Possible generalization of the superstring action to eleven dimensions

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We suggest a $D=11$ super Poincaré invariant action for the superstring which has free dynamics in the physical variables sector. Instead of the standard approach based on the searching for an action with local κ symmetry (or, equivalently, with corresponding first class constraints), we propose a theory with fermionic constraints of second class only. Then the κ symmetry and the well known Γ-matrix identities are not necessary for the construction. Thus, at the classical level, the superstring action of the type described can exist in any spacetime dimensions and the known brane scan can be reexamined. [S0556-2821(98)05814-7]

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I. INTRODUCTION

A revival of interest in the problem of the covariant formulation of eleven-dimensional superstring is due to the search for M theory (see Refs. [1–5] and references therein) which is expected to be the underlying quantum theory for the known extended objects. In the strong coupling limit of M theory $R^{11}\rightarrow\infty$, where R^{11} is the radius of the 11th dimension, the vacuum is an eleven-dimensional Minkowski vacuum and the effective field theory is $D=11$ supergravity. Up to date, $D=11$ supergravity is viewed as the strong coupling limit of the ten-dimensional type-IIA superstring [1]. Since $D=11$ super Poincaré symmetry survives in this special point in the moduli space of M-theory vacua (''uncompactified M theory'' according to Ref. [5]), one may ask about the existence of a consistent $D11$ quantum theory with $D=11$ supergravity being its low energy limit. One possibility is the supermembrane action [6–8], but in this case one faces the problem of a continuous spectrum for the first quantized supermembrane [9,10]. By analogy with the ten-dimensional case, where the known supersymmetric field theories can be obtained as the low energy limit of the corresponding superstrings [5], the other natural candidate might be a $D=11$ superstring theory. But the problem is that a covariant formulation for the $D=11$ superstring action is unknown even at the classical level. The classical Green-Schwarz (GS) superstring (with manifest space-time supersymmetry and local κ symmetry) can propagate in three, four, six, and ten spacetime dimensions [11] and the standard approach fails to construct a $D=11$ superstring action.

The crucial ingredient in the construction of the GS superstring action is the Γ-matrix identity

$$\Gamma^{\mu}_{\alpha(\beta}(C\Gamma^{\mu})_{\gamma\delta)}=0. \tag{1}$$

It provides the existence of both global supersymmetry and local κ symmetry for the action [11,12]. The κ symmetry, in its turn, eliminates half of the initial θ variables as well as provides free dynamics in the physical variable sector. In this paper we discuss a possibility to construct a classical superstring action with those two properties in eleven dimensions. Subsequent development of our method may shed light on the problem of constructing the corresponding quantum theory. To elucidate the construction which will be suggested below let us discuss the problem in the Hamiltonian framework, where one finds the well-known fermionic constraints $L_\alpha=0$ (see, for example, Refs. [11,12]) which obey the Poisson brackets

$$\{L_\alpha,L_\beta\}=2i(\hat{p}^{\mu}+\Pi_1^{\mu})\Gamma^{\mu}_{\alpha\beta}\delta(\sigma-\sigma')$$
$$-2\overline{\theta}^{\gamma}\partial_1\theta^{\delta}\Gamma^{\mu}_{\gamma(\delta}(C\Gamma^{\mu})_{\alpha\beta)}\delta(\sigma-\sigma'). \tag{2}$$

By virtue of Eq. (1), the last term in Eq. (2) vanishes for $D=3,4,6,10$. The resulting equation then means that half of the constraints are first class, which exactly corresponds to the κ symmetry presented in the Lagrangian framework.

The next step is to impose an appropriate gauge. Then the set of functions

$$L_\alpha=0, \tag{3}$$

$$\Gamma^{+}\theta=0 \tag{4}$$

is a system of second class [even though Eq. (1) has not been used].

The situation changes drastically for the $D=11$ case, where instead of Eq. (1) one finds [13–15]

$$10\Gamma^{\mu}_{\alpha(\beta}(C\Gamma^{\mu})_{\gamma\delta)}+\Gamma^{\mu\nu}_{\alpha(\beta}(C\Gamma^{\mu\nu})_{\gamma\delta)}=0. \tag{5}$$

Being appropriate for the construction of the supermembrane action [6], this identity does not allow one to formulate a $D=11$ superstring with desirable properties. As was shown by Curtright [13], the globally supersymmetric action based on this identity involves, additional to x^i, θ_a, $\overline{\theta_{\dot a}}$ degrees of freedom in the physical sector. Moreover, it does not possess a κ symmetry that could provide free dynamics [13,14].

In this paper we suggest a $D=11$ super Poincaré invariant action for the classical superstring which has free dynamics in the physical variable sector. Instead of the standard approach based on the searching for an action with local κ symmetry (or, equivalently, with corresponding first class constraints), we present a theory in which covariant constraints like Eqs. (3), (4) arise among others. Since it is a

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system of second class constraints, κ symmetry and the identity (5) are not necessary for the construction. Thus, at the classical level, a superstring of the type described can exist in any spacetime dimension and the known brane scan [4] can be reexamined. For definiteness, in this paper we discuss the $D=11$ case only.

Two comments are in order. First, one needs to covariantize Eq. (4). The simplest possibility is to introduce an auxiliary variable $\Lambda^\mu(\tau,\sigma)$ subject to $\Lambda^2=0$ and replace Eq. (4) by $\Lambda_\mu\Gamma^\mu\theta=0$. The most preferable formulation seems to be that in which the gauge $\Lambda^-=1$ is possible. Then Eq. (4) is reproduced. Unfortunately, it seems to be impossible to introduce a pure gauge variable with the desired properties [16–20]. Below, we present a formulation in which only zero modes of auxiliary variables survive in the sector of physical degrees of freedom. Since the state spectrum of a string is determined by the action on the vacuum of oscillator modes only, one can expect that the presence of the zero modes will be inessential for the case. This fact will be demonstrated within the canonical quantization framework in Secs. II and IV.

Second, one expects that a model with constraints like Eqs. (3), (4) will possess (if any) off-shell super Poincaré symmetry in a nonstandard realization. Actually, global supersymmetry which does not spoil the equation $\Lambda_\mu\Gamma^\mu\theta=0$ looks like $\delta\theta\sim\Lambda_\mu\Gamma^\mu\epsilon$. On shell, where $\Lambda^2=0$, only half of the supersymmetry parameters ϵ^α are essential.

It is worth mentioning another motivation for this work. As was shown in Refs. [21–25], an action for the super D-brane allowing for the local κ symmetry is very complicated. One can hope that our method, being applied to that case, will lead to a more simple formulation.

The work is organized as follows. In Sec. II we present and discuss an action for the auxiliary variable Λ^μ, which proves to be a necessary ingredient of our construction. In Sec. III a covariant action for the eleven-dimensional superstring and its local symmetries are presented. In Sec. IV within the framework of the Hamiltonian approach we prove that it has free dynamics. In Sec. V the role of the Wess-Zumino term presented in the action is elucidated. In Sec. VI off-shell realization of the super Poincaré algebra is derived and discussed. The Appendix contains our spinor convention for $D=11$.

