Vishniac-type contribution to the polarization of the CMBR?

T. R. Seshadri
Mehta Research Institute, Chhatnag Road, Jhusi, Allahabad 211019, India

K. Subramanian*
Astronomy Centre, University of Sussex, Falmer, Brighton BN1 9QH, United Kingdom
and Max Planck Institute for Astrophysics, Karl Schwarzschild Strasse 1, D-85748 Garching, Germany

(Received 10 February 1998; published 26 August 1998)

Radiation which has a quadrupole component of anisotropy can get polarized by Thomson scattering from charged particles. In the cosmological context, the microwave background photons develop significant quadrupole anisotropy as they free stream away from the epoch of standard recombination. Reionization in the post recombination era can provide free electrons to Thomson scatter the incident anisotropic cosmic microwave background radiation photons. We compute the resulting polarization anisotropy on small (arc minute) angular scales. We look for significant nonlinear contributions, as in the case of the Vishniac effect in temperature anisotropy, due to the coupling of small-scale electron density fluctuations, at the new last scattering surface, and the temperature quadrupole. We show that while in cold dark matter type models this does not lead to very significant signals ($\sim$0.02–0.04 μK) a larger small angular scale polarization anisotropy, ($\sim$0.1–0.5 μK) can result in isocurvature-type models. [S0556-2821(98)07616-4]

PACS number(s): 98.70.Vc, 98.80.Es

I. INTRODUCTION

The cosmic microwave background radiation (CMBR) has a wealth of information about the parameters which govern the dynamics and the physical processes in the universe. Three important aspects of the CMBR in which the information about the universe is encoded are its temperature anisotropy, spectral distortion, and anisotropy in polarization. A study of these aspects will allow one to constrain the evolution of both the background universe and the large scale structures.

Polarization of the CMBR arises from the Thomson scattering of radiation from free electrons. An analysis of polarization properties of the CMBR can reveal valuable information about the ionization history of the universe. Much of the current work on polarization anisotropy is in the context of linear perturbation theory. The expectation generally is that higher order terms are much smaller. However, for the temperature anisotropy, on very small angular scales, of order of arc minutes, it was shown by Vishniac that nonlinear effects can also make significant contributions [1]. These arise through mode coupling of the electron density perturbations on small scales with source terms which vary over larger scales. The Vishniac effect is especially important in models where there is significant early reionization [2,3].

Zaldarriaga [4] studied the effects of such early reionization on first order polarization anisotropy in a semianalytical fashion. It turns out that the CMBR can develop a significant quadrupole, by the epoch of reionization, due to the free streaming of the monopole at recombination. The Thomson scattering of this quadrupole off the electrons at the reionized epoch can lead to additional polarization signals. In this paper we wish to follow suit and examine whether this quadrupole coupling to fluctuations in the electron density at the new last scattering surface can also lead to significant Vishniac-type second order effects and result in a polarization anisotropy of the CMBR at small angular scales.

In the next section, we give the basic equations and their formal solution. Section III presents an analytical estimate of the Vishniac-type contribution to the small scale polarization anisotropy in reionized models. Section IV gives numerical values for some illustrative models of structure formation. We summarize our conclusions in Sec. V.

II. BASIC EQUATIONS AND THEIR FORMAL SOLUTION

The equations governing the evolution of polarization perturbation $\Delta_P(\mathbf{x},\gamma,\tau)$ and temperature perturbation $\Delta_T(\mathbf{x},\gamma,\tau)=\Delta T/T$ for scalar modes can be derived from the moments of the Boltzmann equation for photons. In the conformal Newtonian gauge they are given by [5]

$$\dot{\Delta}_P+\gamma_i\partial_i\Delta_P=n_e\sigma_T a(\tau)\left(-\Delta_P+\frac{1}{2}[1-P_2(\mu)]\Pi\right), \tag{2.1}$$

$$\dot{\Delta}_T+\gamma_i\partial_i\Delta_T=\dot{\phi}-\gamma_i\partial_i\psi+n_e\sigma_T a(\tau)\times\left(-\Delta_T+\Delta_{T0}+\gamma_i v_i-\frac{1}{2}P_2(\mu)\Pi\right). \tag{2.2}$$

Here $\mathbf{x}$ is the comoving coordinate, τ is conformal time, n_e the electron density, $\mathbf{v}$ the fluid velocity, γ is the direction of photon propagation, ϕ and ψ are the conformal Newtonian potentials, and the overdot represents derivative with respect to conformal time. We have also defined

*On leave from National Center for Radio Astrophysics, TIFR Pune, Poona University Campus, Ganeshkhind, Pune 411 007, India.

$$\Pi(\mathbf{x},\tau) = -\Delta_{T2}(\mathbf{x},\tau) - \Delta_{P2}(\mathbf{x},\tau) + \Delta_{P0}(\mathbf{x},\tau) \quad (2.3)$$

with Δ_{T0}, Δ_{P0} the monopole and Δ_{T2}, Δ_{P2} the quadrupole temperature and polarization perturbations, respectively. We define these angular moments by

$$\Delta_{P,T}(\mathbf{x},\gamma,\tau) = \sum_l (2l+1) P_l(\mu) \Delta_{Pl,Tl}(\mathbf{x},\tau);$$

$$\Delta_{Pl,Tl} = \int \frac{d\mu}{2} P_l(\mu) \Delta_{P,T}(\mu). \quad (2.4)$$

(Here, we make the usual assumption that after Fourier transformation, the evolution equations for temperature and polarization perturbations depend on γ only in the combination $\gamma \cdot \mathbf{k}$, where $\mathbf{k}$ is the wave vector; so we define $\mu = \gamma \cdot k/k$, where $k = |\mathbf{k}|$ (cf. Ref. [5]).)

In order to treat inhomogeneities in the electron density, we take $n_e(\mathbf{x},\tau) = \overline{n_e}(\tau)[1+\delta_e(\mathbf{x},\tau)]$, where δ_e is the fractional perturbation of electron density about the space averaged mean. In general we will have $\delta_e \ll 1$. We may note at this point that spatial perturbations in the number density of the electrons is precisely the feature that gives rise to the Vishniac effect in second order temperature perturbations. We will investigate a similar effect for polarization perturbations.

