Quantum fluctuations in brane-world inflation without an inflaton on the brane

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A Randall-Sundrum-type brane-cosmological model in which slow-roll inflation on the brane is driven solely by a bulk scalar field was recently proposed by Himemoto and Sasaki. We analyze their model in detail and calculate the quantum fluctuations of the bulk scalar field ϕ with $m^2 = V''(\phi)$. We decompose the bulk scalar field into the infinite mass spectrum of four-dimensional fields; the field with the smallest mass square, called the zero mode, and the Kaluza-Klein modes above it with a mass gap. We find the zero-mode dominance of the classical solution holds if $|m^2|\overline{l}^2 \ll 1$, where $\overline{l}$ is the curvature radius of the effectively anti–de Sitter bulk, but it is violated if $|m^2|\overline{l}^2 \gg 1$, though the violation is very small. Then we evaluate the vacuum expectation value $\langle \delta\phi^2 \rangle$ on the brane. We find the zero-mode contribution completely dominates if $|m^2|\overline{l}^2 \ll 1$, similar to the case of classical background. In contrast, we find the Kaluza-Klein contribution is small but non-negligible if the value of $|m^2|\overline{l}^2$ is large.

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I. INTRODUCTION

Recent progress in string theory suggests that our spacetime is not four-dimensional but higher dimensional. Horava and Witten showed that a desirable gauge theory appears on the 10-dimensional boundaries of the 11-dimensional spacetime $M^{10} \times S^1/Z_2$ [1,2]. This opened up the possibility that we may live on a brane embedded in a higher-dimensional spacetime. Arkani-Hamed *et al.* [3] investigated such a possibility and showed that the size of the extra dimensions may be as large as 1 mm. But gravity was not fully taken into account in their work. Then taking the gravity into account, Randall and Sundrum showed the existence of five-dimensional models that realize the Horava-Witten theory if the five-dimensional cosmological constant is negative [4,5].

In their first paper [4], Randall and Sundrum found that two flat four-dimensional Minkowski branes can be naturally embedded in five-dimensional anti–de Sitter (AdS_5) space for appropriately tuned values of the brane tension: $\sigma = \pm\sqrt{-6\Lambda_5}/\kappa_5^2$ where $\Lambda_5 (<0)$ is the five-dimensional cosmological constant and κ_5^2 is the five-dimensional gravitational constant. They argued that the mass-hierarchy problem may be solved if we live on the negative tension brane. However, it was shown that there exists the so-called radion mode that describes the relative motion between the two branes and this mode causes an unacceptable modification of the effective gravity on the negative tension brane [6,7] unless the radion is somehow stabilized [8].

In their second paper [5], they investigated the case of a single positive tension brane in the AdS_5 bulk and showed that gravity is confined within the curvature radius $l = \sqrt{6/(-\Lambda_5)}$ of the AdS_5 and the four-dimensional Einstein gravity is recovered on the brane on scales greater than l despite the fact that the extra dimension is infinite. This raised an intriguing possibility that our universe is indeed a brane world in a higher-dimensional spacetime, and a lot of attention has been paid to the cosmology of a Randall-Sundrum-type brane world [9–35]. Among these studies of brane-world cosmology, Himemoto and Sasaki recently investigated the possibility of brane-world inflation induced by a dynamical bulk scalar field without introducing an inflaton field on the brane [35]. The reheating after inflation based on this scenario has been discussed by Yokoyama and Himemoto [32]. Possible realizations of such brane-world inflation models have been discussed in [21,26,27].

In [35], Himemoto and Sasaki found a perturbative solution of the bulk scalar field in the effective AdS_5 background (with a modified curvature radius $\overline{l}$ larger than the original l) and showed that slow-roll inflation is realized on the brane if $|m^2| \ll H^2$, where m is the mass of the bulk scalar field and H is the Hubble parameter on the brane, irrespective of the value of $m^2\overline{l}^2$. This result is interesting in that the dynamics of inflation on the brane is unaffected by the curvature scale of the bulk. Naively one would expect otherwise if $|m^2|\overline{l}^2 \gg 1$, since this implies $H^2\overline{l}^2 \gg 1$ for $|m^2| \ll H^2$, which is the case when the gravity on the brane on the Hubble horizon scale is significantly affected.