II. ACTION FOR AUXILIARY VARIABLES AND THEIR DYNAMICS

As was mentioned in the Introduction, we need to get at our disposal an auxiliary lightlike variable. So as a preliminary step of our construction, let us discuss the $D=11$ Poincaré invariant action

$$S=-\int d^2\sigma\left[\Lambda^\mu\varepsilon^{ab}\partial_a A^\mu_b+\frac{1}{\phi}\Lambda^\mu\Lambda^\mu\right], \tag{6}$$

which turns out to be a building block of the eleven-dimensional superstring action considered below. Here $\Lambda^\mu(\sigma^a)$ is a $D=11$ vector and a $d2$ scalar, and $A^\mu_a(\sigma^b)$ is a $D=11$ and $d2$ vector, while $\phi(\sigma^a)$ is a scalar field. In Eq. (6) we have set $\varepsilon^{ab}=-\varepsilon^{ba}$, $\varepsilon^{01}=-1$ and it was also supposed that $\sigma^1\subset[0,\pi]$. From the equation of motion $\delta S/\delta\phi=0$ it follows that Λ^μ is a lightlike vector.

Local symmetries of the action are $d=2$ reparametrizations[1] and the following transformations with the parameters $\xi^\mu(\sigma^a)$, $\omega_a(\sigma^b)$:

$$\delta_\xi A^\mu_a=\partial_a\xi^\mu,$$
$$\delta_\omega A^\mu_a=\omega_a\Lambda^\mu,$$
$$\delta_\omega\phi=\frac{1}{2}\phi^2\varepsilon^{ab}\partial_a\omega_b. \tag{7}$$

These symmetries are reducible because their combination with the parameters of a special form, $\omega_a=\partial_a\omega$, $\xi^\mu=-\omega\Lambda^\mu$, is a trivial symmetry: $\delta_\omega A^\mu_a=-\omega\partial_a\Lambda^\mu$, $\delta_\omega\phi=0$ (note that $\partial_a\Lambda^\mu=0$ is one of the equations of motion). Thus, Eq. (7) includes 12 essential parameters which correspond to the primary first class constraints $p^\mu_0\approx0$, $\pi_\phi\approx0$ in the Hamilton formalism (see below).

Let us consider the theory in the Hamiltonian framework. Momenta conjugate to the variables Λ^μ, A^μ_a, ϕ are denoted by p^μ_Λ, p^μ_a, π_ϕ. All equations for determining the momenta turn out to be the primary constraints

$$\pi_\phi=0,$$
$$p^\mu_0=0, \tag{8}$$
$$p^\mu_\Lambda=0,$$
$$p^\mu_1-\Lambda^\mu=0. \tag{9}$$

The canonical Hamiltonian is

$$H=\int d\sigma^1\Big[\Lambda^\mu\partial_1A^\mu_0+\frac{1}{\phi}\Lambda^2+\lambda_\phi\pi_\phi+\lambda^\mu_\Lambda p^\mu_\Lambda+\lambda^\mu_0p^\mu_0+\lambda^\mu_1(p^\mu_1-\Lambda^\mu)\Big], \tag{10}$$

where λ_* are the Lagrange multipliers corresponding to the constraints. The preservation in time of the primary constraints implies the secondary ones

$$\partial_1\Lambda^\mu=0,$$
$$\Lambda^2=0, \tag{11}$$

and equations for determining some of the Lagrange multipliers:

$$\lambda^\mu_1=\partial_1A^\mu_0+\frac{2}{\phi}\Lambda^\mu,$$
$$\lambda^\mu_\Lambda=0. \tag{12}$$

The tertiary constraints are absent.

[1] Note that the coupling to the $d=2$ metric $g^{ab}(\sigma^c)$ is not necessary due to the presence of the ε^{ab} symbol and the supposition that the variable ϕ transforms as a density $\phi'(\sigma')=\det(\partial\sigma'/\partial\sigma)\phi(\sigma)$ under reparametrizations.

Constraints (9) form a system of second class and can be omitted after introducing the corresponding Dirac brackets (the Dirac brackets for the remaining variables prove to coincide with the Poisson ones). After imposing the gauge-fixing conditions $\phi=2$, $A_0^\mu=0$ for the first class constraints (8), the dynamics of the remaining variables is governed by the equations

$$\dot{A}_1^\mu=p_1^\mu,$$
$$\dot{p}_1^\mu=0, \tag{13}$$

$$(p_1^\mu)^2=0,$$
$$\partial_1 p_1^\mu=0. \tag{14}$$

In order to find a correct gauge for the second constraint in Eq. (14), let us consider Fourier decomposition of functions periodical in the interval $\sigma\subset[0,\pi]$:

$$A_1^\mu(\tau,\sigma)=Y^\mu(\tau)+\sum_{n\neq 0} y_n^\mu(\tau)e^{i2n\sigma},$$

$$p_1^\mu(\tau,\sigma)=P_y^\mu(\tau)+\sum_{n\neq 0} p_n^\mu(\tau)e^{i2n\sigma}. \tag{15}$$

Then the constraint $\partial_1 p_1^\mu=0$ is equivalent to $p_n^\mu=0$, $n\neq 0$, and an appropriate gauge is $y_n^\mu=0$ or, in the equivalent form, $\partial_1 A_1^\mu=0$. Thus, physical degrees of freedom of the model are the zero modes[2] of these variables, and the corresponding dynamics is

$$A_1^\mu(\tau,\sigma)=Y^\mu+P_y^\mu\tau,$$

$$p_1^\mu(\tau,\sigma)=P_y^\mu=\text{const},$$

$$(P_y)^2=0. \tag{16}$$

Since there are no oscillator variables, the action (6) can be considered as describing a pointlike object, which propagates freely according to Eq. (16). The only quantum state is its ground state $|p_{y0}\rangle$ with mass $m_y^2=p_{y0}^2=0$. As a result, these degrees of freedom do not make contributions into the state spectrum of the superstring (see Sec. IV), and manifest themselves in additional degeneracy of the continuous part of the energy spectrum only. The action of such a kind was successfully used before [26,27] in a different context.

Note that in the previous discussion it was assumed that variables of the theory are periodical in the interval $\sigma\subset[0,\pi]$. For an open world sheet, the stationarity condition $\delta S_\Gamma=0$ for the Hamiltonian action,

$$S_\Gamma=\int d^2\sigma[p_A\dot{q}^A-H(q,p)],$$

yields

$$\int d\tau(\Lambda^\mu\delta A_0^\mu|_{\sigma=0}^{\sigma=\pi})=0. \tag{17}$$

Since the variations $\delta A_0^\mu|_{\sigma=0,\pi}$ are arbitrary, this equation requires $\Lambda^\mu|_{\sigma=0,\pi}=0$. By virtue of Eq. (9) it leads to the trivial solution $p_1^\mu|_{\sigma=0,\pi}=0$. In contrast, for a closed world sheet one has $\delta\Gamma^A|_{\sigma=0}=\delta\Gamma^A|_{\sigma=\pi}$ for any variable Γ^A and Eq. (17) is automatically satisfied. Hence, the model (6) has a nontrivial solution being determined on the closed world sheet only.

III. ELEVEN-DIMENSIONAL SUPERSTRING ACTION AND ITS LOCAL SYMMETRIES

The $D=11$ action functional to be examined is

$$S=\int d^2\sigma\Bigg\{\frac{-g^{ab}}{2\sqrt{-g}}\Pi_a^\mu\Pi_b^\mu-i\varepsilon^{ab}\partial_a x^\mu(\bar{\theta}\Gamma^\mu\partial_b\theta)$$
$$-i\Lambda^\mu\bar{\psi}\Gamma^\mu\theta-\frac{1}{\phi}\Lambda^\mu\Lambda^\mu-\Lambda^\mu\varepsilon^{ab}\partial_a A_b^\mu\Bigg\}, \tag{18}$$

where θ, ψ are 32-component Majorana spinors and $\Pi_a^\mu\equiv\partial_a x^\mu-i\bar{\theta}\Gamma^\mu\partial_a\theta$. Let us mention the origin of the terms presented in Eq. (18). The first two terms are exactly GS-type superstring action written in eleven dimensions. The meaning of the last two terms has been explained in the previous section. The third and the fourth terms will supply the appearance of the equations $\Lambda_\mu\Gamma_\mu\theta=0$ and $\Lambda^2=0$. Thus, the variables $\bar{\psi}^\alpha$ and ϕ are, in fact, the Lagrange multipliers for these constraints.

