\Pi}^{(1)}_{3,\rm REM}(q^2)=&\frac{M_\pi^2}{\pi^4}\left[\frac{49}{13824}-\frac{C}{192}+\log\frac{M_K^2}{\mu^2}\left(\frac{1}{576}+\frac{1}{1536}\log\frac{M_K^2}{\mu^2}\right)\right.\\&\left.+\log\frac{M_\pi^2}{\mu^2}\left(\frac{1}{288}-\frac{\pi^2}{2}L_{10}^{(0)}-\frac{1}{768}\log\frac{M_\pi^2}{\mu^2}-\frac{1}{768}\log\frac{M_K^2}{\mu^2}\right)\right]\\&+\frac{M_K^2}{\pi^4}\left[\frac{5}{36864}-\frac{17C}{3072}+\log\frac{M_K^2}{\mu^2}\left(\frac{17}{3072}-\frac{\pi^2}{4}L_{10}^{(0)}-\frac{1}{1024}\log\frac{M_K^2}{\mu^2}\right)\right]\\&+\frac{q^2}{\pi^4}\left[-\frac{283}{294912}+\frac{3C}{4096}-\frac{1}{3072}\log\frac{M_\pi^2}{\mu^2}-\frac{5}{12288}\log\frac{M_K^2}{\mu^2}\right].\end{aligned} \tag{65}$$

The counterterm part $F_\pi^2\hat{\Pi}^{(1)}_{3,\rm CT}$ shown below is specific to the $\overline{\lambda}$ renormalization. Like the remnant contribution, it consists of terms proportional to M_π^2, M_K^2 and q^2 but now multiplied by $\mathcal{O}(p^6)$ counterterms:

$$F_\pi^2\hat{\Pi}^{(1)}_{3,\rm CT}(q^2)=-q^2P_{\rm A}-8M_K^2R_{\rm A}-4M_\pi^2(Q_{\rm A}+R_{\rm A}). \tag{66}$$

There are finally the terms which arise from the individual 3π, $\overline{K}K\pi$, etc. intermediate states of sunset diagrams. Since all the renormalized polarization functions $\hat{\Pi}^{(1,0)}_{3,8}$ are finite, only the finite sunset integrals (defined in the Appendix as $\{Y_c^{(n)}\}$ and $\{Z_c^{(n)}\}$) will contribute to these quantities. As a reminder of this fact, a ''YZ'' suffix accompanies such contributions:

$$\begin{aligned}F_\pi^2\hat{\Pi}^{(1)}_{3,\rm SUN}(q^2)=&\frac{4}{9}\mathcal{H}^{qq}_{\rm YZ}(q^2,M_\pi^2,M_\pi^2)\\&+\frac{1}{6}\mathcal{H}^{qq}_{\rm YZ}(q^2,M_\eta^2,M_\pi^2)\\&+\frac{1}{18}\mathcal{H}^{qq}_{\rm YZ}(q^2,M_\pi^2,M_K^2)\\&+\frac{1}{3}\mathcal{K}_{1,_{\rm YZ}}(q^2,M_\pi^2,M_K^2)-R_{3,_{\rm YZ}}(q^2).\end{aligned} \tag{67}$$

The functions $\mathcal{K}_{1,_{\rm YZ}}$, $\mathcal{H}^{qq}_{\rm YZ}$, $R_{3,_{\rm YZ}}$ are defined respectively in Eqs. (A15),(A17),(A19) of the Appendix. Since all quantities in Eq. (67) occur in integral form, evaluation of $\hat{\Pi}^{(1)}_{3,\mathrm{SUN}}(q^2)$ must proceed via numerical integration; e.g., we find, at $q^2=0$,

$$F_\pi^2 \hat{\Pi}^{(1)}_{3,\mathrm{SUN}}(0) = 1.927 \times 10^{-6} \ \ \mathrm{GeV}^2. \tag{68}$$

Next comes the spin-zero isospin polarization function $\hat{\Pi}^{(0)}_3$. The remnant and counterterm contributions have structures analogous to their spin-one counterparts,

$$F_\pi^2 \hat{\Pi}^{(0)}_{3,\mathrm{REM}}(q^2) = \frac{M_\pi^4}{\pi^4}\left[-\frac{361}{294912} + \frac{49C}{36864} - \frac{1}{3072}\log\frac{M_\pi^2}{\mu^2} - \frac{11}{12288}\log\frac{M_K^2}{\mu^2} - \frac{1}{9216}\log\frac{M_\eta^2}{\mu^2}\right],$$

$$\hat{\Pi}^{(0)}_{3,\mathrm{CT}} = -4 M_\pi^4 B_{11}^{(0)}, \tag{69}$$

and though we defer the complicated expression for $\hat{\Pi}^{(0)}_{3,\mathrm{SUN}}$ to the Appendix [see Eq. (A20)], we record here its numerical value at $q^2=0$:

$$F_\pi^2 \hat{\Pi}^{(0)}_{3,\mathrm{SUN}}(0) = -2.954 \times 10^{-9} \ \ \mathrm{GeV}^4. \tag{70}$$

The hypercharge polarization functions $\hat{\Pi}^{(1,0)}_8$ are dealt with in like manner. Explicit expressions are given in Ref. [22]. In the $SU(3)$ limit of equal light-quark mass, the isospin and hypercharge polarizations must become equal. In the course of verifying this equality, we have found it is the *sum* of remnant and sunset terms, and not the individual contributions, which obeys the $SU(3)$ constraint.

B. Isospin polarization functions and $\overline{\mathrm{MS}}$ renormalization

The discussion in Sec. IV C provides a means for converting the isospin polarization functions $\hat{\Pi}^{(1,0)}_3$ from $\bar{\lambda}$ renormalization to $\overline{\mathrm{MS}}$ renormalization. As an example, consider the relation between the renormalization constants P_{A} and $P_{\mathrm{A}}^{\overline{\mathrm{MS}}}$. Starting from Eq. (64) and applying the relations in Eq. (60), we find

$$P_{\mathrm{A}}^{\overline{\mathrm{MS}}} = P_{\mathrm{A}} + \frac{F_\pi^2 C}{\pi^2}(-2B_{32}^{(1)} + B_{33}^{(1)}) - \frac{F_\pi^2 C^2}{(2\pi)^4}(-2B_{32}^{(2)} + B_{33}^{(2)}), \tag{71}$$

which together with Eqs. (50),(53) implies

$$P_{\mathrm{A}}^{\overline{\mathrm{MS}}} = P_{\mathrm{A}} - \frac{3}{16}\frac{C}{(4\pi)^4}. \tag{72}$$

This result is entirely consistent with the form obtained earlier in this section for $\hat{\Pi}^{(1)}_3$. That is, from Eqs. (65),(66) we have

$$\hat{\Pi}^{(1)}_3(q^2) = \frac{q^2}{F_\pi^2}\left[-P_{\mathrm{A}} + \frac{3C}{4096\pi^4} + \dots \right]. \tag{73}$$

But this is just the combination of factors appearing in Eq. (72) and we conclude

$$\hat{\Pi}^{(1)\overline{\mathrm{MS}}}_3(q^2) = \frac{q^2}{F_\pi^2}[-P_{\mathrm{A}}^{\overline{\mathrm{MS}}} + \dots]. \tag{74}$$

Analogous steps lead to the further relations

$$Q_{\mathrm{A}}^{\overline{\mathrm{MS}}} = Q_{\mathrm{A}} + 5C/32(4\pi)^4,$$

$$R_{\mathrm{A}}^{\overline{\mathrm{MS}}} = R_{\mathrm{A}} + 17C/96(4\pi)^4,$$

$$B_{11,r}^{\overline{\mathrm{MS}}} = B_{11}^{(0)} - 49C/576(4\pi)^4 F_\pi^2,$$

$$B_{13,r}^{\overline{\mathrm{MS}}} = B_{13}^{(0)} + 173C/5184(4\pi)^4 F_\pi^2. \tag{75}$$

To summarize, the forms for $\hat{\Pi}^{(1,0)}_{3,8}$ obtained in $\bar{\lambda}$-subtraction convert to $\overline{\mathrm{MS}}$ renormalization upon applying the simple rule: *omit all C dependence from the polarization functions and employ the* $\overline{\mathrm{MS}}$ *finite counterterms*.