In this paper, taking up the brane-inflation scenario proposed in [35], we discuss the m^2 dependence of the classical background more carefully and investigate the effect of quantum fluctuations of the bulk scalar field on the dynamics of the brane. In particular, we are interested in whether there appears a significant effect on the brane if $|m^2|\overline{l}^2 \gg 1$. Recently Kobayashi, Koyama, and Soda [33] have considered the quantum fluctuations of a massless bulk scalar field ϕ in the AdS_5 background. The bulk field ϕ may be decomposed into the zero mode and the Kaluza-Klein modes. The former corresponds to a massless four-dimensional field and the latter to massive four-dimensional fields. They evaluated the contribution of the Kaluza-Klein modes to $\langle \phi^2 \rangle$ on the de Sitter (inflating) brane. They found that the Kaluza-Klein contribution is small relative to the zero-mode contribution even in the case $H^2 l^2 \gg 1$. Technically, our analysis is an extension of theirs to a bulk scalar field of arbitrary mass. The effect of quantum corrections in the brane-world scenario has also been discussed by Gilkey, Kirsten, and Vassilevich [34].

The paper is organized as follows. In Sec. II, following [9,35], we formulate the effective gravitational equations and the Friedmann equation on the brane when a bulk scalar field is present. In Sec. III, we discuss in depth the features of the brane-inflation model of [35] which we use as the background. We find an indication that the quantum fluctuations of the bulk scalar field may cause a non-negligible contribution to the dynamics of the brane when $|m^2|\overline{l^2}\gg 1$. In Sec. IV, we calculate the quantum fluctuations of the bulk scalar field and evaluate the extra-dimensional effect on the brane. We find the Kaluza-Klein contribution is indeed non-negligible if $|m^2|\overline{l^2}\gg 1$. Section V is denoted to summary and discussions.

II. FORMULATION

We write the bulk metric in the $(4+1)$ form as

$$ds^2=g_{ab}\,dx^a\,dx^b=n_a n_b\,dx^a\,dx^b+q_{ab}\,dx^a\,dx^b, \quad (2.1)$$

where n_a is the unit normal to the four-dimensional timelike hypersurfaces, one of which is the brane, and $q_{ab}=g_{ab}-n_a n_b$ is the induced metric on the hypersurfaces. We introduce the coordinates $\{\chi,x^\mu\}$ $(\mu=0,1,2,3)$ such that $n_a dx^a=d\chi$ and assume the brane is located at $\chi=\chi_0$.

Following the spirit of the Horava-Witten theory [1,2], we assume the Z_2 symmetry with respect to the brane and that the gravitational equations in the five-dimensional spacetime take the following form:

$$G_{ab}+\Lambda_5 g_{ab}=\kappa_5^2[T_{ab}+S_{ab}\delta(\chi-\chi_0)], \quad (2.2)$$

where G_{ab} is the five-dimensional Einstein tensor and T_{ab} is the effective energy momentum tensor in the five-dimensional bulk which may consist of higher-order curvature terms as well as dilatonlike gravitational scalar fields, and S_{ab} is the energy-momentum tensor of the matter confined on the brane.

As in the Randall-Sundrum model [4,5], we consider S_{ab} of the form

$$S_{ab}=-\sigma q_{ab}+\tau_{ab}; \qquad \tau_{ab}n^a=0, \quad (2.3)$$

where σ is the brane tension and τ_{ab} describes the four-dimensional matter excitations. Then the effective gravitational equations on the brane are found as [9]

$${}^{(4)}G_{\mu\nu}+\Lambda_4 q_{\mu\nu}=\kappa_4^2(T^{(b)}_{\mu\nu}+\tau_{\mu\nu})+\kappa_5^4\pi_{\mu\nu}-E_{\mu\nu}. \quad (2.4)$$