It will be convenient to express Eq. (2.1) in terms of the Fourier modes as follows,

$$\dot{\Delta}_P + ik\mu\Delta_P = \overline{n_e}\sigma_T a \left[\frac{1}{2}[1-P_2(\mu)]\Pi(\mathbf{k},\tau) - \frac{1}{2}[1-P_2(\mu)]S(\mathbf{k},\tau) - \Delta_P - R(\mathbf{k},\tau) \right]. \quad (2.5)$$

We have retained the same symbols for the Fourier transformed quantities and defined the mode coupling source term $S(\mathbf{k},\tau)$ by

$$S(\mathbf{k},\tau) = -\int \frac{d^3\mathbf{p}}{(2\pi)^3} \delta_e(\mathbf{k}-\mathbf{p},\tau)\Pi(\mathbf{p},\tau), \quad (2.6)$$

$$R(\mathbf{k},\tau) = +\int \frac{d^3\mathbf{p}}{(2\pi)^3} \delta_e(\mathbf{k}-\mathbf{p},\tau)\Delta_P(\mathbf{p},\tau). \quad (2.7)$$

The formal solution of Eq. (2.5) is given by

$$\Delta_P(\mathbf{k},\tau) = \int_0^\tau \left\{ \frac{1}{2}[1-P_2(\mu)][\Pi(\mathbf{k},\tau') - S(\mathbf{k},\tau')] - R(\mathbf{k},\tau) \right\} g(\tau,\tau') e^{ik\mu(\tau'-\tau)} d\tau', \quad (2.8)$$

where $g(\tau,\tau')$ which is called the visibility function is given by

$$g(\tau,\tau') = \overline{n_e}(\tau')\sigma_T a(\tau') e^{-\int_{\tau'}^{\tau} \overline{n_e}(\tau'')\sigma_T a(\tau'') d\tau''}. \quad (2.9)$$

In the above equations, the value of the polarization perturbation at the epoch τ is determined by the entire history from $\tau'=0$ to $\tau'=\tau$. The visibility function $g(\tau,\tau')$ determines the probability that a photon last scattered at the epoch τ' reaches us at the epoch τ. The exact form of the visibility function is determined by the reionization history of the universe. Operationally, the role of the visibility function is to give different weightages for the integrand for different epochs.

In this paper we consider a model in which the universe underwent a phase of standard recombination and got reionized completely at a later epoch τ_*. We will only be concentrating on the second order polarization perturbations arising from the $S(\mathbf{k},\tau)$ term in Eq. (2.8). The $S(\mathbf{k},\tau)$ contribution to the right-hand side (RHS) of Eq. (2.8) involves a convolution in Fourier space, which couples the first order temperature (polarization) perturbations with the first order perturbations in the electron density. A very similar situation exists in the case of Vishniac effect in second order temperature perturbations. So we expect a Vishniac-type effect in second order polarization perturbations as well.

Further, the coupling of Δ_P, Δ_{P_0}, and Δ_{P_2} with δ_e in S and R is likely to be much smaller than the coupling of δ_e and Δ_{T2} in S. This is because, first the temperature perturbations generically dominate the polarization. Also the quadrupole temperature anisotropy $\Delta_{T2}(\mathbf{k},\tau)$ will grow to a larger value between the epochs of recombination and reionization, due to free streaming of the monopole at recombination (cf. Ref. [4]). We therefore neglect R and retain in S only the Δ_{T2} term and neglect the other terms. We then have

$$S(\mathbf{k},\tau) \approx \int \frac{d^3\mathbf{p}}{(2\pi)^3} \delta_e(\mathbf{k}-\mathbf{p},\tau)\Delta_{T2}(\mathbf{p},\tau) \quad (2.10)$$

and we neglect R in comparison to S. We will adopt this approximate form for the mode coupling term in what follows.

III. VISHNIAC-TYPE CONTRIBUTION IN REIONIZED MODELS: ANALYTIC ESTIMATE

The absence of ''Gunn-Peterson'' dips in the spectra of distant quasars indicates that the universe was probably reionized at some redshift $z = z_* > 5$. [6]. The value of z_* is not known observationally, while different theoretical models have different predictions for this redshift. In the model which we consider, the universe underwent standard recombination at τ_r, and was reionized completely at a later epoch τ_*. In this case as shown in Ref. [4], the visibility function has two peaks, one around τ_r and another peak around τ_*. We wish to consider here a Vishniac-type second order contribution to Δ_P. This comes dominantly from the value of τ' around the latter peak. It is then convenient to separate the integral over τ' in Eq. (2.8) in two parts: $0 < \tau' < \tau_*$ and $\tau_* < \tau' < \tau_0$, where τ_0 is the present conformal time. We write $\Delta_P = \Delta_P^a + \Delta_P^b$, with

$$\Delta_P^a(\mathbf{k},\tau)=\int_0^{\tau_*}\frac{1}{2}[1-P_2(\mu)]\times[\Pi(\mathbf{k},\tau')-S(\mathbf{k},\tau')]g(\tau,\tau')e^{ik\mu(\tau'-\tau)}d\tau', \tag{3.1}$$

$$\Delta_P^b(\mathbf{k},\tau)=\int_{\tau_*}^{\tau_0}\frac{1}{2}[1-P_2(\mu)]\times[\Pi(\mathbf{k},\tau')-S(\mathbf{k},\tau')]g(\tau,\tau')e^{ik\mu(\tau'-\tau)}d\tau'. \tag{3.2}$$

The first contribution in Eq. (3.1) is simply $\Delta_P^a \equiv \exp(-\kappa_*)\Delta_P^{\rm NR}$. Here $\Delta_P^{\rm NR}$ is the polarization that would be measured if there was no reionization and κ_* is the optical depth to Thomson scattering between now and recombination. This contribution is reduced by the fact that only a fraction $\exp(-\kappa_*)$ of the photons that arrive at the observer come directly from recombination without further scattering. [Also the second order contributions from the S term are much smaller than the first order term because the electron density fluctuations at recombination $\delta_e(\tau_r)\sim 10^{-4}-10^{-3} \ll 1$ for the relevant k value.]