Note also that the Wess-Zumino term in the $D=10$ GS action provides the appearance of the local κ symmetry [9]. In our model it plays a different role, as will be discussed below.

Let us make a comment on the local symmetry structure of the action (18). Local bosonic symmetries are $d=2$ reparametrizations [with the standard transformation laws for all variables except for the variable ϕ, which transforms as a density: $\phi'(\sigma')=\det(\partial\sigma'/\partial\sigma)\phi(\sigma)$], Weyl symmetry, and the transformations with parameters $\xi^\mu(\sigma^a)$ and $\omega_a(\sigma^b)$ described in the previous section.

There is also a fermionic symmetry with parameters $\chi^\alpha(\sigma^a)$,

$$\delta\bar{\psi}=\bar{\chi}\Gamma^\mu\Lambda_\mu,$$

$$\delta\phi=-\phi^2(\bar{\chi}\theta), \tag{19}$$

from which only 16 are essential on shell since $\Lambda^2=0$. As shown below, the reducibility of this symmetry produces no special problem for covariant quantization.

Let us present arguments that the action constructed describes a free theory. The equations of motion for the theory (18) are

$$\Pi_a^\mu\Pi_b^\mu-\frac{1}{2}g_{ab}(g^{cd}\Pi_c^\mu\Pi_d^\mu)=0, \tag{20a}$$

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[2]We are grateful to N. Berkovits and J. Gates for bringing this fact to our attention.

$$\partial_a\left(\frac{g^{ab}}{\sqrt{-g}}\Pi_b^\mu+i\varepsilon^{ab}\bar{\theta}\Gamma^\mu\partial_b\theta\right)=0, \tag{20b}$$

$$4i\Pi_b^\mu(\Gamma^\mu P^{-ba}\partial_a\theta)_\alpha+\varepsilon^{ab}\theta^\beta\partial_a\theta^\gamma\partial_b\theta^\delta\Gamma^\mu_{\alpha(\beta}C\Gamma^\mu_{\gamma\delta)}$$
$$+i\Lambda^\mu(\Gamma^\mu\psi)_\alpha=0, \tag{20c}$$

$$\Lambda^\mu\Gamma^\mu\theta=0,$$

$$\Lambda^2=0, \tag{20d}$$

$$\partial_a\Lambda^\mu=0,$$

$$\varepsilon^{ab}\partial_a A_b^\mu+\frac{2}{\phi}\Lambda^\mu+i\bar{\psi}\Gamma^\mu\theta=0, \tag{20e}$$

where

$$P^{-ba}=\frac{1}{2}\left(\frac{g^{ba}}{\sqrt{-g}}-\varepsilon^{ba}\right).$$

Multiplying Eq. (20c) by $\Lambda_\mu\Gamma^\mu$ one gets

$$(\Lambda^\mu\Pi_b^\mu)P^{-ba}\partial_a\theta=0. \tag{21}$$

In the coordinate system where $\Lambda^-=1$, supplemented by the conformal gauge, it can be rewritten as

$$(\partial_0+\partial_1)\theta=0, \tag{22}$$

from which it follows that any solution $\theta(\sigma)$ of the system (20) obeys this free equation.

Thus, Eqs. (20a)–(20c) for the g^{ab}, x^μ, θ^α variables in fact coincide with those of the GS string and are accompanied by $\Lambda_\mu\Gamma^\mu\theta=0$. The latter reduces to $\Gamma^+\theta=0$ in the coordinate system chosen. As a result, one expects free dynamics in this sector provided that the conformal gauge has been assumed. In the next section we will rigorously prove this fact by direct calculations in the Hamiltonian framework.

IV. ANALYSIS OF DYNAMICS

From the explicit form of the action functional (18) it follows that the variable Λ^μ can be excluded by making use of its equation of motion. The Hamiltonian analogue of the situation is a pair of second class constraints $p_\Lambda{}^\mu=0$, $p_1{}^\mu-\Lambda^\mu=0$, which can be omitted after introducing the associated Dirac bracket (see Sec. II). The Dirac brackets for the remaining variables prove to coincide with the Poisson ones and the Hamiltonian looks like

$$H=\int d\sigma^1\left\{-\frac{N}{2}(\hat{p}^2+\Pi_{1\mu}\Pi_1^\mu)-N_1\hat{p}_\mu\Pi_1^\mu\right.$$
$$+p_{1\mu}(\partial_1A_0^\mu+i\bar{\psi}\Gamma^\mu\theta)+\frac{1}{\phi}(p_1^\mu)^2+\lambda_\phi\pi_\phi+\lambda_{0\mu}p_0^\mu$$
$$\left.+\lambda^{ab}(\pi_g)_{ab}+\lambda_\psi{}^\alpha p_{\psi\alpha}+L_\alpha\lambda_\theta{}^\alpha\right\}, \tag{23}$$

where p^μ, p_0^μ, p_1^μ, $p_{\psi\alpha}$, $(\pi_g)_{ab}$ are momenta conjugate to the variables x^μ, A_0^μ, A_1^μ, ψ_α, g_{ab}, respectively; λ_* are Lagrange multipliers corresponding to the primary constraints. In Eq. (23) we also denoted

$$N=\frac{\sqrt{-g}}{g^{00}},$$

$$N_1=\frac{g^{01}}{g^{00}},$$

$$\hat{p}^\mu=p^\mu-i\bar{\theta}\Gamma^\mu\partial_1\theta,$$

$$L_\alpha\equiv p_{\theta\alpha}-i(p^\mu+\Pi_1^\mu)(\bar{\theta}\Gamma^\mu)_\alpha=0. \tag{24}$$

It is interesting to note that the fermionic constraints $L_\alpha=0$ obey the algebra (2) and, being considered on their own (without taking into account the constraints $\bar{\theta}\Gamma^\mu p_{1\mu}=0$ which will arise below), form a system which has no definite class (this corresponds to the lack of κ symmetry in the GS action written in eleven dimensions).