Finally, we point out that for the renormalization constants P, Q and R [cf. Eq. (63)] which appeared in our two-loop analysis of vector-current propagators [9], there is no difference between the $\bar{\lambda}$-subtraction and $\overline{\mathrm{MS}}$ schemes. This can be traced to the fact that the quantities $P^{(1)}$, $Q^{(1)}$ and $R^{(1)}$ have only contributions from $L_9^{(0)}$ and $L_{10}^{(0)}$.

VI. SPECTRAL FUNCTIONS

As remarked in Sec. II, there will generally exist *two* spectral functions, $\rho^{(1)}_{\mathrm{A}a}(q^2)$ and $\rho^{(0)}_{\mathrm{A}a}(q^2)$ for the system of axial vector propagators. In the following, we determine the 3π contribution to the spectral functions for the isospin case $a=3$. The $K\bar{K}\pi$ and $K\bar{K}\eta$ components will have thresholds at higher energies.

A. Three-pion contribution to isospin spectral functions

Spectral functions can be determined from the imaginary parts of polarization functions [cf. Eqs. (3),(4)]:

$$\rho^{(1)}_{\mathrm{A}a}(q^2) = \frac{1}{\pi}\,\mathrm{Im}\,\Pi^{(1)}_a(q^2),$$

$$\rho^{(0)}_{\mathrm{A}a}(q^2) = \frac{1}{\pi}\,\mathrm{Im}\,\Pi^{(0)}_a(q^2). \tag{76}$$

In two-loop ChPT the imaginary parts arise solely from sunset graphs, and in terms of the notation established in the Appendix, the 3π component of $\rho^{(1)}_{\mathrm{A}3}$ is

$$\rho_{3\pi}^{(1)}(q^2)=-\frac{2}{\pi F_\pi^2}\frac{1}{(16\pi^2)^2}\,\mathrm{Im}\Big[q^2(2\overline{Y}_0^{(3)}-3\overline{Y}_0^{(2)}+\overline{Y}_0^{(1)})+M_\pi^2(4\overline{Y}_0^{(1)}-3\overline{Y}_0^{(2)}-\overline{Y}_0^{(0)})+4M_\pi^2(2\overline{Z}_0^{(2)}-\overline{Z}_0^{(1)})+\frac{M_\pi^4}{q^2}(\overline{Y}_0^{(1)}-\overline{Y}_0^{(0)}-4\overline{Z}_0^{(1)})\Big], \tag{77}$$

where the above finite functions are evaluated with 3π mass values. There is a comparable, but rather more complicated, expression for $\rho_{3\pi}^{(0)}(q^2)$ which we do not display here.

B. Unitarity determination of $\rho_{A3}^{(0,1)}[3\pi]$

Unitarity provides an alternative determination of the three-pion component of the isospin spectral functions $\rho_{A3}^{(1)}[3\pi]$ and $\rho_{A3}^{(0)}[3\pi]$. The first step is to relate the spectral functions to the fourier transform of a non-time-ordered product:

$$\rho_{A3}^{(1)}(q^2)(q^\mu q^\nu-q^2g^{\mu\nu})+\rho_{A3}^{(0)}(q^2)q^\mu q^\nu=\frac{1}{2\pi}\int d^4x e^{iq\cdot x}\langle 0|A_3^\mu(x)A_3^\nu(0)|0\rangle. \tag{78}$$

The three-pion component is obtained by simply inserting the $3\pi^0$ and $\pi^+\pi^0\pi^-$ intermediate states in the above integral. The relevant $\mathcal{S}$-matrix elements are determined from the $\mathcal{O}(p^2)$ chiral Lagrangian:

$$\langle a_3(q,\lambda)|\mathcal{S}|\pi^0(p_1)\pi^0(p_2)\pi^0(p_3)\rangle=i(2\pi)^4\delta^{(4)}(q-p_1-p_2-p_3)\epsilon_\mu^*(q,\lambda)\mathcal{M}_{000}^{\mu},$$

$$\langle a_3(q,\lambda)|\mathcal{S}|\pi^+(p_1)\pi^-(p_2)\pi^0(p_3)\rangle=i(2\pi)^4\delta^{(4)}(q-p_1-p_2-p_3)\epsilon_\mu^*(q,\lambda)\mathcal{M}_{+-0}^{\mu}, \tag{79}$$

where the invariant amplitudes are

$$\mathcal{M}_{000}^{\mu}=\frac{i}{\sqrt{6}F_\pi}q^\mu\frac{M_\pi^2}{q^2-M_\pi^2},$$

$$\mathcal{M}_{+-0}^{\mu}=\frac{i}{F_\pi}\left[2p_0^\mu+q^\mu\frac{M_\pi^2-2q\cdot p_0}{q^2-M_\pi^2}\right]. \tag{80}$$

Observe that $\mathcal{M}_{+-0}^{\mu}$ has two distinct contributions, a direct coupling and a pion pole term, whereas $\mathcal{M}_{000}^{\mu}$ has only a pion pole term. In the chiral limit of massless pions, the above amplitudes are conserved ($q_\mu\mathcal{M}_{3\pi}^{\mu}=0$) as required by chiral symmetry. Finally, evaluation of the phase space integral gives

$$\rho_{3\pi}^{(1)}(q^2)=\frac{1}{768\pi^4}\frac{q^2}{F_\pi^2}\theta(q^2-9M_\pi^2)\overline{I}^{(1)}(x), \tag{81}$$

where $x\equiv M_\pi/\sqrt{q^2}$ and $\overline{I}^{(1)}(x)$ is the dimensionless integral:

$$\overline{I}^{(1)}(x)=\left[\frac{1}{2}(1-x)^2-2x^2\right]^{3/2}\int_0^\pi d\phi\,\sin\phi(1+\cos\phi)^{1/2}\frac{[-4x^2+\{\frac{1}{2}(1-x)^2+x^2-1+[\frac{1}{2}(1-x)^2-2x^2]\cos\phi\}^2]^{3/2}}{\{\frac{1}{2}(1-x)^2+2x^2+[\frac{1}{2}(1-x)^2-2x^2]\cos\phi\}^{1/2}}. \tag{82}$$

A similar treatment yields the three-pion component of the spin-zero isospin spectral function:

$$\rho_{3\pi}^{(0)}(q^2)=\frac{1}{512\pi^4}\frac{q^2}{F_\pi^2}\frac{M_\pi^4}{(q^2-M_\pi^2)^2}\left[\frac{7}{3}\overline{I}_0^{(0)}(x)-4\overline{I}_1^{(0)}(x)+2\overline{I}_2^{(0)}(x)\right]\theta(q^2-9M_\pi^2), \tag{83}$$

where

$$\overline{I}_n^{(0)}(x)=\left[\frac{1}{2}(1-x)^2-2x^2\right]^{3/2}\int_0^\pi d\phi\,\sin\phi(1+\cos\phi)^{1/2}\left\{-4x^2+\left[\frac{1}{2}(1-x)^2+x^2-1+\left(\frac{1}{2}(1-x)^2-2x^2\right)\cos\phi\right]^2\right\}^{1/2}\times\frac{\{1-x^2-\frac{1}{2}(1-x)^2-[\frac{1}{2}(1-x)^2-2x^2]\cos\phi\}^n}{\{\frac{1}{2}(1-x)^2+2x^2+[\frac{1}{2}(1-x)^2-2x^2]\cos\phi\}^{1/2}}. \tag{84}$$

The spectral functions obtained in the unitarity approach agree precisely with those obtained earlier in this section from the imaginary parts of the sunset amplitudes.

VII. CHIRAL SUM RULES

In previous sections, we have determined the isospin and hypercharge axial vector propagators to two-loop order in ChPT. Essential to the success of this program is the renormalization procedure by which the results are rendered finite. As a consequence of the renormalization paradigm, however, the physical results contain a number of undetermined finite counterterms. The real parts of the polarization functions $\Pi^{(1)}_{\mathrm{A}a}$ and $\Pi^{(0)}_{\mathrm{A}a}$ ($a=3,8$) contain two such constants from one-loop order ($L^{(0)}_{10}$ and $H^{(0)}_1$) and five independent combinations of the $\{B^{(0)}_i\}$ from two-loop order. Of these, the constants $H^{(0)}_1$ and $B^{(0)}_{11}$ are related to contact terms which are regularization dependent and thus physically unobservable. The remaining counterterms [called "low energy constants" (LECs)] must be extracted from the data. In this section, we describe how some of the $\mathcal{O}(p^6)$ counterterm coupling constants are obtainable from chiral sum rules, and as an example we study a specific case involving broken $SU(3)$.