In order to calculate the second contribution, one has to determine the form of the visibility function after the standard recombination epoch, that is, $g(\tau_0,\tau')$ for $\tau'>\tau_r$. Using the exact form for $g(\tau_0,\tau')$, to solve for the Δ_P is not analytically tractable. So we resort to an approximation for $g(\tau_0,\tau')$ in this work which, while preserving its main features, also allows analytical results to be derived. We will return to a full numerical treatment of the problem elsewhere. In particular, we choose the form of the visibility function after standard recombination to be a truncated exponential given by

$$g(\tau_0,\tau')=N\frac{1}{\sigma}e^{-(\tau'-\tau_*)/\sigma}\theta(\tau'-\tau_*). \tag{3.3}$$

Here the Heavyside $\theta(x)$ function, is zero for $x<0$ and 1 for $x>0$. It takes account of the fact that before reionization, $n_e=0$. Further, N is a normalization constant and σ gives the spread of the exponential. By appropriately choosing σ, we can set the width of the reionized last scattering surface. Also note that $g(\tau_0,\tau')$ has the interpretation of probability so its integral over τ' from $\tau'=0$ to $\tau'=\tau_0$ should be normalized to unity. This determines the normalization factor N. For a sufficiently early epoch of reionization, we generally have $(\tau_0-\tau_*)/\sigma_2\gg 1$. In this case, the condition that the integral of $g(\tau_0,\tau')$ over τ' should be unity implies $N+e^{-\kappa_*}=1$ or $N=1-e^{-\kappa_*}$. So N measures the probability of at least one scattering between τ_0 and τ_*, due to the reionization. Another feature to note is that in our approximation, τ_0 does not appear at all. This is because for the models we will consider the major contribution to the scattering optical depth comes from epochs much before τ_0.

In Eq. (3.2) for Δ_P^b, the first order contribution to the polarization due to a reionized universe has already been considered in detail by Zaldarriaga [4]. So here we concentrate on purely the second order Vishniac-type effect, due to S, which we call Δ_P^V. This can be written as

$$\Delta_P^V=-\frac{1}{2}[1-P_2(\mu)]\int_{\tau_*}^{\tau_0}\exp^{ik\mu(\tau'-\tau)}g(\tau_0,\tau')S(\mathbf{k},\tau')d\tau', \tag{3.4}$$

where retaining only the Δ_{T2} term the mode coupling term is given by Eq. (2.10). In evaluating the τ' integral in Eq. (3.4) we assume the visibility function to be given by the truncated exponential form of Eq. (3.3).

Let us look at the mode coupling term $S(\mathbf{k},\tau)$, given by Eq. (2.10), in a little more detail. This term involves a coupling of the quadrupole temperature perturbation at $\tau>\tau_*$ and the electron density perturbation at the same epoch. Note that the temperature quadrupole at late times can have a significant contribution due to the free streaming of the monopole. For example, in a flat universe at large enough wavelengths the first order quadrupole temperature perturbation is related to the temperature perturbations at recombination by [7],

$$\Delta_{T2}(p,\tau)=[\Delta_{T_0}(p,\tau_r)+\psi(p,\tau_r)]j_2[p(\tau-\tau_r)], \tag{3.5}$$

where j_2 is the second order spherical Bessel function and $p=|\mathbf{p}|$. (Here we have assumed that p is small enough that the doppler velocity term makes little contribution to the free-streamed quadrapole, cf. Ref. [7]. We have also assumed that the integrated Sachs-Wolfe effect makes a negligible contribution to the free streamed quadrapole at the new last scattering surface.) The mode coupling term can then be written as

$$S(\mathbf{k},\tau)=\int\frac{d^3\mathbf{p}}{(2\pi)^3}\delta_e(\mathbf{k}-\mathbf{p},\tau)[\Delta_{T_0}(p,\tau_r)+\psi(p,\tau_r)]\times j_2[p(\tau-\tau_r)]. \tag{3.6}$$

The spherical Bessel function $j_2(\xi)$ is well approximated by a Gaussian peaked at $\xi\sim 3.345$ and with a spread of ~ 1.4. Therefore $j_2[p(\tau-\tau_r)]$ is peaked at wave numbers around $p=p_0\sim 3.345/(\tau-\tau_r)$. Note that for the small scale polarization anisotropy which we wish to calculate, $k=|\mathbf{k}|\gg p_0$; in general we will have $k\sim(10\ \mathrm{Mpc})^{-1}$ to $(1\ \mathrm{Mpc})^{-1}$, where as $p_0\sim(300\ \mathrm{Mpc})^{-1}$ to $(1000\ \mathrm{Mpc})^{-1}$ for $\tau>\tau_*$ (cf. Ref. [4] and see below). So in the mode coupling integral, for a fixed $k\gg p_0$, the electron density perturbation $\delta_e[(\mathbf{k}-\mathbf{p}),\tau]$ varies negligibly with $\mathbf{p}$ in the range of $\mathbf{p}$ for which the Bessel function makes a significant contribution. Therefore, we can evaluate δ_e at $p=p_0$ and pull it out of the p integral. Also since $k\gg p_0$, one can approximate $(\mathbf{k}-\mathbf{p}_0)\sim\mathbf{k}$. The mode coupling integral for large $k\gg p_0$ then simplifies to the uncoupled form

$$S(\mathbf{k},\tau)=\delta_e(\mathbf{k},\tau)\int\frac{d^3\mathbf{p}}{(2\pi)^3}\Delta_{T2}(\mathbf{p},\tau)\equiv\delta_e(\mathbf{k},\tau)Q_2(\tau), \tag{3.7}$$

where we have used Eq. (3.5) to rewrite the resulting expressions in terms of Δ_{T_2} again and defined for later convenience

$$Q_2(\tau) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \Delta_{T2}(\mathbf{p},\tau). \tag{3.8}$$