The conservation in time of the primary constraints implies the secondary ones

$$\partial_1 p_1^\mu=0,$$

$$(p_1^\mu)^2=0,$$

$$(\bar{\theta}\Gamma^\mu)_\alpha p_1^\mu=0,$$

$$(\hat{p}^\mu\pm\Pi_1^\mu)^2=0, \tag{25}$$

$$(\bar{\lambda}_\theta\Gamma^\mu)_\alpha(\hat{p}^\mu+\Pi_1^\mu)+i\bar{\theta}^\gamma\partial_1\theta^\delta\lambda_\theta^\beta\Gamma^\mu_{\gamma(\delta}C\Gamma^\mu_{\beta\alpha)}+\frac{1}{2}(\bar{\psi}\Gamma^\mu)_\alpha\Lambda^\mu$$
$$-(\partial_1\bar{\theta}\Gamma^\mu)_\alpha(N+N_1)(\hat{p}^\mu+\Pi_1^\mu)$$
$$-\frac{1}{2}(\bar{\theta}\Gamma^\mu)_\alpha\partial_1(N\hat{p}^\mu+N_1\Pi_1^\mu)=0. \tag{26}$$

At the next step, there arises only one nontrivial equation. From the condition $\{\bar{\theta}\Gamma^\mu p_1^\mu,H\}=0$ one gets

$$(\bar{\lambda}_\theta\Gamma^\mu)_\alpha p_1^\mu=0. \tag{27}$$

Equations (26), (27) are equivalent to

$$\bar{\lambda}_\theta=(N+N_1)\partial_1\bar{\theta}+\frac{\tilde{\xi}}{2}\bar{\theta}, \tag{28}$$

$$\tilde{S}_\alpha\equiv(\bar{\psi}\Gamma^\mu)_\alpha p_1^\mu+(\bar{\theta}\Gamma^\mu)_\alpha\tilde{D}^\mu=0, \tag{29}$$

where we denoted

$$\tilde{D}^\mu=\tilde{\xi}(\hat{p}^\mu+\Pi_1^\mu)-\partial_1(N\hat{p}^\mu+N_1\Pi_1^\mu),$$

$$\tilde{\xi}=\frac{\partial_1(N\hat{p}^\mu+N_1\Pi_1^\mu)p_1^\mu}{(\hat{p}^\mu+\Pi_1^\mu)p_1^\mu}.$$

Thus, we have Eq. (28) for determining the Lagrange multiplier λ_θ and the tertiary constraint $\tilde{S}_\alpha=0$. One can check that there are no more constraints in the problem.

Hamiltonian equations of motion for the variables $(g^{ab},(\pi_g)_{ab})$, (ϕ,π_ϕ), (A_0^μ,p_0^μ), $(\psi^\alpha,p_\psi^\alpha)$ look like $\partial_0 q=\lambda_q$, $\partial_0 p_q=0$, while for other variables one has

$$\partial_0 A_1^\mu=\partial_1 A_0^\mu+\frac{2}{\phi}p_1^\mu+i\bar{\psi}\Gamma^\mu\theta,$$

$$\partial_0 p_1^\mu=0, \tag{30a}$$

$$\partial_0 x^\mu=-N\hat{p}^\mu-N_1\Pi_1^\mu-i\bar{\theta}\Gamma^\mu\lambda_\theta,$$

$$\partial_0 p^\mu=-\partial_1(N\Pi_1^\mu+N_1\hat{p}^\mu)+i\bar{\theta}\Gamma^\mu\lambda_\theta, \tag{30b}$$

$$\partial_0\theta^\alpha=-\lambda_\theta^\alpha. \tag{30c}$$

Note that equations $\partial_0 p_{\theta\alpha}=\cdots$ have been omitted since they follow from the constraints $L_\alpha=0$ and other equations.

To go further, note that the constraints $(\pi_g)_{ab}=0$ form nonvanishing Poisson brackets with the $\tilde{S}_\alpha$ from Eq. (29). A modification which splits them out of other constraints is

$$(\tilde{\pi}_g)_{ab}\equiv(\pi_g)_{ab}+\frac{1}{2(\hat{p}+\Pi_1)p_1}(p_\psi\Gamma^\mu\Gamma^\nu\theta)(\hat{p}^\mu+\Pi_1^\mu)T_{ab}^\nu,$$

with T_{ab}^ν being defined by the equality $\{(\pi_g)_{ab},\tilde{S}_\alpha\}=T_{ab}^\mu(\bar{\theta}\Gamma^\mu)_\alpha$. Hence, the constraints $(\tilde{\pi}_g)_{ab}=0$ are first class and one can adopt the gauge choice $g^{ab}=\eta^{ab}$. The full set of constraints can now be rewritten in a more simple form

$$\pi_\phi=0,$$

$$p_0^\mu=0, \tag{31a}$$

$$(p_1^\mu)^2=0,\partial_1 p_1^\mu=0,(\hat{p}^\mu\pm\Pi_1^\mu)^2=0,$$

$$L_\alpha=0,$$

$$\bar{\theta}\Gamma^\mu p_{1\mu}=0,$$

$$p_{\psi\alpha}=0,$$

$$S_\alpha\equiv\bar{\psi}\Gamma^\mu p_{1\mu}+(\bar{\theta}\Gamma^\mu)_\alpha D_\mu=0, \tag{31b}$$

where

$$D^\mu\equiv\xi(\hat{p}^\mu+\Pi_1^\mu)-\partial_1 p^\mu,$$

$$\xi\equiv\frac{\partial_1\hat{p}^\mu p_{1\mu}}{(\hat{p}^\nu+\Pi_1^\nu)p_{1\nu}}. \tag{32}$$

Now, let us impose gauge fixing conditions to the first class constraints (31a). The choice consistent with the equations of motion is

$$\phi=2,$$

$$A_0^\mu=-i\int_0^\sigma d\sigma'\,\bar{\psi}\Gamma^\mu\theta.$$

After that, dynamics for the remaining variables looks like

$$\partial_0\psi^\alpha=\lambda_\psi^\alpha,$$

$$\partial_0 p_{\psi\alpha}=0,$$

$$p_{\psi\alpha}=0,$$

$$S_\alpha=0, \tag{33a}$$

$$\partial_0 A_1^\mu=p_1^\mu,$$

$$\partial_0 p_1^\mu=0,$$

$$(p_1^\mu)^2=0,$$

$$\partial_1 p_1^\mu=0, \tag{33b}$$

$$\partial_0 x^\mu=-p^\mu,$$

$$\partial_0 p^\mu=-\partial_1\partial_1 x^\mu,$$

$$(\hat{p}^\mu\pm\Pi_1^\mu)^2=0, \tag{33c}$$

$$\partial_0\theta=-\partial_1\theta-\frac{\xi}{2}\theta,\quad L_\alpha=0,\quad(\bar{\theta}\Gamma^\mu)_\alpha p_{1\mu}=0. \tag{33d}$$

The sector (33a) includes $32+16$ independent constraints from which the first class ones can be picked out as follows:

$$(p_\psi\Gamma^\mu)_\alpha p_{1\mu}=0. \tag{34}$$

As was mentioned above, the reducibility of the constraints does not spoil the covariant quantization program. Actually, let us impose the following covariant (and redundant) gauge-fixing conditions for the constraints (34):

$$S_\alpha^1\equiv\frac{1}{(\hat{p}+\Pi_1)p_1}\bar{\psi}\Gamma^\mu(\hat{p}_\mu+\Pi_{1\mu})=0. \tag{35}$$

Then the set of equations $S_\alpha=0$, $S^1{}_\alpha=0$ is equivalent to

$$S'\equiv\bar{\psi}-\frac{1}{2(\hat{p}+\Pi_1)p_1}\bar{\theta}\Gamma^\mu D_\mu\Gamma^\nu(\hat{p}_\nu+\Pi_{1\nu}), \tag{36}$$

the latter forming nondegenerate Poisson brackets together with the constraints $p_{\psi\alpha}=0$:

$$\{p_{\psi\alpha},S'_\beta\}=-C_{\alpha\beta}. \tag{37}$$

After passing to the Dirac brackets associated with the second class functions $p_{\psi\alpha}$, S'_α, the variables ψ, p_ψ can be dropped.