A derivation of the chiral sum rules together with an application of one of them appears in Ref. [26]. We refer the reader to that article for a general orientation. For the purpose of writing dispersion relations it suffices to note that at low energies the polarization functions are already determined from the results obtained in previous sections, although some care must be taken with the kinematic poles at $q^2=0$ in the individual functions $\Pi^{(1)}_{\mathrm{A}a}$ and $\Pi^{(0)}_{\mathrm{A}a}$. As regards high energy behavior, the large-s limit of spectral functions relevant to our analysis can be read off from the work in Refs. [27,28]. However, since we are calculating up to order p^6, terms up to and including the quadratic dependence in the light quark masses must be included. Thus, for example, the asymptotic expansion for the isospin axial vector spectral function summed over spin-one and spin-zero reads

$$\rho^{(1+0)}_{\mathrm{A}3}(s)=\frac{1}{8\pi^2}\left(1+\frac{\alpha_s}{\pi}\left[1+12\,\frac{\hat{m}^2}{q^2}\right]+\mathcal{O}(\alpha_s^2,1/s^2)\right). \tag{85}$$

Similar expressions hold for the other components, and we summarize their collective leading asymptotic behavior by

$$\rho^{(1)}_{\mathrm{A}a}(s)\sim\mathcal{O}(1),\quad \rho^{(0)}_{\mathrm{A}a}(s)\sim\mathcal{O}(s^{-1}),\quad (\rho^{(1)}_{\mathrm{A}3}-\rho^{(1)}_{\mathrm{A}8})(s)\sim\mathcal{O}(s^{-1}). \tag{86}$$

The real parts of the polarization functions show exactly the same asymptotic behavior as the imaginary parts; i.e., expressions analogous to Eq. (86) hold. The Källén-Lehmann spectral representation of two-point functions then implies the following dispersion relations for the axial vector polarization functions of a given flavor $a=3,8$:

$$q^2\Pi^{(0)}_{\mathrm{A}a}(q^2)-\lim_{q^2=0}[q^2\Pi^{(0)}_{\mathrm{A}a}(q^2)]=q^2\int_0^\infty ds\,\frac{\rho^{(0)}_{\mathrm{A}a}(s)}{s-q^2-i\epsilon}, \tag{87}$$

$$q^2\Pi^{(1)}_{\mathrm{A}a}(q^2)-\lim_{q^2=0}[q^2\Pi^{(1)}_{\mathrm{A}a}(q^2)]-\lim_{q^2=0}\frac{d}{dq^2}[q^2\Pi^{(1)}_{\mathrm{A}a}(q^2)]=q^4\int_0^\infty ds\,\frac{\rho^{(1)}_{\mathrm{A}a}(s)}{s(s-q^2-i\epsilon)}. \tag{88}$$

We work with $q^2\Pi^{(0),(1)}_{\mathrm{A}a}(q^2)$ due to the presence of $q^2=0$ kinematic poles. Moreover, the subtraction constants have been placed on the left hand side in Eqs. (87),(88) in order to equate only physically observable quantities. Dispersion relations involving $SU(3)$-breaking combinations have an improved asymptotic behavior, such as

$$(\Pi^{(1)}_{\mathrm{A}3}+\Pi^{(0)}_{\mathrm{A}3}-\Pi^{(1)}_{\mathrm{A}8}-\Pi^{(0)}_{\mathrm{A}8})(q^2)=\int_0^\infty ds\,\frac{(\rho^{(1)}_{\mathrm{A}3}+\rho^{(0)}_{\mathrm{A}3}-\rho^{(1)}_{\mathrm{A}8}-\rho^{(0)}_{\mathrm{A}8})(s)}{s-q^2-i\epsilon} \tag{89}$$

and

$$q^2(\Pi^{(1)}_{\mathrm{A}3}-\Pi^{(1)}_{\mathrm{A}8})(q^2)-\lim_{q^2=0}[q^2(\Pi^{(1)}_{\mathrm{A}3}-\Pi^{(1)}_{\mathrm{A}8})(q^2)]=q^2\int_0^\infty ds\,\frac{(\rho^{(1)}_{\mathrm{A}3}-\rho^{(1)}_{\mathrm{A}8})(s)}{s-q^2-i\epsilon}. \tag{90}$$

Sum rules are obtained by evaluating arbitrary derivatives of such relations at $q^2=0$. For the sum rules inferred from Eqs.(87),(88) it is preferable to express the left hand side in terms of $\hat{\Pi}^{(0,1)}_{\mathrm{A}a}$:

$$\frac{1}{n!}\left[\frac{d}{dq^2}\right]^n\hat{\Pi}^{(0)}_{\mathrm{A}a}(0)=\int_0^\infty ds\,\frac{\overline{\rho^{(0)}_{\mathrm{A}a}}(s)}{s^n}\quad(n\geqslant 1), \tag{91}$$

$$\frac{1}{(n-1)!}\left[\frac{d}{dq^2}\right]^{n-1}\hat{\Pi}^{(1)}_{\mathrm{A}a}(0)-\frac{1}{n!}\left[\frac{d}{dq^2}\right]^n\hat{\Pi}^{(0)}_{\mathrm{A}a}(0)=\int_0^\infty ds\,\frac{\rho^{(1)}_{\mathrm{A}a}(s)}{s^n}\quad(n\geqslant 2), \tag{92}$$

where $\overline{\rho^{(0)}_{\mathrm{A}a}}(s)$ is defined in Eq. (5). Finally, Eq. (90) leads directly to the following sequence of sum rules explicitly involving broken $SU(3)$:

$$\frac{1}{n!}\left[\frac{d}{dq^2}\right]^n(\Pi^{(1)}_{A3}+\Pi^{(0)}_{A3}-\Pi^{(1)}_{A8}-\Pi^{(0)}_{A8})(0)$$
$$=\int_0^\infty ds\,\frac{(\rho^{(1)}_{A3}+\rho^{(0)}_{A3}-\rho^{(1)}_{A8}-\rho^{(0)}_{A8})(s)}{s^{n+1}}, \qquad (93)$$

where $n\geqslant 0$.

For this last sum rule, let us consider in some detail the case $n=0$. An equivalent form, better suited for phenomenological analysis, is given by

$$(\hat{\Pi}^{(1)}_{A3}-\hat{\Pi}^{(1)}_{A8})(0)-\frac{d}{dq^2}(\hat{\Pi}^{(0)}_{A3}-\hat{\Pi}^{(0)}_{A8})(0)$$
$$=\int_0^\infty ds\,\frac{(\rho^{(1)}_{A3}-\rho^{(1)}_{A8})(s)}{s}. \qquad (94)$$

Evaluation of the left hand side (LHS) of this sum rule yields

$$\text{LHS}=\frac{16}{3}\frac{M_K^2-M_\pi^2}{F_\pi^2}Q_A(\mu)+0.001053+\frac{1}{F_\pi^2(16\pi^2)^2}\left[M_\pi^2\log\frac{M_\pi^2}{\mu^2}\left(\frac{8}{9}-\frac{1}{3}\log\frac{M_\pi^2}{\mu^2}-\frac{1}{3}\log\frac{M_K^2}{\mu^2}\right)\right.$$
$$\left.+M_\pi^2\log\frac{M_K^2}{\mu^2}\left(-\frac{1}{18}+\frac{1}{6}\log\frac{M_K^2}{\mu^2}\right)+M_K^2\log\frac{M_K^2}{\mu^2}\left(-\frac{5}{6}+\frac{1}{2}\log\frac{M_K^2}{\mu^2}\right)+128\pi^2L_{10}^{(0)}\left(M_K^2\log\frac{M_K^2}{\mu^2}-M_\pi^2\log\frac{M_\pi^2}{\mu^2}\right)\right]. \qquad (95)$$

Recall from the discussion at the beginning of Sec. V that there are three distinct sources for the finite low energy terms: (i) $\mathcal{O}(p^6)$ CTs $\{B_l^{(0)}\}$, (ii) the ''remnant'' contributions, and (iii) the finite sunset integrals [cf. Eq. (62)]. It is a combination of the latter two which give rise to the numerical term [which is scale independent and vanishes in the $SU(3)$ limit of equal masses] in the first line. We have displayed all chiral logarithms explicitly, and $Q_A(\mu)$ is the $\mathcal{O}(p^6)$ counterterm defined earlier in Eq. (64). Since the full expression is scale independent, this allows one to directly read off the variation of the contributing counterterm combination at renormalization scale μ.