We now evaluate Δ_P^V using Eqs. (3.4) and (3.3). Let us assume that the reionization epoch is late enough, and that the electron density perturbation trace the perturbations in total matter density. Then, in a flat universe, the electron density perturbations grow as $a(\tau)\propto\tau^2$. For an open model or one involving a cosmological constant Λ, the growth law is given as

$$\delta_e(\mathbf{k},\tau) = \left(\frac{\tau}{\tau_0}\right)^2 \delta_e(\mathbf{k},\tau_0)\frac{f(\Omega(z(\tau)))}{f(\Omega_0)}, \tag{3.9}$$

Here, τ_0 is present conformal time and the functions f takes into account the reduced growth in the open or flat Λ models at late times, compared to a flat dark matter dominated model. In a flat universe $\Omega(z)=\Omega_0=1$, $f(1)=1$, and we recover the $\delta_e\propto\tau^2$ growth law. The expressions for f and $\Omega(z)$ for Λ dominated and the open models can be found in Refs. [8] and [9], respectively [in these papers, our $f(\Omega)$ is called $g(\Omega)$]. For $z\gg1$, or $\tau/\tau_0\ll1$ and $f[\Omega(z)]\to1$. So the approximation $\delta_e(k,\tau)=(\tau/\tau_0)^2\delta_e(k,\tau_0)f^{-1}(\Omega_0)$ works very well for these other models as well at sufficiently early times.

Further, as displayed explicitly in Ref. [7], the power in the CMB monopole, per unit logarithmic interval of p space $p^3[\Delta_{T_0}(p,\tau_r)+\psi(p,\tau_r)]$ is roughly constant on scales $p<(100\ \mathrm{Mpc})^{-1}$. (This reflects the fact that perturbations on scales larger than the Hubble radius at recombination have not evolved and are laid out with constant power for a scale invariant initial power spectrum.) Recall that the presence of the j_2 term in the integral over $\mathbf{p}$ picks out dominantly contributions from $p\sim p_0<(300\ \mathrm{Mpc})^{-1}$, at any time $\tau>\tau_*$. Now in any realization, one does expect some variation of the monopole as p is varied. Nevertheless, because of the constancy of monopole power with p for $p\sim p_0$ one expects the integral term $Q_2(\tau)$ in Eq. (3.7) to vary much slower with τ than the electron density perturbation $\delta_e(\mathbf{k},\tau)$ or the visibility function. This will especially be so if the visibility function is sufficiently peaked around the reionization redshift (for σ small enough). So when evaluating Δ_P^V, we will assume that $Q_2(\tau)$ can be evaluated at some effective $\tau_e\sim\tau_*$ and pulled out of the integral over conformal time τ'.

The remaining integral, which can be done analytically, then gives

$$\Delta_P^V = -\frac{1}{2}[1-P_2(\mu)]\exp^{-ik\mu(\tau-\tau_*)} \times \frac{NF(k,\mu)}{(1-ik\mu\sigma)}\delta_e(\mathbf{k},\tau_*)Q_2(\tau_e), \tag{3.10}$$

where we have taken $(\tau_0-\tau_*)/\sigma\gg1$, as before, and defined the factor $F(k,\mu)$

$$F(k,\mu) = 1+\frac{2\sigma}{\tau_*}\frac{1}{(1-ik\mu\sigma)}+\frac{2\sigma^2}{\tau_*^2}\frac{1}{(1-ik\mu\sigma)^2}. \tag{3.11}$$

[We have also assumed τ_*/τ_0 is small enough that $\Omega(\tau_*)\approx1$.]

We are now in a position to compute the Vishniac-type contribution to the polarization power spectrum. We define this simply by analogy to the temperature power spectrum (cf. Ref. [10]). Recall that for temperature perturbations one first expands in spherical harmonics, with

$$\frac{\Delta T}{T} = \Delta_T(\mathbf{x},\gamma,\tau_0) = \sum_{lm} a_{lm}Y_{lm}(\gamma). \tag{3.12}$$

Then the mean square temperature perturbation is

$$\left\langle\left(\frac{\Delta T}{T}\right)^2\right\rangle = \sum_l \frac{(2l+1)}{4\pi}C_{Tl} \equiv \int Q_T(k)\frac{dk}{k}, \tag{3.13}$$

where

$$C_{Tl} = \langle|a_{lm}|^2\rangle = 4\pi\int\frac{k^2dk}{2\pi^2}\langle|\Delta_{Tl}(k,\tau_0)|^2\rangle \tag{3.14}$$

and

$$Q_T(k) = \frac{k^3}{2\pi^2}\frac{1}{2}\int_{-1}^{+1}\langle|\Delta_T|^2\rangle d\mu. \tag{3.15}$$

(Note that, as before, we have taken the normalization volume, over which periodic boundary conditions are assumed to be $V=1$.) $Q_T(k)$ gives the power in temperature perturbations in any logarithmic interval of k.

We define, therefore, a corresponding Vishniac-type contribution to the polarization power spectrum $Q_P(k)$ given by (cf. Ref. [3])

$$Q_P^V(k) = \frac{k^3}{2\pi^2}\frac{1}{2}\int_{-1}^{+1}\langle|\Delta_P^V|^2\rangle d\mu. \tag{3.16}$$

We calculate Q_P^V below. For this, first take the ensemble average of $|\Delta_P^V|^2$. We get

$$\langle|\Delta_P^V|^2\rangle = \frac{1}{4}[1-P_2(\mu)]^2\frac{N^2|F|^2}{[1+k^2\mu^2\sigma^2]}P_e(k,\tau_*) \times\left[\frac{1}{5}\left(\frac{\Delta T}{T}\right)^2_Q(\tau_e)\right]. \tag{3.17}$$

In the above we have assumed that the δ_e and Δ_{T2} are uncorrelated with each other, defined the power spectrum of electron density fluctuation as $P_e(k,\tau_*)\equiv\langle|\delta_e(\mathbf{k},\tau_*)|^2\rangle$, and also defined

$$\left(\frac{\Delta T}{T}\right)^2_Q(\tau_e)\equiv\frac{5C_{T2}(\tau_e)}{4\pi}=5\int\frac{d^3\mathbf{p}}{(2\pi)^3}\langle|\Delta_{T2}(\mathbf{p},\tau_e)|^2\rangle. \tag{3.18}$$

Here, $(\Delta T/T)_Q(\tau_e)$ is the quadrupole temperature anisotropy as seen by an observer at the conformal time τ_e.