To proceed further, we impose the gauge $\partial_1 A_1^\mu=0$ for the constraints in Eq. (33b), and pass to an appropriately chosen coordinate system. By making use of the Lorentz transformation one can consider a coordinate system where $P^\mu_y=(1,0,\ldots,0,1)$ (note that this is an admissible procedure within the canonical quantization approach since the Lorentz transformation is a particular example of a canonical one). To get the dynamics in the final form, we pass to the light-cone coordinates $x^\mu \to (x^+,x^-,x^i)$, $i=1,2,\ldots,8,10$, $\theta^\alpha \to (\theta_a, \overline{\theta'}_{\dot a}, \theta'_a, \overline{\theta}_{\dot a})$, $a,\dot a=1,\cdots,8$ and impose the gauge-fixing conditions

$$x^+=P^+\tau,$$

$$p^+=-P^+=\text{const}, \tag{38}$$

to the Virasoro first class constraints remaining in Eqs. (33c). The equation $\overline{\theta}\Gamma^\mu p_{1\mu}=0$ acquires now the form $\Gamma^+\theta=0$ and it is easy to show that $32+16$ constraints $L_\alpha=0$, $\Gamma^+\theta=0$ are second class. A solution is $\theta^\alpha=(\theta_a,0,0,\overline{\theta}_{\dot a})$ with θ_a and $\overline{\theta}_{\dot a}$ being $SO(8)$ spinors of opposite chirality. In the gauge chosen, the relation $(\hat p^\mu+\Pi^\mu_1)p_{1\mu}\neq 0$ holds which correlates with the assumption made above in Eqs. (32), (35). For the remaining variables one gets the free field equations

$$\partial_0 x^i=-p^i,$$

$$\partial_0 p^i=-\partial_1\partial_1 x^i,$$

$$(\partial_0+\partial_1)\theta_a=0,$$

$$(\partial_0+\partial_1)\overline{\theta}_{\dot a}=0. \tag{39}$$

Moreover, θ_a and $\overline{\theta}_{\dot a}$ form two pairs of self-conjugate variables under the Dirac brackets associated with the constraints from Eq. (33d):

$$\{\theta_a,\theta_b\}=\frac{i}{\sqrt{8}P^+}\delta_{ab},$$

$$\{\overline{\theta}_{\dot a},\overline{\theta}_{\dot b}\}=\frac{i}{\sqrt{8}P^+}\delta_{\dot a\dot b}. \tag{40}$$

Let us look shortly at the spectrum of the theory. The ground state of the full theory $|p_{y0},p_0,0\rangle=|p_{y0}\rangle|p_0\rangle|0\rangle$ is a direct product of vacua, where $P^2_y|p_{y0}\rangle=0$, $|p_0\rangle$ is a vacuum for zero modes of the variables x^μ,p^μ, while through $|0\rangle$ are denoted vacua for bosonic and fermionic oscillator modes. From Eq. (40) it follows that zero modes of the $\theta_a,\overline{\theta}_{\dot a}$ variables form the Clifford algebra which is also the symmetry algebra of a ground state. A representation space is 256 dimensional which corresponds to the spectrum of the $D=11$ supergravity [29]. The excitation levels are then obtained by acting with oscillators on the ground state. One notes that zero modes Y^μ,P^μ_y manifest themselves in additional degeneracy of the continuous energy spectrum only.

V. COMMENT ON THE WESS-ZUMINO TERM IN THE $D=11$ SUPERSTRING ACTION

For the $D=10$ GS superstring the Wess-Zumino term provides the local κ symmetry [11,12], which leads to free dynamics for physical variables. Since there is no κ symmetry in our construction, it is interesting to elucidate the meaning of this term in the $D=11$ action suggested. Let us consider the action (18) with the second term omitted. Canonical analysis for this model turns out to be very similar to that made above and we present results only.

Instead of Eqs. (24), (28), (29) one finds

$$L_\alpha\equiv p_{\theta\alpha}-i(\overline{\theta}\Gamma^\mu)_\alpha p^\mu=0,$$
$$\overline{\lambda}_\theta=\frac{N(\Pi_1 p_1)+N_1(pp_1)}{(pp_1)}\partial_1\overline{\theta},$$
$$\tilde S\equiv[(pp_1)\overline{\psi}-\partial_1\overline{\theta}\Gamma^\rho(N\Pi^\rho_1+N_1p^\rho)\Gamma^\nu p^\nu]\Gamma^\mu p^\mu_1=0. \tag{41}$$

In the coordinate system where $P^\mu_y=(1,0,\ldots,0,1)$ the analogue of Eqs. (33c), (33d) reads

$$\partial_0 x^\mu=-p^\mu-i\frac{\partial_1 x^+}{p^+}(\overline{\theta}\Gamma^\mu\partial_1\theta),$$

$$\partial_0 p^\mu=-\partial_1\Pi^\mu_1,$$

$$(p^\mu\pm\Pi^\mu_1)^2=0,$$

$$\partial_0\theta=-\frac{\partial_1 x^+}{p^+}\partial_1\theta,$$

$$L_\alpha=0,\Gamma^+\theta=0, \tag{42}$$

provided that the conformal gauge has been chosen.

To impose a gauge for the first class constraints $(p^\mu\pm\Pi^\mu_1)^2=0$, consider a one-parameter set of equations[3]

$$x^+=P^+(\tau+c\sigma),$$

$$p^+=-P^+=\text{const},$$

$$c=\text{const}\neq\pm 1, \tag{43}$$

which leads to the following dynamics for variables of the physical sector:

$$\partial_0 x^i=-p^i,$$

$$\partial_0 p^i=-\partial_1\partial_1 x^i,$$

$$(\partial_0-c\partial_1)\theta=0. \tag{44}$$

[3]The value $c=\pm 1$ is not admissible since in that case the Poisson brackets of the constraints $(p^\mu\pm\Pi^\mu_1)^2=0$ and the gauges (43) vanish.

One can check that it is impossible to get rid of the number c by making use of some other gauge choice for the g^{ab} and A_1^{μ} variables.

Thus, omitting the Wess-Zumino term in Eq. (18) one arrives at the theory which possesses all the properties of the model (18) with the only modification in the last equation of Eqs. (39): $(\partial_0 - c\partial_1)\theta = 0$ with c a constant. Depending on the gauge chosen it can take any value except $c = \pm 1$. Hence, the dynamics is not manifestly $d=2$ Poincaré covariant, provided that θ is a $d=2$ scalar. It is the Wess-Zumino term which corrects this inconsistency.

VI. OFF-SHELL REALIZATION OF THE $D=11$ SUPER POINCARÉ ALGEBRA

It is convenient first to recall some facts relating to the $D=10$ GS superstring. Off-shell realization of the super Poincaré algebra for that case includes the Poincaré transformations accompanied by the supersymmetries

$$\delta\theta^{\alpha} = \epsilon^{\alpha},$$

$$\delta x^{\mu} = -i\bar{\theta}\Gamma^{\mu}\epsilon. \tag{45}$$

Being considered on their own, in the gauge $\Gamma^{+}\theta = 0$ these transformations are reduced to trivial shifts for variables of the physical sector:

$$\delta\overline{\theta_{\dot{a}}} = \overline{\epsilon_{\dot{a}}},$$

$$\delta x^{i} = 0. \tag{46}$$

To get on-shell realization of the supersymmetry algebra, one needs to consider a combination of the ϵ and κ transformations $\delta_{\epsilon} + \delta_{\kappa(\epsilon)}$, which does not violate the gauge $\Gamma^{+}\theta = 0$. These transformations are (see, for example, Ref. [28])