Here

$$\Lambda_4=\frac{1}{2}\left(\Lambda_5+\frac{1}{6}\kappa_5^4\sigma^2\right), \quad (2.5)$$

$$\kappa_4^2=\frac{1}{6}\kappa_5^4\sigma, \quad (2.6)$$

$$\pi_{\mu\nu}=-\frac{1}{4}\tau_{\mu\alpha}\tau^{\alpha}_{\nu}+\frac{1}{12}\tau\tau_{\mu\nu}+\frac{1}{8}q_{\mu\nu}\tau_{\alpha\beta}\tau^{\alpha\beta}-\frac{1}{24}q_{\mu\nu}\tau^2, \quad (2.7)$$

$$T^{(b)}_{\mu\nu}=\frac{4}{\sigma\kappa_5^2}\left[T_{cd}q^c_\mu q^d_\nu+\left(T_{cd}n^c n^d-\frac{1}{4}T\right)q_{\mu\nu}\right], \quad (2.8)$$

$$E_{\mu\nu}={}^{(5)}C_{abcd}n^a n^c q^b_\mu q^d_\nu, \quad (2.9)$$

where Λ_4 and κ_4^2 describe the four-dimensional cosmological constant and the four-dimensional gravitational constant, respectively, and ${}^{(5)}C_{abcd}$ is the five-dimensional Weyl tensor. Note that the projected Weyl tensor $E_{\mu\nu}$ is traceless by definition. Note also that the bulk energy momentum tensor contributes to a four-dimensional energy-momentum tensor. Here and in what follows $q_{\mu\nu}$ denotes the metric on the brane unless otherwise noted. We consider a scenario in which the Randall-Sundrum brane is recovered in the ground state. Thus we assume the brane tension as

$$\sigma=\sigma_c:=\frac{\sqrt{-6\Lambda_5}}{\kappa_5^2}=\frac{6}{\kappa_5^2 l}, \quad (2.10)$$

which gives $\Lambda_4=0$.

For the brane cosmology, we assume the spatial homogeneity and isotropy and express the metric on the brane as

$$ds^2|_{\rm brane}=q_{\mu\nu}\,dx^\mu\,dx^\nu=-dt^2+a(t)^2\gamma_{ij}\,dx^i\,dx^j, \quad (2.11)$$

where γ_{ij} is the three metric of a constant curvature space with unit or zero curvature, $K=\pm 1,0$. The matter energy-momentum tensor takes a perfect fluid form on this brane:

$$\tau_{\mu\nu}=(\rho^{(m)}+P^{(m)})t_\mu t_\nu+P^{(m)}q_{\mu\nu}, \quad (2.12)$$

where $t_\mu dx^\mu=-dt$. With these, the time-time component of Eq. (2.4) gives the Friedmann equation on the brane:

$$3\left[\left(\frac{\dot a}{a}\right)^2+\frac{K}{a^2}\right]=\kappa_4^2\rho^{(m)}+\frac{\kappa_5^4}{12}\rho^{(m)2}+\kappa_4^2 T^{(b)}_{tt}-E_{tt}, \quad (2.13)$$

where the dot denotes the derivative with respect to t, and the trace of the space-space components gives

$$-\left[2\frac{\ddot a}{a}+\left(\frac{\dot a}{a}\right)^2+\frac{K}{a^2}\right]=\kappa_4^2 P^{(m)}+\frac{\kappa_5^4}{12}\rho^{(m)}(\rho^{(m)}+2P^{(m)})+\kappa_4^2\frac{T^{(b)}{}_i{}^i}{3}-\frac{E_{tt}}{3}. \quad (2.14)$$

The second equation is not necessary if the Bianchi identities are taken into account. We give it here for later convenience. The Friedmann equation (2.13) is not closed on the brane because of the contributions $T^{(b)}_{tt}$ and E_{tt} which depend on the bulk dynamics. However, under certain situations as described below, the scalar field dynamics in the bulk may be solved perturbatively and E_{tt} may be expressed in terms of the scalar field relatively easily.