Now turn to the integral of $\langle|\Delta_P^V|^2\rangle$ over μ. The factor $1-P_2$ can be expressed as a sum of even powers in μ:

$$[1-P_2]^2=\frac{9}{4}(1+\mu^4-2\mu^2). \tag{3.19}$$

We are generally interested in the behavior of the power spectrum for large values of k or small angular scales. In this case the dominant contribution to the integral over μ in determining Q_P^V will come from the vicinity of $\mu=0$. It then suffices to retain only the first term in the above expression for $(1-P_2)^2$. The integral over μ can be done analytically to give the remarkably simple expression

$$Q_P^V(k,\tau_0)=\frac{9\pi N^2}{160}G\left(\frac{\sigma}{\tau_*}\right)\left[\frac{\Delta_e^2(k,\tau_*)}{k\sigma}\right]\left(\frac{\Delta T}{T}\right)^2_Q(\tau_e). \tag{3.20}$$

In doing the integral we have assumed that $k\sigma\gg1$ is large enough that $\tan^{-1}(k\sigma)\approx\pi/2$, and defined the factor $G(y)=1+2y+3y^2+5y^3/2+35y^4/32$, where $y=\sigma/\tau_*$. Further,

$$\Delta_e^2(k,\tau_*)=\frac{k^3P_e(k,\tau_*)}{2\pi^2}=\Delta_e^2(k,\tau_0)\left(\frac{\tau_*}{\tau_0}\right)^4\left(\frac{f[\Omega(z_*)]}{f(\Omega_0)}\right)^2 \tag{3.21}$$

is the power per until logarithmic interval in k space of the electron density perturbations at the epoch $\tau=\tau_*$.

We see that the contribution to polarization anisotropy due to the second order Vishniac-type effect for reionized models is basically proportional to the product of the temperature quadrupole and the power in electron density perturbations at last scattering. For small angular scales or large k, Q_P^V is suppressed because of the finite thickness of the last scattering surface (σ) by a factor $k\sigma$. We note that this suppression is much milder than estimated in Ref. [3], essentially because in that paper, the first order temperature quadrupole contribution due to free streaming of the monopole at recombination was not included. The power spectrum of electron density perturbations, of course, depends on the model for structure formation. Also the parameters σ,τ_*,N depend on the reionization history. However the power in the temperature quadrupole at $\tau_e\sim\tau_*$ is likely to be of the order of the observed quadrupole; for a large enough τ_e and if it arises due to the free streaming of the monopole at recombination. This is again because of the slow variation of $Q_2(\tau)$ mentioned previously. We now use Eq. (3.20) to make numerical estimates of the polarization due to Vishniac-type effect in cold dark matter (CDM) and other models of structure formation.

IV. NUMERICAL ESTIMATES IN DIFFERENT MODELS

A. CDM and variants

Consider first the case of a standard CDM model (SCDM), with matter density equal to critical density ($\Omega_0=1$), a baryonic contribution $\Omega_b=0.05$, and a Hubble constant $h=(H_0/100\ \text{km s}^{-1}\ \text{Mpc}^{-1})=0.5$. The optical depth to Thomson scattering in a fully ionized matter dominated flat universe is given by $\kappa(z)=0.0418\Omega_B h[(1+z)^{3/2}-1]$. In general, for a universe with matter density Ω_0 and assuming $z_*\gg1$, an optical depth κ_* is obtained at a redshift

$$z_*\approx97.1\kappa_*^{2/3}\Omega_0^{1/3}\left(\frac{\Omega_B h}{0.025}\right)^{-2/3}. \tag{4.1}$$

Therefore, to have $\kappa_*=1$, in standard CDM, we need $z_*\approx97.1$ and $\kappa_*=0.2$ needs $z_*\approx33.2$. Also in a flat matter dominated universe, the conformal time is related to the redshift by $\tau=\tau_0/(1+z)^{1/2}$, where $\tau_0=2H_0^{-1}=6000h^{-1}$ Mpc. (Note we adopt units with c=1.) So given z_*, this fixes τ_*; for $\kappa_*=1$, we have $\tau_*\approx605.8h^{-1}$ Mpc, while for $\kappa_*=0.2$, we have $\tau_*\approx1025.8h^{-1}$ Mpc.

In order to estimate the parameter σ in the model visibility function (3.3), we proceed as follows. Let $\tau_m=\tau_*+\sigma$. From Eq. (3.3) at epochs after reionization we have $g(\tau_0,\tau_*)/g(\tau_0,\tau_m)=e$. For the exact visibility function the same ratio is given by $g(\tau_0,\tau_*)/g(\tau_0,\tau_m)=[a^2(\tau_m)/a^2(\tau_*)]\exp(\kappa_m-\kappa_*)$, where $\kappa_m=\kappa(\tau_m)$. Equating these two expressions gives an estimate of σ. In particular, using $\kappa\propto(1+z)^{3/2}$ valid for large $z\gg1$ and $a\propto\tau^2\propto(1+z)^{-1}$ we have the implicit equation for τ_m/τ_*, $1=4ln(\tau_m/\tau_*)-\kappa_*[1-(\tau_*/\tau_m)^3]$. For $\kappa_*=1$, this gives $\sigma\approx0.54\tau_*=327h^{-1}$ Mpc, while for $\kappa_*=0.2$, one gets $\sigma=0.32\tau_*=328.3h^{-1}$ Mpc.