$$\delta\overline{\theta_{\dot{a}}} = \overline{\epsilon_{\dot{a}}} + \frac{1}{P^{+}}\partial_{-}x^{i}\overline{\gamma^{i}}_{\dot{a}a}\epsilon_{a},$$

$$\delta x^{i} = -i\sqrt{2}(\overline{\theta\gamma^{i}}\epsilon). \tag{47}$$

We turn now to the $D=11$ case. Off-shell realization of the super Poincaré algebra for the action (18) includes the Poincaré transformations in the standard realization and the following supersymmetries with a 32-component spinor parameter ϵ^{α}:

$$\delta\theta = \tilde{\Lambda}\epsilon,$$

$$\delta x^{\mu} = -i\bar{\theta}\Gamma^{\mu}\tilde{\Lambda}\epsilon,$$

$$\delta A^{\mu}{}_{a} = -2i\epsilon_{ab}\frac{g^{bc}}{\sqrt{-g}}(\bar{\theta}\tilde{\Pi}_{c}\Gamma^{\mu}\epsilon) - 2i\partial_{a}x^{\nu}(\bar{\theta}\Gamma^{\nu}\Gamma^{\mu}\epsilon) - 2(\bar{\theta}\epsilon)(\bar{\theta}\Gamma^{\mu}\partial_{a}\theta),$$

$$\delta\bar{\psi} = i\epsilon^{ab}[\bar{\epsilon}\Gamma^{\mu}(\partial_{a}\bar{\theta}\Gamma^{\mu}\partial_{b}\theta) - 2\partial_{a}\bar{\theta}(\partial_{b}\bar{\theta}\epsilon)],$$

$$\delta\phi = -i\phi^{2}(\bar{\psi}\epsilon), \tag{48}$$

where $\tilde{\Lambda} \equiv \Lambda_{\mu}\Gamma^{\mu}$, $\tilde{\Pi}_{c} \equiv \Pi_{c}{}^{\mu}\Gamma^{\mu}$. The action is invariant up to total derivative terms. These transformations are the analogue of Eq. (45) since in the physical sector they are reduced to $\delta\theta_{a} = \sqrt{2}\epsilon'_{a}$, $\delta\overline{\theta_{\dot{a}}} = -\sqrt{2}\overline{\epsilon'_{\dot{a}}}$, $\delta x^{i} = 0$.

To find a global supersymmetry of the action (18) corresponding to Eq. (47) let us consider the following ansatz:

$$\delta\theta = \tilde{\Lambda}\tilde{\Pi}_{c}\epsilon^{c},$$

$$\delta\phi = -i\phi^{2}(\bar{\psi}\tilde{\Pi}_{c}\epsilon^{c}),$$

$$\delta x^{\mu} = 4i(\Lambda\Pi_{c})(\bar{\theta}\Gamma^{\mu}\epsilon^{c}) + 2i(\bar{\theta}\tilde{\Pi}_{c}\epsilon^{c})\Lambda^{\mu}, \tag{49}$$

where we denoted

$$\epsilon^{a}_{\alpha} \equiv P^{-ab}\epsilon_{\alpha\, b},$$

$$P^{-ab} = \frac{1}{2}\left(\frac{g^{ab}}{\sqrt{-g}} - \varepsilon^{ab}\right),$$

$$(\Lambda\Pi_{c}) \equiv \Lambda^{\mu}\Pi_{c}{}^{\mu}. \tag{50}$$

Variation of the GS part of the action (18) under these transformations looks like

$$\begin{aligned}\delta S_{GS} = \varepsilon^{ab}[&-8(\bar{\theta}\Gamma^{\mu}\epsilon^{c})(\partial_{a}\bar{\theta}\Gamma^{\mu}\partial_{b}\theta)(\Lambda\Pi_{c})\\ &-4(\bar{\theta}\tilde{\Pi}_{c}\epsilon^{c})(\partial_{a}\bar{\theta}\tilde{\Lambda}\partial_{b}\theta) + 2(\partial_{a}\bar{\theta}\Gamma^{\mu}\tilde{\Lambda}\tilde{\Pi}_{c}\epsilon^{c})\\ &\times(\bar{\theta}\Gamma^{\mu}\partial_{b}\theta) + (\bar{\theta}\Gamma^{\mu}\tilde{\Lambda}\tilde{\Pi}_{c}\epsilon^{c})(\partial_{a}\bar{\theta}\Gamma^{\mu}\partial_{b}\theta)]\\ &-2iP^{-ba}[4(\bar{\theta}\tilde{\Pi}_{c}\epsilon^{c})(\partial_{a}\Lambda\Pi_{b})\\ &+2(\partial_{a}\bar{\theta}\tilde{\Lambda}\epsilon^{c})(\Pi_{b}\Pi_{c}) - (\bar{\theta}\tilde{\Lambda}\partial_{a}\tilde{\Pi}_{b}\tilde{\Pi}_{c}\epsilon^{c})].\end{aligned} \tag{51}$$

After integrating by parts, reordering the $\tilde{\Lambda}$ and $\tilde{\Pi}$ terms, and making use of the identities

$$P^{-ab}P^{-cd} = P^{-cb}P^{-ad},$$

$$(\partial_{a}\bar{\theta}\Gamma^{\mu}\partial_{b}\theta)(\Lambda\Pi_{c}) = -\frac{1}{2}\partial_{a}\bar{\theta}\Gamma^{\mu}\{\tilde{\Lambda},\tilde{\Pi}_{c}\}\partial_{b}\theta, \tag{52}$$

it proves to be possible to represent all the terms in Eq. (51) either as $K\tilde{\Lambda}\theta$ or $\partial_{a}\Lambda^{\mu}T^{\mu a}$ with K and T being certain coefficients. These terms can evidently be canceled by appropriate variations of the $\bar{\psi}$ and A^{a}_{μ} variables. The final form for these variations is

$$\delta A_a^\mu = 8(\bar{\theta}\Gamma^\rho \epsilon^c)(\bar{\theta}\Gamma^\mu \Pi_c^\nu \Gamma^{\nu\rho}\partial_a\theta)$$
$$-5(\bar{\theta}\tilde{\Pi}_c\epsilon^c)(\bar{\theta}\Gamma^\mu\partial_a\theta)-3(\bar{\theta}\Gamma^\mu\Gamma^\nu\tilde{\Pi}_c\epsilon^c)(\bar{\theta}\Gamma^\nu\partial_a\theta)$$
$$-4i\varepsilon_{ad}P^{-bd}[(\bar{\theta}\Gamma^\mu\epsilon^c)(\Pi_b\Pi_c)-2(\bar{\theta}\tilde{\Pi}_c\epsilon^c)\Pi_b^\mu],$$

$$\delta\bar{\psi}=i\varepsilon^{ab}\{2(\partial_a\bar{\theta}\tilde{\Pi}_c\epsilon^c)\partial_b\bar{\theta}-8(\partial_a\bar{\theta}\tilde{\Pi}_c\partial_b\theta)\bar{\epsilon^c}$$
$$-8\partial_a[(\bar{\theta}\Gamma^\mu\epsilon^c)\partial_b\bar{\theta}\Gamma^{\mu\nu}\Pi_\nu^c]+5(\bar{\theta}\partial_a\tilde{\Pi}_c\epsilon^c)\partial_b\theta$$
$$+3(\bar{\theta}\Gamma^\mu\partial_b\theta)\bar{\epsilon^c}\partial_a\tilde{\Pi}_c\Gamma^\mu+(\partial_a\bar{\theta}\Gamma^\mu\partial_b\theta)\bar{\epsilon^c}\tilde{\Pi}_c\Gamma^\mu\}$$
$$-2iP^{-ba}[\bar{\epsilon^c}\partial_a\tilde{\Pi}_c\tilde{\Pi}_b-2\bar{\epsilon^c}\Pi_b\partial_a\Pi_c]. \tag{53}$$

Note that the complicated transformation law for the ψ variable might be predicted, since one of the Lagrangian equations of motion is

$$(\tilde{\Lambda}\psi)_\alpha=-4\tilde{\Pi}_bP^{-ba}\partial_a\theta_\alpha+i\varepsilon^{ab}\theta^\beta\partial_a\theta^\gamma\partial_b\theta^\delta\Gamma^\mu_{\alpha(\beta}(C\Gamma^\mu)_{\gamma\delta)}. \tag{54}$$

Thus, transformation of the $\tilde{\Lambda}\psi$ part of the ψ variable is dictated by this equation and the transformation laws for the x and θ variables.