We can use the sum rule of Eq. (95) to numerically estimate $Q_A(\mu)$. One needs to evaluate the spectral integral on the right hand side (RHS) of Eq. (94). For our purposes, it is sufficient to approximate the contribution of the isospin spectral function in terms of the a_1 resonance taken in narrow width approximation, $\rho^{(1)\text{res}}_{A3}(s)\simeq g_{a_1}\delta(s-M^2_{a_1})$. Employing resonance parameters as obtained from the fit in Ref. [30], we obtain

$$\int_0^\infty ds\,\frac{\rho^{(1)\text{res}}_{A3}}{s}\simeq 0.0189. \qquad (96)$$

Although consistency with QCD dictates that we also include the large-s continuum [31], the leading-order contributions would cancel in Eq. (94) and the remaining mass corrections are small. As regards the hypercharge spectral function $\rho^{(1)}_{A8}$, little is presently known. The lowest lying resonances which contribute are $f_1(1285)$ and $f_1(1510)$ but the couplings of these resonances to the axial vector current have not been determined. Since the corresponding sum rule for vector current spectral functions [14] exhibits large cancellations between the contributions from $\rho(770)$, $\omega(782)$ and $\Phi(1020)$, we expect a similar cancellation to be at work in the axial vector sector. To obtain a rough estimate, we assume that the two resonances $f_1(1285)$ and $f_1(1510)$ can be approximated by a single effective resonance with spectral function $\rho^{(1)\text{eff}}_{A8}(s)\simeq g_{a_8}\delta(s-M^2_{a_8})$. Assuming further $g_{a_8}\simeq g_{a_1}$ and $M_{a_8}\simeq 1.4$ GeV we estimate the hypercharge contribution to the RHS of Eq. (94) as 0.012. Allowing for a 50% error in this estimate places the RHS in the range $0.001\leqslant\text{RHS}\leqslant 0.013$ and leads finally to

$$0.000043\leqslant Q_A(M_{a_1})\leqslant 0.000130, \qquad (97)$$

where the renormalization scale $\mu=M_{a_1}$ has been adopted. This is clearly to be taken as just a rough estimate. Only experimental determination of the missing coupling constants can provide a more reliable estimate. In addition, a more thorough phenomenological analysis will involve use of the entire spectrum. However, this example serves to illustrate the general procedure.

We have not touched on chiral sum rules involving *both* vector and axial vector spectral functions. The most prominent example of this type is the Das-Mathur-Okubo (DMO) sum rule [21] which, in modern terminology, has been employed to determine the LEC $L_{10}^{(0)}$ [3,29]. In Ref. [26] we have shown how the DMO sum rule must be modified to be valid to second order in the light quark masses. Recently, τ-decay data have renewed interest in this sum rule from the experimental side [32]. Our phenomenological study of the DMO sum rule using the two-loop results of polarization functions obtained both here and in Ref. [9] is reported in Ref. [20].

Finally, there are also those sum rules involving *no* $\mathcal{O}(p^6)$ counterterm coupling constants, i.e. those obtained by taking appropriately many derivatives of the dispersion relations Eqs. (87)–(90). From experience with the corresponding inverse moment sum rules of vector current spectral functions

[14], we expect these sum rules in general not to be verified. This is because the relevant physics (which involves the low-lying resonances) enters the relations only in higher order of the chiral expansion. A quantitative study of such sum rules is deferred to a forthcoming publication.

VIII. CONCLUSIONS

Our analysis of axial vector current propagators in two-loop ChPT has led to a complete two-loop renormalization of the pion and eta masses and decay constants as well as the real-valued parts of the isospin and hypercharge polarization functions. It has yielded predictions for axial vector spectral functions and has allowed the derivation of spectral function sum rules.

Despite the complexity of many of the individual steps and results, the sum of tree, one-loop and two-loop contributions to the axial vector propagator yields a simple overall structure

$$\Delta_{\mathrm{A},\mu\nu}(q^2)=[F^2+\hat{\Pi}_{\mathrm{A}}^{(0)}(q^2)]g_{\mu\nu}-\frac{F^2}{q^2-M^2}q_\mu q_\nu +[2L_{10}^{(0)}-4H_1^{(0)}+\hat{\Pi}_{\mathrm{A}}^{(1)}(q^2)](q_\mu q_\nu-q^2 g_{\mu\nu}), \tag{98}$$

where flavor labelling is suppressed. Comparing this to the general decomposition of Eq. (4) yields

$$\Pi_{\mathrm{A}}^{(1)}(q^2)=2L_{10}^{(0)}-4H_1^{(0)}+\hat{\Pi}_{\mathrm{A}}^{(1)}(q^2)-\frac{F^2+\hat{\Pi}_{\mathrm{A}}^{(0)}(q^2)}{q^2},$$

$$\Pi_{\mathrm{A}}^{(0)}(q^2)=\frac{\hat{\Pi}_{\mathrm{A}}^{(0)}(q^2)}{q^2}-\frac{F^2M^2}{q^2(q^2-M^2)}. \tag{99}$$

As noted earlier, there are kinematic poles at $q^2=0$ in both the spin-one and spin-zero polarization functions, but the sum $\Pi_{\mathrm{A}}^{(1)}+\Pi_{\mathrm{A}}^{(0)}$ is free of such singularities.

A large number of $\mathcal{O}(p^6)$ counterterms entered the axial vector calculation, and many constraints among them were obtained from the subtraction procedure. Thus given the total of 23 $\mathcal{O}(p^6)$ counterterms which appeared by employing the basis of Ref. [24], each of the $\overline{\lambda}^2$ and $\overline{\lambda}$ subtractions were found to yield 14 constraints. The analysis of vector propagators in Ref. [9] yielded another 3 conditions for each of the $\overline{\lambda}^2$ and $\overline{\lambda}$ subtractions. This total of 17 conditions constraining the $\{B_l^{(2)}\}$ and $\{B_l^{(1)}\}$ counterterms is of course universal and can be used together with results of other two-loop studies. We can summarize the remaining nine finite $\mathcal{O}(p^6)$ counterterms as

$$\begin{aligned}
&\text{polarization amplitudes:} && P_{\mathrm{A}},Q_{\mathrm{A}},R_{\mathrm{A}},B_{11}^{(0)},B_{13}^{(0)}; \\
&\text{decay constants:} && \{\tilde{B}_l\} \quad (l=1,\dots,4); \\
&\text{masses:} && \{\tilde{B}_l\} \quad (l=1,\dots,9),
\end{aligned} \tag{100}$$

where $P_{\mathrm{A}},Q_{\mathrm{A}},R_{\mathrm{A}}$ are defined in Eq. (64) and the $\{\tilde{B}_l\}$ in Eqs. (B10), (B20) of Ref. [22]. We have made preliminary numerical estimates for Q_{A} in this paper [cf. Eq. (97)] and for P_{A} in Ref. [26]. The counterterm $B_{11}^{(0)}$ is related to a contact term and is regularization dependent, much the same as the constant $H_1^{(0)}$ appearing in Eq. (98). However, these terms always drop out when physical observable quantities are considered. The constant R_{A} is seen to contribute equally to all flavor components of the axial vector polarization functions. It cannot therefore be accessed by the chiral sum rules involving broken $SU(3)$ considered in the previous section. However, by combining the results obtained here with the two-loop analysis of the vector current two-point functions, the combination $R-R_{\mathrm{A}}$ is seen to constitute a mass correction to the Das-Mathur-Okubo sum rule [20]. Finally, little is known about the nine constants $\{\tilde{B}_l\}$ $(l=1,\dots,9)$ which determine (together with the calculated loop contributions) the p^6 corrections to masses and decay constants. We have not attempted to estimate these constants (e.g. by the resonance saturation hypothesis) since as far as the axial vector two-point function is concerned, their contribution is implicit. However, the explicit expressions (given in Ref. [22]) can be used in further ChPT studies when expressing bare masses and decay constants in terms of fully renormalized physical quantities.