It remains to fix Δ_e and the temperature quadrupole. We take the $\Delta_e(k,\tau_0)=\Delta(k)$, the matter power spectrum. [This assumes that z_* is small enough that baryons (and electrons) no longer feel a significant drag due to the CMB photons and so their density fluctuations on the relevant scales have caught up with any dark matter fluctuation. It also assumes that there are no nongravitational forces which have pushed the baryons (and electrons) relative to the dark matter component.] The matter power spectrum is in turn given by

$$\Delta^2(k)=\frac{k^3P(k)}{2\pi^2}=\left(\frac{k}{H_0}\right)^4\delta_H^2(k)T^2(k) \tag{4.2}$$

with the transfer function $T(k)$ in the form given by Bardeen *et al.* [11]

$$T(q)=\frac{ln(1+2.34q)}{2.34q[1+3.89q+(16.1q)^2+(5.64q)^3+(6.71q)^4]^{1/4}}, \tag{4.3}$$

where $q=k/(h\Gamma)$ (cf. Refs. [12,13]). The parameter Γ is referred to as the shape parameter, and is given in Ref. [13] (Eqs. D28 and E12). It is of order 0.48 for standard CDM. The four-year COBE normalization gives $\delta_H(k=H_0)=1.94\times10^{-5}$ (cf. Ref. [14]), for a scale invariant initial power spectrum (with $n=1$). For such initial conditions, the COBE data also give the $(\Delta T/T)_Q(\tau_0)=18\pm1.6\,\mu$K [15]. The value of $\tau_*>1000$ Mpc that we generally obtain is likely to be large enough so that $(\Delta T/T)_Q(\tau_e)\sim(\Delta T/T)_Q(\tau_0)$ with reasonable accuracy. So we will scale the quadrupole at τ_e with the present day observed value. From the above considerations and normalizing all quantities to a value of $k=k_c=1h$ Mpc^{-1}, we get

$$\sqrt{Q_P^V(k)}=2.2\times10^{-2}\ \ \mu\mathrm{K}\left(\frac{\Delta_e^2(k)/k}{\Delta^2(k_c)/k_c}\right)^{1/2}\left(\frac{(\Delta T/T)_Q(\tau_e)}{18\ \ \mu\mathrm{K}}\right)$$

$$\text{for}\ \ \kappa_*=1. \quad (4.4)$$

(In the above equation and in what follows, we give the numerical value of $\sqrt{Q_P^V}$ in temperature units by multiplying $\sqrt{Q_P^V}$ by T_0, the present day CMBR temperature.) In the case where $\kappa_*=0.2$, one has to replace the numerical value in Eq. (4.4) by $\sqrt{Q_P^V(k_c)}\approx1.5\times10^{-2}\,\mu$K. Lower values of $\kappa_*<1$ may already be implied by the observed tentative rise in the CMB anisotropy on degree scales. Note that decreasing κ_*, means a decrease in the fraction of photons last scattered from the reionized epochs and so a decrease in Q_P^V. But at the same time since z_* is decreased, the electron density perturbations at last scattering are larger than the $\kappa_*=1$ case, which is partially compensated for by increasing Q_P^V. If the power spectrum can be approximated locally as a power law $\Delta^2(k)\propto k^{3+n}$, then $Q_P^V(k)\propto k^{2+n}$. Recall that on galactic scales with $k\sim k_c$, $n\sim-2$, while for $k\ll k_c$, $n\sim-1$ and for $k\gg k_c$, $n\to-3$. So the polarization anisotropy $Q_P^V(k)\propto k^{2+n}$ will increase with k at $k\ll k_c$ and decrease for $k\gg k_c$, attaining a maximum at $k\sim k_c$. This was one of the reasons for normalizing our estimate to $k=k_c$.

For small angular scales, one can also set up an approximate correspondence between the wave number k and the angular multipole number l using $l=kR_*$. Here R_* translates comoving distance at the surface of last scattering (roughly the epoch when $\tau=\tau_*$) to angle on the sky and for a flat model is given by $R_*=\tau_0-\tau_*$. In case $\sigma/\tau_0\ll1$, this approximation will be reasonable, but for a thick last scattering surface the above correspondence is less accurate. In the model discussed above, where $\kappa_*=1$, a wave number $k=k_c=1h$ Mpc, corresponds to $l\approx5394$, and an angular scale $\sim1/l\sim0.64$ arc min.

We briefly discuss the predicted small angular scale polarization anisotropy Q_P^V, for some variations on the standard CDM (SCDM) model, by using the same method of calculation as above. For example, increasing the baryon density to $\Omega_b=0.1$, leads to a smaller redshift of last scattering with a given κ_* but also a smaller Γ. For $\kappa_*=1$, one gets a slightly larger value $\sqrt{Q_P^V(k_c)}\sim2.9\times10^{-2}\,\mu$K, than in SCDM. Also if the primordial spectrum is tilted to $n=1.2$, the best fit slope determined by the cosmic Background Explorer (COBE) [15], (keeping all other parameters of SCDM same), then also one has a larger value $\sqrt{Q_P^V(k_c)}\sim4.0\times10^{-2}\,\mu$K.

Suppose we adopt a Λ+CDM type model, as discussed for example in Ref. [16], with $\Omega_0=0.35$, $\Omega_\Lambda=0.65$, $h=0.7$, $n=1$, $\Omega_b=0.04$. Then we get $z_*\sim63$ for $\kappa_*=1$. Assuming that τ_* is early enough that σ can be estimated as for the flat universe, we have

$$\sigma=\frac{6000h^{-1}\ \ \mathrm{Mpc}}{\Omega_0^{1/2}(1+z_*)^{1/2}}\left(\frac{\tau_m}{\tau_*}-1\right). \quad (4.5)$$

Therefore, for the Λ+CDM model, we get $\sigma\sim687.5h^{-1}$ Mpc. For this model $f^{-1}(\Omega_0)\sim1.24$ and adopting a normalization of the power spectrum as in Refs. [8,14] we then get $\Delta_e(k_c,z_*)\sim3.56/(1+z_*)$, which leads to $\sqrt{Q_P^V}\sim1.9\times10^{-2}\,\mu\mathrm{K}[(\Delta T/T)_Q(\tau_e)/18\,\mu\mathrm{K}]$. Note that in this model the integrated Sachs-Wolfe effect, which makes a contribution to the present day quadrupole, will make little contribution at redshifts $\sim z^*$. In fact using Eq. (10) of Ref. [17], we estimate that this will cause only a 10% reduction in the above value. For an open CDM model (OCDM), with $\Lambda=0$, but all other parameters as for the above Λ+CDM model, z_* is the same and $\Delta_e(k_c)$ is of the same order, using the power spectrum as determined by Ref. [9]. However, the quadrupole at last scattering is likely to be smaller because the integrated Sachs-Wolfe effect contributes a larger part of present day quadrupole. So the predicted $\sqrt{Q_P^V(k_c)}$ is likely to be smaller than that for the above models.