Being reduced to the physical sector, Eq. (49) looks as follows:

$$\delta\theta_a=-\sqrt{2}(P^+\epsilon_a-\partial_-x^i\gamma^i{}_{a\dot{a}}\bar{\epsilon}'_{\dot{a}}+\partial_-x^{10}\epsilon'_a,$$
$$\delta\bar{\theta}_{\dot{a}}=-\sqrt{2}(P^+\bar{\epsilon}_{\dot{a}}+\partial_-x^i\bar{\gamma}^i_{\dot{a}a}\epsilon'_a-\partial_-x^{10}\bar{\epsilon}'_{\dot{a}}),$$
$$\delta x^i=2\sqrt{2}iP^+(\theta\gamma^i\bar{\epsilon}'-\bar{\theta}\bar{\gamma}^i\epsilon'), \tag{55}$$

and seems to be the analogue of Eq. (47).

To summarize, in this paper we have suggested a super Poincaré invariant action for the superstring which classically exists in any spacetime dimension. As compare with the GS formulation for the $N=1,D=10$ superstring action, the only difference is an additional infinite degeneracy in the continuous part of the energy spectrum, related to the zero modes Y^μ,P_y^μ. Since supersymmetry is realized in the physical subspace (55), one also gets the corresponding representation in the space of functions on that subspace. This allows one to expect a supersymmetric spectrum of quantum states. Analysis of this situation in terms of oscillator variables as well as the critical dimension will be presented in a separate publication.

Note added: After this work was completed, there appeared a paper by Bars and Deliduman [30] where a covariant action for a superstring in a space with a nonstandard signature $(D-2,2)$ was suggested.

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APPENDIX

In this appendix we describe the minimal spinor representation of the Lorentz group $SO(1,10)$ which is known to have dimension $2^{[D/2]}$. For this aim, it suffices to find eleven 32×32 Γ^μ matrices satisfying the equation $\Gamma^\mu\Gamma^\nu+\Gamma^\nu\Gamma^\mu=-2\eta^{\mu\nu}$, $\mu,\nu=0,1,\dots,10$, $\eta^{\mu\nu}=(+,-,\dots,-)$. A convenient way is to use the well-known 16×16 Γ matrices of the $SO(1,9)$ group which we denote as $\Gamma^m_{\alpha\beta}$, $\tilde{\Gamma}^{m\alpha\beta}$, $m=0,1,\dots,9$. Their explicit form is

$$\Gamma^0=\begin{pmatrix}\mathbf{1}_8 & 0\\ 0 & \mathbf{1}_8\end{pmatrix},$$
$$\tilde{\Gamma}^0=\begin{pmatrix}-\mathbf{1}_8 & 0\\ 0 & -\mathbf{1}_8\end{pmatrix},$$
$$\Gamma^i=\begin{pmatrix}0 & \gamma^i{}_{a\dot{a}}\\ \bar{\gamma}^i{}_{\dot{a}a} & 0\end{pmatrix},$$
$$\tilde{\Gamma}^i=\begin{pmatrix}0 & \gamma^i{}_{a\dot{a}}\\ \tilde{\gamma}^i{}_{\dot{a}a} & 0\end{pmatrix},$$
$$\Gamma^9=\begin{pmatrix}\mathbf{1}_8 & 0\\ 0 & -\mathbf{1}_8\end{pmatrix},$$
$$\tilde{\Gamma}^9=\begin{pmatrix}\mathbf{1}_8 & 0\\ 0 & -\mathbf{1}_8\end{pmatrix}, \tag{A1}$$

where $\gamma^i{}_{a\dot{a}}$, $\bar{\gamma}^i{}_{\dot{a}a}\equiv(\gamma^i{}_{a\dot{a}})^{\mathrm{T}}$ are real $SO(8)\,\gamma$ matrices [29],

$$\gamma^i\bar{\gamma}^j+\gamma^j\bar{\gamma}^i=2\delta^{ij}\mathbf{1}_8, \tag{A2}$$

where $i,a,\dot{a}=1,\dots,8$. As a consequence, the matrices Γ^m, $\tilde{\Gamma}^m$ are real and symmetric and obey the algebra

$$\{\Gamma^m,\tilde{\Gamma}^n\}=-2\eta^{mn}\mathbf{1}, \tag{A3}$$

where $\eta^{mn}=(+,-,\dots,-)$. Then a possible realization for the $D=11$ Γ matrices is

$$\Gamma^\mu=\left\{\begin{pmatrix}0 & \Gamma^m\\ \tilde{\Gamma}^m & 0\end{pmatrix},\begin{pmatrix}\mathbf{1}_{16} & 0\\ 0 & -\mathbf{1}_{16}\end{pmatrix}\right\}, \tag{A4}$$

where $\mu=0,1,\dots,10$. The properties of Γ^m, $\tilde{\Gamma}^m$ induce the following relations for Γ^μ:

$$(\Gamma^0)^{\mathrm{T}}=-\Gamma^0,$$
$$(\Gamma^i)^{\mathrm{T}}=-\Gamma^i,$$
$$(\Gamma^\mu)^*=\Gamma^\mu, \tag{A5}$$

$$\{\Gamma^\mu,\Gamma^\nu\}=-2\,\eta^{\mu\nu}\mathbf{1}_{32},$$

where $\eta^{\mu\nu}=(+,-,\ldots,-)$. The charge conjugation matrix

$$C\equiv\Gamma^0,$$

$$C^{-1}=-C,$$

$$C^2=-\mathbf{1} \tag{A6}$$

can be used to construct the symmetric matrices: $(C\Gamma^\mu)^{\mathrm{T}}=C\Gamma^\mu$.