In view of the length and difficulty of the calculation, it is reassuring that a broad range of independent checks was available to gauge the correctness of the results. We list the most important of them here:

(1) Because the calculation involved *independent* determinations of isospin and hypercharge channels at each stage, the $SU(3)$ limit of equal masses provided numerous tests among the set of isospin and hypercharge decay constants, masses and polarization functions. As a by-product, it also revealed the presence of previously unnoticed identities among the sunset amplitudes [cf. Eqs. (A40)–(A42)].

(2) As shown in Sec. VI, it was possible to determine spectral functions directly from the two-loop analysis or equivalently from a unitarity approach which employed one-loop amplitudes as input. In this way, both the specific sunset integrals as well as the structural relations of Eq. (99) were able to be tested successfully.

(3) It turned out that although most of the divergent terms could be subtracted away with counterterms, there occurs no m_K^6 counterterm contribution in $M_\pi^{(6)2}$. To avoid disaster, there must thus be a cancellation between sunset and non-sunset numerical terms. Such a cancellation indeed occurs and constitutes a non-trivial check on our determination of the sunset contribution.

(4) Given that $\hat{\Pi}_{\mathrm{A}}^{(0)}(q^2)$ can be shown to vanish in the limit of zero quark mass, it follows that our final result in Eq. (98) has the correct chiral limit.

(5) Last, we have explicitly verified in $\overline{\mathrm{MS}}$ renormalization that the constant C is absent, as must be the case.

Yet more on this subject remains to be done. This is especially true of the material composing Sec. VII, where an application of $SU(3)$-breaking sum rules to determine finite $\mathcal{O}(p^6)$ counterterms was discussed and additional points were raised. Future work will be needed to carefully analyze the axial vector sum rules, particularly the role of existing data to provide as precise a determination of the counterterms as experimental uncertainties allow. We can, of course, combine the results of the present axial vector study with the vector results of Ref. [9] to study an even wider range of sum rules (e.g. as with the proposed determination of $R_{\rm A}$ discussed above). On an even more ambitious level, our experience with such relations makes us optimistic about the possibility of describing a possible framework for interpreting chiral sum rules to *arbitrary* order in the chiral expansion. Finally, it will be of interest to reconsider the phenomenological extraction of spectral functions such as $\rho^{(0)}_{\rm A33}[3\pi]$ and to stimulate experimental efforts to extract spectral function information involving non-pionic particles such as kaons and etas.

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APPENDIX: SUNSET INTEGRALS

This appendix compiles mathematical details related to the sunset contributions. In the first section, we give integral representations for all sunset functions which appear in the two-loop analysis and also extract the finite sunset functions (called $\{Y_c^{(n)}\}$ and $\{Z_c^{(n)}\}$) which remain after the singular parts have been identified. The second section provides expressions for the sunset contribution to the isospin polarization amplitudes. This is followed in the third section by a detailed discussion of the finite-valued $\{Y_c^{(n)}(q^2,m^2,M^2)\}$ and $\{Z_c^{(n)}(q^2,m^2,M^2)\}$ functions, and in the final subsection certain identities which relate various sunset integrals are listed. Additional work on sunset integrals can be found in Refs. [33,10,12,17] for the equal mass case and in Refs. [34–36,11] for the general mass case. As a matter of notation, for sunset amplitudes containing unequal masses we denote the mass occurring twice as M and the third mass as m (e.g. for $\bar{K}K\pi$ amplitudes, $M\to m_K$ and $m\to m_\pi$).

1. Sunset integrals

The sunset amplitudes integrals appear in all sectors of the two-loop analysis (in the 1PI amplitudes and in the vertex, self-energy 1PR amplitudes). They are singular, and so their finite parts must be systematically extracted from the divergent parts.

a. Definitions

All sunset contributions in the calculation can ultimately be expressed in terms of the integral A of Eq. (11) and six additional functions S, $\bar{S}$, S_1, S_2, $\mathcal{K}_1$ and $\mathcal{K}_2$. The first four of these occur via the integrals

$$\{S;S_\mu;S_{\mu\nu}\}\equiv\int d\tilde{t}\,\frac{\{1;t_\mu;t_\mu t_\nu\}}{t^2-m^2}\times\int d\tilde{b}\,\frac{1}{(b^2-M^2)[(Q-b)^2-M^2]}, \tag{A1}$$

where $Q\equiv q-t$. From covariance, we have

$$S_\mu(q^2,m^2,M^2)\equiv q_\mu\bar{S}(q^2,m^2,M^2),$$
$$S_{\mu\nu}(q^2,m^2,M^2)\equiv q_\mu q_\nu S_1(q^2,m^2,M^2)+g_{\mu\nu}S_2(q^2,m^2,M^2). \tag{A2}$$

Explicit expressions for S, $\bar{S}$, S_1, S_2 and for $\mathcal{K}_1$, $\mathcal{K}_2$ are given respectively in Eqs. (A11)–(A14) and Eqs. (A15), (A16) below.

We now summarize sunset integrals which contribute to the 1PI and 1PR amplitudes. The isospin 1PI sunset amplitude of Fig. 5 is given by

$$\mathcal{M}^{\rm SUN}_{\mu\nu 3}=\frac{4}{9}\,\mathcal{H}_{\mu\nu}(q^2,m_\pi^2,m_\pi^2)+\frac{1}{6}\,\mathcal{H}_{\mu\nu}(q^2,m_\eta^2,m_K^2)+\frac{1}{18}\,\mathcal{H}_{\mu\nu}(q^2,m_\pi^2,m_K^2)+\mathcal{L}_{\mu\nu}(q^2,m_\pi^2,m_K^2), \tag{A3}$$

where

$$F_0^2\mathcal{H}_{\mu\nu}(q^2,m^2,M^2)\equiv\int d\tilde{t}\,\frac{(q-3t)_\mu(q-3t)_\nu}{t^2-m^2}\int d\tilde{b}\,\frac{1}{(b^2-M^2)[(Q-b)^2-M^2]}=q_\mu q_\nu S-3q_\mu S_\nu-3q_\nu S_\mu+9S_{\mu\nu} \tag{A4}$$

and

[4] Laboratoire de Recherche des Universités Paris XI et Paris VI, associé au CNRS.

$$F_0^2\mathcal{L}_{\mu\nu}(q^2,m^2,M^2)\equiv\int d\tilde{t}\,\frac{1}{t^2-m^2}\int d\tilde{b}\,\frac{(Q-2b)_\mu(Q-2b)_\nu}{(b^2-M^2)[(Q-b)^2-M^2]}$$
$$\equiv\frac{1}{d-1}\left[q_\mu q_\nu\mathcal{K}_1(q^2,m^2,M^2)+g_{\mu\nu}\mathcal{K}_2(q^2,m^2,M^2)+4g_{\mu\nu}\frac{3-(4-d)}{2}A(m^2)A(M^2)\right]. \quad \text{(A5)}$$

The sunset contribution to the isospin vertex amplitude corresponding to Fig. 6(e) is

$$\Gamma_{\mu 3}^{(4)\mathrm{SUN}}=\frac{i}{F_0^3}\left[\frac{2}{9}I_{1\mu}(q^2;m_\pi^2;m_\pi^2;m_\pi^2)+\frac{1}{36}I_{1\mu}(q^2;m_\pi^2;m_K^2;2(m_\pi^2+m_K^2))\right.$$
$$\left.+\frac{1}{12}I_{1\mu}\left(q^2;m_\eta^2;m_K^2;\frac{2}{3}\,(m_\pi^2-m_K^2)\right)+\frac{1}{2}I_{2\mu}(q^2;m_\pi^2;m_K^2)\right], \quad \text{(A6)}$$

where

$$I_{1\mu}(q^2;m^2;M^2;\Lambda)\equiv q_\mu I_1(q^2;m^2;M^2;\Lambda)=\int d\tilde{t}\,\frac{(q-3t)_\mu}{t^2-m^2}\int d\tilde{b}\,\frac{Q^2-2q\cdot t+2b\cdot(Q-b)+\Lambda}{(b^2-M^2)[(Q-b)^2-M^2]}$$
$$=-2q_\mu[2A^2(M^2)+A(m^2)A(M^2)]-6[q_\mu q_\nu S^\nu(q^2;m^2;M^2)-3q^\nu S_{\mu\nu}(q^2;m^2;M^2)]$$
$$+[q_\mu S(q^2;m^2;M^2)-3S_\mu(q^2;m^2;M^2)][2(q^2+m^2-M^2)+\Lambda] \quad \text{(A7)}$$

and

$$I_{2\mu}(q^2;m^2;M^2)\equiv q_\mu I_2(q^2;m^2;M^2)=\int d\tilde{t}\,\frac{(q+t)^\nu}{t^2-m^2}\int d\tilde{b}\,\frac{(Q-2b)_\mu(Q-2b)_\nu}{(b^2-M^2)[(Q-b)^2-M^2]}$$
$$=q_\mu\frac{2(3+d-4)}{d-1}A(m^2)A(M^2)+\frac{2F_0^2}{d-1}q_\mu[q^2\mathcal{K}_1(q^2;m^2;M^2)+\mathcal{K}_2(q^2;m^2;M^2)]. \quad \text{(A8)}$$