As noted before, the bulk energy-momentum tensor may contain the contributions from higher-order curvature terms as well as dilatonlike fields. However, lacking the detailed knowledge of their explicit forms, we choose the simplest case of a minimally coupled scalar field,

$$T_{ab}=\partial_a\phi\partial_b\phi-g_{ab}\left[\frac{1}{2}g^{cd}\partial_c\phi\partial_d\phi+V(\phi)\right]. \tag{2.15}$$

As discussed in Ref. [35], the assumption of the Z_2-symmetry and the energy-momentum conservation for the matter field on the brane, $D_\mu\tau^{\mu\nu}=0$, implies [36]

$$\partial_\chi\phi|_{\rm brane}=0. \tag{2.16}$$

Hence together with the spatial homogeneity of ϕ on the brane, $T^{(b)}_{\mu\nu}$ reduces to

$$T^{(b)}_{\mu\nu}=(\rho^{(b)}+P^{(b)})t_\mu t_\nu+P^{(b)}q_{\mu\nu}, \tag{2.17}$$

where

$$\rho^{(b)}=T^{(b)}_{tt}=\frac{3}{\kappa_5^2\sigma_c}\left[\frac{1}{2}\dot{\phi}^2+V(\phi)\right]=\frac{l}{2}\left[\frac{1}{2}\dot{\phi}^2+V(\phi)\right], \tag{2.18}$$

$$P^{(b)}=\frac{1}{3}T^{(b)}{}_i{}^i=\frac{3}{\kappa_5^2\sigma_c}\left[\frac{5}{6}\dot{\phi}^2-V(\phi)\right]=\frac{l}{2}\left[\frac{5}{6}\dot{\phi}^2-V(\phi)\right]. \tag{2.19}$$

The form of the projected Weyl tensor on the brane is determined from its traceless nature,

$$E_{\mu\nu}=\left(\frac{4}{3}t_\mu t_\nu+\frac{1}{3}q_{\mu\nu}\right)E_{tt}. \tag{2.20}$$

Then, from the four-dimensional Bianchi identities and the spatial homogeneity of the brane, we obtain [9]

$$D^\nu E_{\mu\nu}=\kappa_4^2 D^\nu T^{(b)}_{\mu\nu}, \tag{2.21}$$

which gives

$$\frac{1}{a^4}\partial_t(a^4E_{tt})=-\frac{\kappa_5^2}{6}[\ddot{\phi}-4\partial_\chi^2\phi+V'(\phi)]\dot{\phi}. \tag{2.22}$$

Further, using the scalar field equation at the location of the brane,

$$\partial_\chi^2\phi-\frac{1}{\sqrt{-q}}\partial_\mu(\sqrt{-q}q^{\mu\nu}\partial_\nu\phi)=0, \tag{2.23}$$

and Eq. (2.22) may be rewritten as

$$\frac{1}{a^4}\partial_t(a^4E_{tt})=\frac{\kappa_5^2}{2}\left(\partial_\chi^2\phi+\frac{\dot{a}}{a}\dot{\phi}\right)\dot{\phi}. \tag{2.24}$$

Formally integrating the above, we obtain

$$E_{tt}=\frac{\kappa_5^2}{2a^4}\int^t a^4\dot{\phi}\left(\partial_\chi^2\phi+\frac{\dot{a}}{a}\dot{\phi}\right)dt. \tag{2.25}$$

Thus E_{tt} is expressed in terms of ϕ apart from the integration constant which gives a term $\propto a^{-4}$ and which should be determined by the initial condition of the five-dimensional universe.

III. MODEL

Himemoto and Sasaki proposed a model for brane-world inflation induced by a bulk scalar field [35]. Here we adopt their model. Namely, we assume the potential of the form

$$V(\phi)=V_0+\frac{1}{2}m^2\phi^2, \tag{3.1}$$

and consider the region $|m^2|\phi^2/2\ll V_0$. In [35], it was further assumed that $m^2<0$ in order for a regular solution in the separable form to exist. Here we basically follow [35] and assume $m^2<0$ for the moment. However, one can argue that the singularity may not harm the brane if the action is still finite [12]. There is yet another argument that supports the admissibility of the case $m^2>0$ [37]. Hence, we relax this assumption in the end and extend our analysis to positive values of m^2.