The next step is to introduce the antisymmetrized products

$$\Gamma^{\mu\nu}=\frac{1}{2}(\Gamma^\mu\Gamma^\nu-\Gamma^\nu\Gamma^\mu), \tag{A7}$$

which have the following explicit form in terms of the corresponding $SO(1,9)$ and $SO(8)$ matrices:

$$\Gamma^{0i}=\begin{pmatrix}\Gamma^{0i} & 0\\ 0 & \tilde{\Gamma}^{0i}\end{pmatrix}=\left(\begin{array}{cc|cc} 0 & \gamma^i & & \\ \bar{\gamma}^i & 0 & \multicolumn{2}{c}{0} \\ \hline & & 0 & -\gamma^i \\ \multicolumn{2}{c|}{0} & -\bar{\gamma}^i & 0\end{array}\right),$$

$$\Gamma^{09}=\begin{pmatrix}\Gamma^{09} & 0\\ 0 & \tilde{\Gamma}^{09}\end{pmatrix}=\left(\begin{array}{cc|cc} 1 & 0 & & \\ 0 & -1 & \multicolumn{2}{c}{0} \\ \hline & & -1 & 0 \\ \multicolumn{2}{c|}{0} & 0 & 1\end{array}\right),$$

$$\Gamma^{ij}=\begin{pmatrix}\Gamma^{ij} & 0\\ 0 & \tilde{\Gamma}^{ij}\end{pmatrix}=\left(\begin{array}{cc|cc} \gamma^{ij} & 0 & & \\ 0 & \bar{\gamma}^{ij} & \multicolumn{2}{c}{0} \\ \hline & & \gamma^{ij} & 0 \\ \multicolumn{2}{c|}{0} & 0 & \bar{\gamma}^{ij}\end{array}\right),$$

$$\Gamma^{i9}=\begin{pmatrix}\Gamma^{i9} & 0\\ 0 & \tilde{\Gamma}^{i9}\end{pmatrix}=\left(\begin{array}{cc|cc} 0 & -\gamma^i & & \\ \bar{\gamma}^i & 0 & \multicolumn{2}{c}{0} \\ \hline & & 0 & -\gamma^i \\ \multicolumn{2}{c|}{0} & \bar{\gamma}^i & 0\end{array}\right), \tag{A8}$$

$$\Gamma^{0,10}=\begin{pmatrix}0 & -\Gamma^{0}\\ \tilde{\Gamma}^{0} & 0\end{pmatrix}=\left(\begin{array}{cc|cc} & & \mathbf{1} & 0 \\ \multicolumn{2}{c|}{0} & 0 & \mathbf{1} \\ \hline \mathbf{1} & 0 & & \\ 0 & \mathbf{1} & \multicolumn{2}{c}{0}\end{array}\right),$$

$$\Gamma^{i,10}=\begin{pmatrix}0 & -\Gamma^{i}\\ \tilde{\Gamma}^{i} & 0\end{pmatrix}=\left(\begin{array}{cc|cc} & & 0 & -\gamma^i \\ \multicolumn{2}{c|}{0} & -\bar{\gamma}^i & 0 \\ \hline 0 & \gamma^i & & \\ \bar{\gamma}^i & & \multicolumn{2}{c}{0}\end{array}\right),$$

$$\Gamma^{9,10}=\begin{pmatrix}0 & -\Gamma^{9}\\ \tilde{\Gamma}^{9} & 0\end{pmatrix}=\left(\begin{array}{cc|cc} & & -\mathbf{1} & 0 \\ \multicolumn{2}{c|}{0} & 0 & \mathbf{1} \\ \hline \mathbf{1} & 0 & & \\ 0 & -\mathbf{1} & \multicolumn{2}{c}{0}\end{array}\right), \tag{A9}$$

where $i=1,2,\ldots,8$ and Γ^{0i}, Γ^{09}, $\Gamma^{0,10}$ are symmetric, whereas Γ^{ij}, Γ^{i9}, $\Gamma^{i,10}$, $\Gamma^{9,10}$ are antisymmetric. Besides, these matrices are real and, as a consequence of Eq. (A5), obey the commutation relations of the Lorentz algebra.

Under the action of the Lorentz group a $D=11$ Dirac spinor is transformed as

$$\delta\theta=\frac{1}{4}\omega_{\mu\nu}\Gamma^{\mu\nu}\theta. \tag{A10}$$

Since the $\Gamma^{\mu\nu}$ matrices are real, the reality condition $\theta^*=\theta$ is compatible with Eq. (A10) which defines a Majorana spinor. To construct Lorentz-covariant bilinear combinations, note that

$$\delta\bar{\theta}=-\frac{1}{4}\omega_{\mu\nu}\bar{\theta}\Gamma^{\mu\nu},$$

$$\bar{\theta}\equiv\theta^{\mathrm{T}}C. \tag{A11}$$

Then the combination $\bar{\psi}\Gamma^\mu\theta$ is a vector under the action of the $D=11$ Lorentz group

$$\delta(\bar{\psi}\Gamma^\mu\theta)=-\omega^\mu{}_\nu(\bar{\psi}\Gamma^\mu\theta). \tag{A12}$$

In various calculations the properties

$$\bar{\psi}\Gamma^\mu\theta=-\bar{\theta}\Gamma^\mu\psi,$$

$$\bar{\psi}\Gamma^\mu\Gamma^\nu\theta=\bar{\theta}\Gamma^\nu\Gamma^\mu\psi,$$

$$\bar{\psi}\Gamma^\mu\Gamma^\nu\Gamma^\rho\theta=-\bar{\theta}\Gamma^\rho\Gamma^\nu\Gamma^\mu\psi \tag{A13}$$

are also useful.

It is possible to decompose a $D=11$ Majorana spinor in terms of its $SO(1,9)$ and $SO(8)$ components. Namely, from Eq. (A8) it follows that the decomposition

$$\theta=(\bar{\theta}_\alpha,\theta^\alpha), \tag{A14}$$

where $\alpha=1,\ldots,16$, holds. Here θ and $\bar{\theta}$ are Majorana-Weyl spinors of opposite chirality with respect to the $SO(1,9)$ subgroup of the $SO(1,10)$ group. Further, from the third equation in Eq. (A8) it follows that in the decomposition

$$\theta=(\theta_a,\overline{\theta'_{\dot a}},\theta'_a,\overline{\theta_{\dot a}}), \tag{A15}$$

where $a,\dot a=1,\ldots,8$, the pairs θ_a, θ'_a and $\overline{\theta'_{\dot a}}$, $\overline{\theta_{\dot a}}$ are $SO(8)$ spinors of opposite chirality.

It is convenient to define the $D=11$ light-cone Γ matrices

$$\Gamma^+=\frac{1}{\sqrt{2}}(\Gamma^0+\Gamma^9)=\sqrt{2}\left(\begin{array}{cc|cc} \multicolumn{2}{c|}{0} & \mathbf{1}_8 & 0 \\ & & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{0} \\ 0 & -\mathbf{1}_8 & & \end{array}\right),$$

$$\Gamma^-=\frac{1}{\sqrt{2}}(\Gamma^0-\Gamma^9)=\sqrt{2}\left(\begin{array}{cc|cc} \multicolumn{2}{c|}{0} & 0 & 0 \\ & & 0 & \mathbf{1}_8 \\ \hline -\mathbf{1}_8 & 0 & \multicolumn{2}{c}{0} \\ 0 & 0 & & \end{array}\right),$$

$$\Gamma^i=\begin{pmatrix} 0 & \Gamma^i \\ \tilde\Gamma^i & 0\end{pmatrix}, \quad \Gamma^{10}=\begin{pmatrix} \mathbf{1}_{16} & 0 \\ 0 & -\mathbf{1}_{16}\end{pmatrix}, \tag{A16}$$

where $i=1,\ldots,8$. Then the equation $\Gamma^+\theta=0$ has a solution

$$\theta=(\theta_a,0,0,\overline{\theta_{\dot a}}). \tag{A17}$$

Besides, under the condition $\Gamma^+\theta=0$ the identities

$$\bar\theta\Gamma^+\partial_1\theta=\bar\theta\Gamma^i\partial_1\theta=\bar\theta\Gamma^{10}\partial_1\theta=0,$$

$$(\bar\theta\Gamma^\mu\partial_1\theta)\Gamma^\mu\theta=0 \tag{A18}$$

hold.

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