Finally, in the calculation of self-energies there appear along with the integral S of Eq. (A1) the additional quantities

$$R(q^2;m^2;M^2;\Lambda)\equiv\int\frac{d\tilde{t}}{t^2-m^2}\int d\tilde{b}\,\frac{[(q-t)^2-2qt+2b(Q-b)+\Lambda]^2}{(b^2-M^2)[(Q-b)^2-M^2]}$$
$$=\Lambda^2S(q^2;m^2;M^2)-4\Lambda[-(q^2+m^2-M^2)S(q^2;m^2;M^2)+3q_\mu S^\mu(q^2;m^2;M^2)-A^2(M^2)$$
$$+A(M^2)A(m^2)]+36q_\mu q_\nu S^{\mu\nu}(q^2;m^2;M^2)+24(M^2-m^2-q^2)q_\mu S^\mu(q^2;m^2;M^2)$$
$$+4(q^2+m^2-M^2)^2S(q^2;m^2;M^2)+A^2(M^2)(4m^2-12q^2)+A(m^2)A(M^2)(8M^2-6q^2-6m^2) \quad \text{(A9)}$$

and

$$U(q^2;m^2;M^2)\equiv\int\frac{d\tilde{t}}{t^2-m^2}\int d\tilde{b}\,\frac{[(q+t)(2b+t-q)]^2}{(b^2-M^2)[(Q-b)^2-M^2]}$$
$$=\frac{4}{d-1}\left[(q^2+m^2)\left(\frac{3}{2}+\frac{d-4}{2}\right)A(m^2)A(M^2)+q^2[q^2\mathcal{K}_1(q^2;m^2;M^2)+\mathcal{K}_2(q^2;m^2;M^2)]\right]. \quad \text{(A10)}$$

b. Identification of the finite parts

The functions S, $\overline{S}$, S_1, S_2, $\mathcal{K}_1$ and $\mathcal{K}_2$ can in turn each be written as the sum of terms (which diverge in the $d\to4$ limit) proportional to gamma functions plus finite-valued functions $\{Y_c^{(n)}(q^2,m^2,M^2)\}$ and $\{Z_c^{(n)}(q^2,m^2,M^2)\}$. Thus, we have

$$S(q^2,m^2,M^2)=\frac{\Gamma^2(2-d/2)}{(4\pi)^d}(M^2)^{d-4}\,\frac{1}{d-2}\left\{-2\left[\left(\frac{m^2}{M^2}\right)^{d/2-1}+\frac{1}{d-3}\right]M^2+\frac{1}{5-d}m^2+\frac{4-d}{d(5-d)}q^2\right\}$$
$$-Y_0^{(0)}m^2+(2Y_0^{(1)}-Y_0^{(0)})q^2, \quad \text{(A11)}$$

$$\overline{S}(q^2,m^2,M^2)=\frac{\Gamma^2(2-d/2)}{(4\pi)^d}(M^2)^{d-4}\,\frac{1}{d}\left(-\frac{2}{d-3}M^2-\frac{4-d}{(5-d)(d-2)}m^2+\frac{4-d}{(5-d)(d+2)}q^2\right)$$
$$+(Y_0^{(0)}-2Y_0^{(1)})m^2+(3Y_0^{(2)}-2Y_0^{(1)})q^2+Z_0^{(1)}, \tag{A12}$$

$$S_1(q^2,m^2,M^2)=\frac{\Gamma^2(2-d/2)}{(4\pi)^d}(M^2)^{d-4}\,\frac{1}{d+2}\left(-\frac{2}{d-3}M^2-\frac{4-d}{d(5-d)}m^2+\frac{4-d}{(d+4)(5-d)}q^2\right)$$
$$+(2Y_0^{(1)}-3Y_0^{(2)})m^2+(4Y_0^{(3)}-3Y_0^{(2)})q^2+2Z_0^{(2)}, \tag{A13}$$

$$S_2(q^2,m^2,M^2)=\frac{\Gamma^2(2-d/2)}{(4\pi)^d}(M^2)^{d-4}\left(\left\{-\frac{2}{d(d-2)}\left[\left(\frac{m^2}{M^2}\right)^{d/2}+\frac{2}{d-2}\right]M^4-\frac{2}{d(d-2)(d-3)}m^2M^2\right.\right.$$
$$\left.\left.+\frac{2}{d(d-3)(d+2)}q^2M^2+\frac{1}{d(5-d)(d-2)}m^4+\frac{2(4-d)}{d(5-d)(d^2-4)}q^2m^2+\frac{d-4}{d(5-d)(d+2)(d+4)}q^4\right\}\right)$$
$$+\frac{1}{2}[(Y_0^{(1)}-Y_0^{(0)})m^4+(2Y_0^{(3)}-3Y_0^{(2)}+Y_0^{(1)})q^4+(4Y_0^{(1)}-3Y_0^{(2)}-Y_0^{(0)})m^2q^2-Z_0^{(1)}m^2+(2Z_0^{(2)}-Z_0^{(1)})q^2] \tag{A14}$$

and

$$\mathcal{K}_1(q^2,m^2,M^2)=\frac{\Gamma^2(2-d/2)}{(4\pi)^d}(M^2)^{d-4}\frac{d-1}{d-2}\left\{-\frac{16}{d(d-3)(5-d)(d+2)}M^2\right.$$
$$\left.+\left[-\frac{2}{3}\left(\frac{m^2}{M^2}\right)^{d/2-2}+\frac{24}{d(5-d)(7-d)(d+2)}\right]m^2+\frac{24(4-d)}{d(7-d)(5-d)(d+2)(d+4)}q^2\right\}$$
$$+(6Y_1^{(1)}-3Y_1^{(2)}-3Y_1^{(0)})m^2+(4Y_1^{(3)}-9Y_1^{(2)}+6Y_1^{(1)}-Y_1^{(0)})q^2+2Z_1^{(2)}-2Z_1^{(1)}, \tag{A15}$$

$$\mathcal{K}_2(q^2,m^2,M^2)=\frac{\Gamma^2(2-d/2)}{(4\pi)^d}(M^2)^{d-4}\frac{d-1}{d(d-2)}\left\{\frac{2(d-1)}{(d-3)(5-d)}m^2M^2-\frac{4(d-1)}{(d-2)(d-3)}M^4\right.$$
$$+\left[\frac{2(d-1)}{3}\left(\frac{m^2}{M^2}\right)^{d/2-2}-\frac{d-1}{(5-d)(7-d)}\right]m^4+\left[\frac{2d}{3}\left(\frac{m^2}{M^2}\right)^{d/2-2}+\frac{2(d^2-5d-8)}{(5-d)(7-d)(d+2)}\right]$$
$$\left.\times q^2m^2+\frac{2(6+3d-d^2)}{(d-3)(5-d)(d+2)}q^2M^2+\frac{(4-d)(d^2-3d-22)}{(5-d)(7-d)(d+2)(d+4)}q^4\right\}$$
$$+\left(-7Y_1^{(3)}+\frac{27}{2}Y_1^{(2)}-\frac{15}{2}Y_1^{(1)}+Y_1^{(0)}\right)q^4+\left(\frac{15}{2}Y_1^{(2)}-12Y_1^{(1)}+\frac{9}{2}\,Y_1^{(0)}\right)m^2q^2$$
$$+\frac{3}{2}(Y_1^{(0)}-Y_1^{(1)})m^4+\left(\frac{7}{2}Z_1^{(1)}-5Z_1^{(2)}\right)q^2+\frac{3}{2}Z_1^{(1)}m^2. \tag{A16}$$

For the sake of simplicity, we have omitted the arguments of the $\{Y_c^{(n)}\}$ and the $\{Z_c^{(n)}\}$. We have verified that Eq. (A11) agrees in the equal mass limit with the explicit expression appearing in Ref. [17].