B. Isocurvature type models

Finally, consider as an alternative to the standard models, the isocurvature model recently discussed by Peebles [18], where density perturbations are provided by CDM that is the remnant of a massive scalar field frozen from quantum fluctuations during inflation. The novel feature of such a picture, as pointed out in Ref. [18], is that the primeval CDM mass distribution is proportional to the square of a random Gaussian process; so prominent upward fluctuations are much larger (by factor $F\sim3$) than for a Gaussian process with the same rms. The merits of such a picture has been discussed by Peebles [18]. We consider two representative models. Model 1 discussed in Ref. [18] adopts $\Omega_0=0.3$, $\Lambda=0.7$, $\Omega_b=0.05$, $h=0.7$, and a matter power spectrum, which on relevant small scales can be approximated as $\Delta_e^2(k)=(k/k_0)^{3+m}$, with $m=-1.8$, $k_0=0.1h$ Mpc^{-1}. For model 2, $\Omega_0=0.1$, $\Lambda=0.9$, $\Omega_b=0.05$, $h=0.7$, and $m=-1.4$.

Note that in these models, due to early structure formation, reionization is expected to occur at large redshifts. The optical depth to electron scattering, measured from the present epoch could then rise to values larger than unity. However, the possible ionization history in these models is largely unexplored. In order to get a preliminary estimate of the anisotropies in polarization that could be generated, we simply use Eq. (3.20) (implicitly making the simplifying assumptions which went into its derivation). So the universe after standard recombination is assumed to be largely neutral, and then reionized after an epoch $\tau=\tau_*$. Again at $\tau \sim \tau_*$, a quadrupole would arise from the free streaming of the large scale entropy perturbation at recombination (the isocurvature effect, cf. [19]). Further, in Eq. (3.3) we assume τ_* to be the epoch with $\kappa_*=1$, but take $N\sim 1$ (to reflect the fact that little of the small angular scale anisotropy is due to conventional last scattering at around the recombination epoch). We hope to return to a better treatment of these models in future work.

For model 1 of Peebles [18], one then gets $z_*\sim 52$, and from Eq. (4.5), $\sigma=821h^{-1}$ Mpc. Also for $z_*\gg 1$, we have $f[\Omega(z_*)]\to 1$, while for $\Omega_0=0.3$, $\Lambda=0.7$, $f^{-1}(\Omega_0)\sim 1.3$. Putting in all the numerical values, we then get

$$\sqrt{Q_P^V(k)} \approx 2.7F\times 10^{-2}\ \mu\mathrm{K}\frac{(\Delta T/T)_Q(\tau_e)}{10\ \mu\mathrm{K}}\left(\frac{k}{1h\ \ \mathrm{Mpc}^{-1}}\right)^{0.1}. \tag{4.6}$$

Here we have scaled $(\Delta T/T)_Q(\tau_e)$ by a smaller value of 10 μK, since these isocurvature models predict a somewhat smaller quadrupole than SCDM models (cf. Fig 1. of Ref. [18]). We have also incorporated a factor F to remind ourselves that the non-Gaussian statistics of the density field may lead to F times larger prominent upward fluctuations. If we wish to convert k to l in this model, we again use $R_*=\tau_0-\tau_*$, with $\tau_0\sim 2.17/(\Omega_0^{1/2}H_0)\sim 11{,}862.2h^{-1}$ Mpc (cf. Eq. (20) of Ref. [20]). Also $\tau_*\sim 1520.6h^{-1}$ and so $k=1h$ Mpc corresponds to $l\sim 10{,}342$, or an angular scale $\sim$0.33 arc min.

A similar analysis for model 2 gives $z_*\sim 36$, $\sigma\sim 1710h^{-1}$ Mpc and a larger polarization signal, with

$$\sqrt{Q_P^V(k)} \approx 5.7F\times 10^{-2}\ \mu\mathrm{K}\frac{(\Delta T/T)_Q(\tau_e)}{10\ \mu\mathrm{K}}\left(\frac{k}{1h\ \ \mathrm{Mpc}^{-1}}\right)^{0.3}. \tag{4.7}$$

Note that Peebles adopts a cosmological constant for convenience of analysis. If we were to consider open versions of these models, z_* and σ are nearly the same (since $z_*\gg 1$, but the power in electron density perturbation at last scattering is much larger, because of a much larger $f^{-1}(\Omega_0)$. The numerical value in Eqs. (4.6) and (4.7) at $k=1h$ Mpc^{-1} gets increased to $\sqrt{Q_P^V(k_c)}\sim 4.6F\times 10^{-2}$ μK (model 1) and $\sqrt{Q_P^V(k_c)}\sim 0.17F$ μK (model 2). The k dependence remains the same. In the open models, the density on scales of $k=k_c$ are already going nonlinear at z_* and so the above numbers provide only a crude estimate. We see in general that these isocurvature models predict much larger polarization signals compared to the SCDM-type models. First, the rms value is larger. Further, because of the non-Gaussian statistics of the density field, one expects prominent upward fluctuations $F\sim 3$ times larger than the rms value (cf. [18]).

We note in passing that the older versions of the baryonic isocurvature models, say, with $\Omega_0\sim 0.2$, $\Omega_b=0.05$, $h=0.8$, $m\sim -0.5$, cf. Ref. [21], leads to even larger signals with $\sqrt{Q_P^V}\sim 0.3$ μK$[(\Delta T/T)_Q(\tau_e)/10\ \mu\mathrm{K})(k/k_c)^{3/4}$. However, these models may already be ruled out by the fact that they result in spectral distortions, larger than the limit implied by the COBE observations [22].