Assuming T_{ab} is dominated by V_0 at the lowest order of approximation,

$$T_{ab}\simeq -V_0 g_{ab}, \tag{3.2}$$

the effective five-dimensional cosmological constant becomes

$$\Lambda_{5,\rm eff}=\Lambda_5+\kappa_5^2V_0. \tag{3.3}$$

Here we assume that $|\Lambda_5|>\kappa_5^2V_0$ so that the background space-time is still effectively AdS$_5$. Then the bulk metric may be written as [12]

$$ds^2=dr^2+(H\bar{l})^2\sinh^2(r/\bar{l})ds^2_{\rm dS}, \tag{3.4}$$

where $ds^2_{\rm dS}$ is the metric of the four-dimensional de Sitter space with radius H^{-1} given by

$$H^{-1}=\bar{l}\sinh(r_0/\bar{l});\quad \bar{l}=\left|\frac{6}{\Lambda_{5,\rm eff}}\right|^{1/2}. \tag{3.5}$$

Note that we have replaced the coordinate χ in the previous section by r in accordance with the standard convention, and the value of r_0 (equivalently of H) is arbitrary for the moment.

As noted in the previous section, we assume the brane tension to be that of the Randall-Sundrum value, given by Eq. (2.10). Then choosing $r=r_0$ to be the location of the brane, we have the de Sitter brane with the metric,

$$ds_{\rm dS}^2 = -dt^2 + \frac{1}{H^2} e^{2Ht} d\boldsymbol{x}^2, \tag{3.6}$$

where we have adopted the spatially flat slicing for simplicity. Neglecting the matter field excitations on the brane, the Friedmann equation at the lowest order determines H as

$$H^2 = \frac{\kappa_5^2 V_0}{6}, \tag{3.7}$$

which in turn determines the location of the brane r_0.

The next order correction is determined by solving the scalar field equation in the bulk,

$$\frac{e^{-3Ht}}{(\bar{l}H)^2 \sinh^2(r/\bar{l})} \partial_t [e^{3Ht} \partial_t \phi] - \frac{1}{\sinh^4(r/\bar{l})} \partial_r [\sinh^4(r/\bar{l}) \partial_r \phi]$$
$$- \frac{e^{-2Ht}}{\bar{l}^2 \sinh^2(r/\bar{l})} {}^{(3)}\Delta \phi + m^2 \phi = 0, \tag{3.8}$$

where ${}^{(3)}\Delta$ is the flat Laplacian with respect to $\boldsymbol{x}$. Assuming that our background solution is in the separable form,

$$\phi(t,r) = \psi(t) u(r), \tag{3.9}$$

the equations for $\psi(t)$ and $u(r)$ become

$$\left[\frac{d^2}{dt^2} + 3H \frac{d}{dt} + \lambda^2 H^2 \right] \psi = 0, \tag{3.10}$$

$$\left[\frac{d^2}{dr^2} + \frac{4}{l} \coth\left(\frac{r}{l} \right) \frac{d}{dr} - \left(m^2 - \frac{\lambda^2}{\bar{l}^2 \sinh^2(r/\bar{l})} \right) \right] u = 0, \tag{3.11}$$

where λ is a separation constant [38].

Equation (3.11) is the eigenvalue equation for λ^2. The Z_2 symmetry implies the boundary condition at the brane,

$$\left. \frac{d}{dr} u \right|_{r=r_0} = 0. \tag{3.12}$$

As for the boundary condition at the origin $r=0$, we impose the regularity of u as in [35],

$$u|_{r=0} = 0. \tag{3.13}$$

Under these conditions, the solution is found as

$$u = u_{\lambda_0}(r) = \frac{\mathrm{P}_{\nu-1/2}^{-\mu_0}[\cosh(r/\bar{l})]}{\sinh^{3/2}(r/\bar{l})}, \tag{3.14}$$

where $\mu_0 := \sqrt{9/4 - \lambda_0^2}$ and $\nu := \sqrt{m^2 \bar{l}^2 + 4}$ with the eigenvalue $\lambda = \lambda_0$ determined by

$$(\nu+2) z_0 \mathrm{P}_{\nu-1/2}^{-\mu_0}(z_0) = (\nu + \mu_0 + 1/2) \mathrm{P}_{\nu+1/2}^{-\mu_0}(z_0);$$
$$z_0 := \cosh(r_0/\bar{l}). \tag{3.15}$$