2. Sunset contribution to renormalized isospin polarization functions

Since the renormalized polarization functions are finite valued, their sunset-related content will consist entirely of the $\{Y_c^{(n)}\}$ and $\{Z_c^{(n)}\}$ integrals introduced in the previous section. Thus, we employ the subscript ''YZ'' in the following.

For the spin-one isospin polarization function, the quantity $\mathcal{H}^{qq}$ appearing in Eq. (67) is defined in terms of the functions in Eq. (A2):

$$\mathcal{H}^{qq}_{\rm YZ}\equiv S_{\rm YZ}-6\overline{S}_{\rm YZ}+9S_{1,\rm YZ}\,. \tag{A17}$$

The qq superscript indicates that the contributions to $\mathcal{H}^{qq}$ come from the $q_\mu q_\nu$ part of $\mathcal{H}_{\mu\nu}$ defined in Eq. (A4).

However, the quantity $R_{3,\rm YZ}$ is rather more complicated, and so before writing it down explicitly we first develop some useful notation. For a quantity $f(q^2,\dots)$, we define the auxiliary functions

$$\overline{f}(q^2,\dots)\equiv f(q^2,\dots)-f(M^2,\dots)$$

$$\check{f}(q^2,\dots)\equiv\overline{f}(q^2,\dots)-(q^2-M^2)f'(M^2,\dots), \tag{A18}$$

where the M^2 quantities become M_π^2 for the case of isospin flavor and M_η^2 for hypercharge flavor. Then we have for R_3 the expression

$$\begin{aligned}R_3(q^2)\equiv&\frac{2}{q^2-M_\pi^2}\left[\frac{2}{9}\,\overline{I}_{1,\mathrm{YZ}}(q^2;M_\pi^2;M_\pi^2;M_\pi^2)+\frac{1}{36}\,\overline{I}_{1,\mathrm{YZ}}(q^2;M_\pi^2;M_K^2;2(M_\pi^2+M_K^2))\right.\\&\left.+\frac{1}{12}\,\overline{I}_{1,\mathrm{YZ}}\left(q^2;M_\eta^2;M_K^2;\frac{2(M_\pi^2-M_K^2)}{3}\right)+\frac{1}{2}\,\overline{I}_{2,\mathrm{YZ}}(q^2;M_\pi^2;M_K^2)\right]\\&-\frac{1}{(q^2-M_\pi^2)^2}\left[\frac{M_\pi^4}{6}\,\check{S}_{\mathrm{YZ}}(q^2;M_\pi^2;M_\pi^2)+\frac{M_\pi^4}{18}\,\check{S}_{\mathrm{YZ}}(q^2;M_\pi^2;M_\eta^2)+\frac{1}{4}\,\check{U}_{\mathrm{YZ}}(q^2;M_\pi^2;M_K^2)\right.\\&\left.+\frac{1}{9}\,\check{R}_{\mathrm{YZ}}(q^2;M_\pi^2;M_\pi^2;M_\pi^2)+\frac{1}{72}\,\check{R}_{\mathrm{YZ}}(q^2;M_\pi^2;M_K^2;2(M_\pi^2+M_K^2))+\frac{1}{24}\,\check{R}_{\mathrm{YZ}}\left(q^2;M_\eta^2;M_K^2;\frac{2(M_\pi^2-M_K^2)}{3}\right)\right].\end{aligned} \tag{A19}$$

For the spin-zero isospin polarization function, the piece coming from the finite sunset functions is

$$\begin{aligned}F_\pi^2\hat{\Pi}^{(0)}_{3,\mathrm{SUN}}(q^2)=&4S_{2,\mathrm{YZ}}(q^2,M_\pi^2,M_\pi^2)+\frac{3}{2}S_{2,\mathrm{YZ}}(q^2,M_\eta^2,M_\pi^2)+\frac{1}{2}S_{2,\mathrm{YZ}}(q^2,M_\pi^2,M_K^2)+\frac{1}{3}\,\mathcal{K}_{2,\mathrm{YZ}}(q^2,M_\pi^2,M_K^2)\\&+q^2\left(\frac{4}{9}\,\mathcal{H}^{qq}_{\mathrm{YZ}}(q^2,M_\pi^2,M_\pi^2)+\frac{1}{6}\,\mathcal{H}^{qq}_{\mathrm{YZ}}(q^2,M_\eta^2,M_\pi^2)+\frac{1}{18}\,\mathcal{H}^{qq}_{\mathrm{YZ}}(q^2,M_\pi^2,M_K^2)\right.\\&\left.+\frac{1}{3}\,\mathcal{K}_{1,\mathrm{YZ}}(q^2,M_\pi^2,M_K^2)-R_{3,\mathrm{YZ}}(q^2)\right)-2\left[\frac{2}{9}I_{1,\mathrm{YZ}}(M_\pi^2;M_\pi^2;M_\pi^2;M_\pi^2)+\frac{1}{2}I_{2,\mathrm{YZ}}(M_\pi^2;M_\pi^2;M_K^2)\right.\\&+\frac{1}{36}I_{1,\mathrm{YZ}}(M_\pi^2;M_\pi^2;M_K^2;2(M_\pi^2+M_K^2))+\frac{1}{12}I_{1,\mathrm{YZ}}\left(M_\pi^2;M_\eta^2;M_K^2;\frac{2}{3}(M_\pi^2-M_K^2)\right)\\&-\frac{M_\pi^4}{12}S'_{\mathrm{YZ}}(M_\pi^2;M_\pi^2;M_K^2)-\frac{M_\pi^4}{36}S'_{\mathrm{YZ}}(M_\pi^2;M_\pi^2;M_\eta^2)-\frac{1}{18}R'_{\mathrm{YZ}}(M_\pi^2;M_\pi^2;M_\pi^2;M_\pi^2)\\&-\frac{1}{144}R'_{\mathrm{YZ}}(M_\pi^2;M_\pi^2;M_K^2;2(M_\pi^2+M_K^2))-\frac{1}{48}R'_{\mathrm{YZ}}\left(M_\pi^2;M_\eta^2;M_K^2;\frac{2}{3}(M_\pi^2-M_K^2)\right)\\&\left.-\frac{1}{8}\,U'_{\mathrm{YZ}}(M_\pi^2;M_\pi^2;M_K^2)\right].\end{aligned} \tag{A20}$$

3. Finite sunset integrals

Having identified the singular parts of the sunset functions by expanding these quantities in a Laurent series about $d=4$, one can express the finite-valued functions $\{Y_c^{(n)}\}$ and $\{Z_c^{(n)}\}$ which remain by means of integral representations

$$Y_c^{(n)}\equiv\frac{1}{(16\pi^2)^2}\int_{4M^2}^\infty\frac{d\sigma}{\sigma}\left(1-\frac{4M^2}{\sigma}\right)^{1/2+c}\times\int_0^1 dx\,x^n\ln(1+\Delta g) \tag{A21}$$

and

$$Z_c^{(n)}\equiv\frac{1}{(16\pi^2)^2}\int_{4M^2}^\infty d\sigma\left(1-\frac{4M^2}{\sigma}\right)^{1/2+c}\times\int_0^1 dx\,x^n[\ln(1+\Delta g)-\Delta g] \tag{A22}$$

where

$$\Delta g\equiv\left(\frac{m^2}{x}-q^2\right)\frac{1-x}{\sigma}. \tag{A23}$$

For convenience, we shall introduce the dimensionless variables

$$\overline{q^2} \equiv \frac{q^2}{4M^2} \quad \text{and} \quad r^2 \equiv \frac{m^2}{4M^2}, \tag{A24}$$

and likewise work with the *reduced functions* $\overline{Y_c^{(n)}}$, $\overline{Z_c^{(n)}}$:

$$Y_c^{(n)}(\overline{q^2},r^2) \equiv \frac{1}{(16\pi^2)^2}\,\overline{Y_c^{(n)}}(\overline{q^2},r^2),$$

$$Z_c^{(n)}(\overline{q^2},r^2) \equiv \frac{4M^2}{(16\pi^2)^2}\,\overline{Z_c^{(n)}}(\overline{q^2},r^2). \tag{A25}$$