V. CONCLUSIONS

In this paper we have explored the possibility of a Vishniac-type contribution to the polarization anisotropy at small angular scales. It is well known that nonlinear effects can make significant contribution to temperature anisotropy on small angular scales through the Vishniac effect, especially in reionized models. This arises due to the mode coupling of large angular scale first-order velocity perturbations with small angular scale electron density perturbations. We have considered here whether a similar effect contributes to the polarization anisotropy by studying the coupling of large angular scale first-order temperature anisotropy (quadrupole) with small angular scale electron density perturbation in reionized models.

We find that in cold dark matter models and its variants, the Vishniac-type effect leads to a fairly small polarization anisotropy, with $\sqrt{Q_P^V}\sim 0.02-0.04$ μK, on scales with $k\sim 1h$ Mpc (or angular scales of arc min or smaller). However in isocurvature type models the Vishniac-type contribution can result in much larger signals. For the models of Ref. [18], the anisotropy on small angular scales is non-Gaussian, with prominent upward fluctuations of order 0.1–0.5 μK, assuming $F\sim 3$. This reflects basically the fact that, the isocurvature type models have much more power on small scales and so produce much larger electron density fluctuations. We note in passing that the suppression factor due to the finite thickness of the last scattering surface on the small scale polarization anisotropy is much milder than that obtained in Ref. [3]. This is because, as mentioned earlier, the first order temperature quadrupole contribution arising due to the free streaming of the monopole at recombination was not included in their analysis.

It is clear that the polarization signals on arc min scales predicted by CDM-type models will be difficult to detect, but those predicted by isocurvature-type models will be much easier. If small scale polarization anisotropy is eventually detected, it will open up the novel prospect of studying directly both the quadrupole anisotropy and small scale electron density fluctuations at high redshifts. As pointed out in Ref. [23], in a different context, one can then also reduce the cosmic variance of the quadrupole significantly. In this paper we have made several approximations to analytically estimate the polarization anisotropy. We plan to return to a better numerical analysis in the near future.

ACKNOWLEDGMENTS

T.R.S. and K.S. thank John Barrow and the Astronomy Center, University of Sussex for hospitality during the course of this work. K.S. was supported by the PPARC at the Astronomy center, University of Sussex. K.S. also thanks Simon White and the Max-Planck Institute for Astrophysics, Garching, for hospitality when the paper was being completed, and Scott Dodelson for a useful discussion.

[1] J. P Ostriker and E. T. Vishniac, Astrophys. J. Lett. Ed. **306**, L51 (1986); E. T. Vishniac, Astrophys. J. **322**, 597 (1987).
[2] W. Hu, D. Scott, and J. Silk, Phys. Rev. D **49**, 648 (1994); S. Dodelson and J. M. Jubas, Astrophys. J. **439**, 503 (1995); observational constraints come from R. Subrahmanyan *et al.*, Mon. Not. R. Astron. Soc. **263**, 416 (1993).
[3] G. Efstathiou, in *Large Scale Motions in the Universe: A Vatican Study Week*, edited by V. C. Rubin and G. V. Coyne (Princeton University Press, Princeton, 1988), p. 299.
[4] M. Zaldarriaga, Phys. Rev. D **55**, 1822 (1997).
[5] C-P Ma and E. Bertschinger, Astrophys. J. **455**, 7 (1995); J. R. Bond, in *Cosmology and Large Scale Structure*, Proceedings of the Les Houches School, edited by R. Schaeffer, J. Silk, M. Spiro, and J. Zinn-Justin (Elsevier, Amsterdam, 1996).
[6] J. E. Gunn and B. A. Peterson, Astrophys. J. **142**, 1633 (1965).
[7] W. Hu and N. Sugiyama, Astrophys. J. **444**, 489 (1995).
[8] A. R. Liddle, D. H. Lyth, P. T. Viana, and M. White, Mon. Not. R. Astron. Soc. **282**, 281 (1996).
[9] A. R. Liddle, D. H. Lyth, D. Roberts, and P. T. P. Viana, Mon. Not. R. Astron. Soc. **278**, 674 (1996).
[10] G. Efstathiou, in *Physics of the Early Universe*, edited by Peacock *et al.* (Edinburgh University Press, Edinburgh, 1990), p. 361.
[11] J. M. Bardeen, J. R. Bond, N. Kaiser, and A. S. Szalay, Astrophys. J. **304**, 15 (1986).
[12] N. Sugiyama, Astrophys. J., Suppl. **100**, 281 (1995).
[13] W. Hu and N. Sugiyama, Astrophys. J. **471**, 542 (1996).
[14] E. F. Bunn and M. White, Astrophys. J. **480**, 6 (1997).
[15] C. L. Bennett *et al.*, Astrophys. J. **464**, L1 (1996); K. M. Gorski *et al.*, *ibid.* **464**, L11 (1996).
[16] J. P. Ostriker and P. J. Steinhardt, Nature (London) **377**, 600 (1995); see also, J. S. Bagla, T. Padmanabhan, and J. V. Narlikar, Comments. Astrophys. **18**, 275 (1996).
[17] W. Hu and M. White, Astron. Astrophys. **315**, 33 (1996).
[18] P. J. E. Peebles, Astrophys. J. **483**, L1 (1997).
[19] N. Sugiyama and J. Silk, Phys. Rev. Lett. **73**, 509 (1994); G. Efstathiou and J. R. Bond, Mon. Not. R. Astron. Soc. **218**, 103 (1986); H. Kodama and M. Sasaki, Prog. Theor. Phys. Suppl. **78**, 1 (1984).
[20] W. Hu and M. White, Astrophys. J. **479**, 568 (1997).
[21] P. J. E. Peebles, Astrophys. J. **315**, L73 (1986); Nature (London) **327**, 210 (1987); R. Cen, J. P. Ostriker, and P. J. E. Peebles, Astrophys. J. **415**, 423 (1994).
[22] W. Hu, E. F. Bunn, and N. Sugiyama, Astrophys. J. **447**, L59 (1996).
[23] M. Kamionkowski and A. Loeb, Phys. Rev. D **56**, 4511 (1997).