One is allowed to express such finite quantities in terms of the physical meson masses, and it is understood we do so in the remainder of this section. For the six flavor configurations which can contribute to the sunset amplitude, the parameter r^2 takes on the numerical values

$$r^2 = \begin{cases} 0.016 & (\eta\eta\pi) \\ 0.020 & (\bar{K}K\pi) \\ 0.25 & (3\pi,3\eta) \\ 0.31 & (\bar{K}K\eta) \\ 3.82 & (\pi\pi\eta). \end{cases} \tag{A26}$$

a. Behavior at $r^2=0$ and near $q^2=0$

In the $r^2=0$ limit (i.e., $m^2=0$), analytic expressions can be obtained for $\overline{Y_c^{(n)}}$ and $\overline{Z_c^{(n)}}$:

$$\overline{Y_c^{(n)}}(\overline{q^2},0) = -\sum_{k=1}^{\infty} \frac{B(k+1;n+1)B(k;c+3/2)}{k}\,\overline{q^2}^k \tag{A27}$$

and

$$\overline{Z_c^{(n)}}(\overline{q^2},0) = -\sum_{k=2}^{\infty} \frac{B(k+1;n+1)B(k-1;c+3/2)}{k}\,\overline{q^2}^k, \tag{A28}$$

where $B(m;n)$ denotes the Euler beta function. Observe in the summations that the indices begin at $k=1$ for $\overline{Y_c^{(n)}}$ and at $k=2$ for $\overline{Z_c^{(n)}}$, i.e. that

$$\overline{Y_c^{(n)}}(0,0) = \overline{Z_c^{(n)}}(0,0) = \overline{Z_c^{(n)}}'(0,0) = 0. \tag{A29}$$

For the more general case of nonzero r^2 but small q^2, it is useful to employ a power series

$$\overline{Y_c^{(n)}}(\overline{q^2},r^2) = \overline{Y_c^{(n)}}(0,r^2) + \overline{Y_c^{(n)}}'(0,r^2)\overline{q^2} + \frac{1}{2}\,\overline{Y_c^{(n)}}''(0,r^2)\overline{q^4} + \dots$$

$$\overline{Z_c^{(n)}}(\overline{q^2},r^2) = \overline{Z_c^{(n)}}(0,r^2) + \overline{Z_c^{(n)}}'(0,r^2)\overline{q^2} + \frac{1}{2}\,\overline{Z_c^{(n)}}''(0,r^2)\overline{q^4} + \dots. \tag{A30}$$

For nonzero r^2, one can obtain numerical values for the above $q^2=0$ derivatives of $\overline{Y_c^{(n)}}(\overline{q^2},r^2)$ and $\overline{Z_c^{(n)}}(\overline{q^2},r^2)$. Of course, the integral representations of Eqs. (A21),(A22) allow also for a straightforward numerical determination of the real part of the sunset amplitudes for arbitrary q^2. However, some care must be taken to obtain accurate values for q^2 close to or above three-particle thresholds.

b. Imaginary parts

For $\overline{q^2}<1$, the finite sunset amplitudes are real valued. However, $Y_c^{(n)}$ and $Z_c^{(n)}$ have a branch point singularity at $\overline{q^2}=(1+r)^2$ [corresponding to $q^2=(2M+m)^2$] and become complex valued for $\overline{q^2}>(1+r)^2$. We shall be concerned here with determining the imaginary parts of these quantities.

Consider first the integral $X^{(n)}(\overline{q^2},r^2)$ defined by

$$X^{(n)}(\overline{q^2},r^2) \equiv \int_0^1 dx\,x^n \ln(1+\Delta g), \tag{A31}$$

which can be rewritten as

$$\begin{aligned} X^{(n)}(\overline{q^2},r^2) &= \int_0^1 dx\,x^n \left\{ \ln\left[x^2 + x\left(\frac{1}{u\overline{q^2}} - 1 - \frac{r^2}{\overline{q^2}}\right) + \frac{r^2}{\overline{q^2}}\right] + \ln(u\overline{q^2}/x)\right\} \\ &= \int_0^1 dx\,x^n [\ln[(x-x_+)(x-x_-)] + \ln(u\overline{q^2}/x)], \end{aligned} \tag{A32}$$

where $x_\pm$ are given by

$$x_\pm = \frac{1}{2}\left[1 - \frac{1}{u\overline{q^2}} + \frac{r^2}{\overline{q^2}} \pm \sqrt{\left(1 - \frac{1}{u\overline{q^2}} + \frac{r^2}{\overline{q^2}}\right)^2 - \frac{4r^2}{\overline{q^2}}}\right]. \tag{A33}$$

The imaginary part of $X^{(n)}(\overline{q^2},r^2)$ will occur when the argument of the first logarithm in the above becomes negative,

$$\begin{aligned} \text{Im}\, X^{(n)}(\overline{q^2},r^2) &= \int_0^1 dx\,x^n \,\text{Im}\,\ln[(x-x_+)(x-x_-)] \\ &= -\frac{\pi}{n+1}(x_+^{n+1} - x_-^{n+1}), \end{aligned} \tag{A34}$$

so that

$$\text{Im}\,\overline{Y_c^{(n)}}(\overline{q^2},r^2) = -\frac{\pi}{n+1}\int_{u_0}^1 \frac{du}{u}(1-u)^{1/2+c}(x_+^{n+1} - x_-^{n+1}), \tag{A35}$$

where

$$u_0 = \frac{1}{(\sqrt{\overline{q^2}} - \sqrt{r^2})^2}. \tag{A36}$$

The lower limit u_0 on the u-integral is simply a reflection of the branch point occurring in the sunset amplitude at $q^2 = (2M+m)^2$. Proceeding in a like manner leads to the following formula for Im $Z_c^{(n)}$:

$$\text{Im}\ \overline{Z}_c^{(n)}(\overline{q^2},r^2) = -\frac{\pi}{n+1}\int_{u_0}^{1}\frac{du}{u^2}(1-u)^{1/2+c}(x_+^{n+1}-x_-^{n+1}). \tag{A37}$$

4. Identities

Given the set of sunset integrals $\{S;S_\mu;S_{\mu\nu}\}$, it is not difficult to infer the following ''trace identity'':

$$S_\mu^\mu(q^2,m^2,M^2) = m^2 S(q^2,m^2,M^2) + A^2(M^2), \tag{A38}$$

which is valid for arbitrary kinematics.

It turns out that several more identities become derivable in the equal mass limit of $SU(3)$ symmetry. This is a consequence of the symmetry constraint that the isospin and hypercharge results agree. Indeed, their direct comparison serves to check the correctness of the calculation. Interestingly, the identities discovered in the $SU(3)$ limit are typically not at all *a priori* obvious. Below, we list and indicate the source of relations:

(1) Relating $\overline{S}$ to S:

$$\overline{S}(q^2,m^2,m^2) = \frac{1}{3}S(q^2,m^2,m^2). \tag{A39}$$

(2) 1PI amplitudes:

$$\mathcal{H}_{\mu\nu}(q^2,m^2,m^2) = 3\mathcal{L}_{\mu\nu}(q^2,m^2,m^2). \tag{A40}$$

(3) Vertex functions:

$$I_{1\mu}(q^2;m^2;m^2;\Lambda) = I_{1\mu}(q^2;m^2;m^2;0)$$
$$I_{1\mu}(q^2;m^2;m^2;0) = 3I_{2\mu}(q^2;m^2;m^2;0). \tag{A41}$$

(4) Self-energies:

$$R(q^2;m^2;m^2;\Lambda) = R(q^2;m^2;m^2;0) + \Lambda^2 S(q^2;m^2;m^2)$$
$$R(q^2;m^2;m^2;0) = 3U(q^2;m^2;m^2). \tag{A42}$$

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