Provided $|m^2|/H^2 \ll 1$, λ_0^2 is approximately given by [35]

$$\lambda_0^2 = \begin{cases} \dfrac{m^2}{2H^2} & \text{for } |m^2| \bar{l}^2 \ll 1, \\ \dfrac{3m^2}{5H^2} & \text{for } |m^2| \bar{l}^2 \gg 1. \end{cases} \tag{3.16}$$

Then the corresponding solution for ψ is given by

$$\psi = \psi_{\lambda_0}(t) = C \exp\left[\left(\mu_0 - \frac{3}{2} \right) Ht \right] + D \exp\left[\left(-\mu_0 - \frac{3}{2} \right) Ht \right], \tag{3.17}$$

which behaves as $\propto e^{-\lambda_0^2 Ht/3}$ at late times. Thus slow-roll inflation is realized on the brane for $|m^2|/H^2 \ll 1$, irrespective of the magnitude of $|m^2| \bar{l}^2$.

Note that the regularity condition (3.13) implies that λ^2 must be nonpositive. This was the reason why only the case $m^2 < 0$ was considered in [35]. However, as mentioned earlier, the singularity at $r=0$ may be harmless as long as u_{λ_0} is square integrable [12,37]. As we shall see in Sec. IV, u_{λ_0} is indeed square integrable even for $m^2 > 0$ and it turns out to be the unique bound-state solution, i.e., the zero-mode solution. Therefore we relax the assumption $m^2 < 0$ and consider both signs of m^2 in the following. The eigenvalue λ_0^2 gives the effective four-dimensional mass of the zero mode,

$$m_{\rm eff}^2 = \lambda_0^2 H^2 = \begin{cases} \dfrac{m^2}{2} & \text{for } |m^2| \bar{l}^2 \ll 1, \\ \dfrac{3m^2}{5} & \text{for } |m^2| \bar{l}^2 \gg 1. \end{cases} \tag{3.18}$$

With the above perturbative solution for the scalar field, let us now consider the correction to the Friedmann equation on the brane. To see the effect on the dynamics of the brane, we evaluate the effective energy density and pressure on the brane. In the present case, Eqs. (2.13) and (2.14) become

$$3\left(\frac{\dot{a}}{a} \right)^2 = \kappa_4^2 (\bar{\rho} + V_0),$$

$$-\left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 \right] = \kappa_4^2 (\bar{P} - V_0), \tag{3.19}$$

where

$$\bar{\rho} = \frac{l}{2} \left[\frac{1}{2} \dot{\phi}^2 + \frac{1}{2} m^2 \phi^2 \right] - \frac{1}{\kappa_4^2} E_{tt}, \tag{3.20}$$

$$\bar{P} = \frac{l}{2} \left[\frac{5}{6} \dot{\phi}^2 - \frac{1}{2} m^2 \phi^2 \right] - \frac{1}{3\kappa_4^2} E_{tt}. \tag{3.21}$$

Using our solution for the scalar field given by Eqs. (3.14) and (3.17), the integral expression (2.25) for E_{tt} can be evaluated as

$$E_{tt}=\frac{\kappa_5^2}{2}\,\frac{\mu_0-3/2}{2\mu_0+1}\left[m^2-\lambda_0^2H^2+\left(\mu_0-\frac{3}{2}\right)H^2\right]\phi^2 \tag{3.22}$$

$$\simeq-\frac{3\kappa_5^2}{8\lambda_0^2H^2}\left(m^2-\frac{4}{3}\lambda_0^2H^2\right)\dot{\phi}^2, \tag{3.23}$$

where we have taken the asymptotic limit $Ht\gg1$ and used the approximation $\mu_0\simeq3/2-\lambda_0^2/3$. Inserting the above to Eqs. (3.20) and (3.21), we find

$$\bar{\rho}\simeq\frac{l}{2}\left[\left(\frac{3m^2}{4\lambda_0^2H^2}-\frac{1}{2}\right)\dot{\phi}^2+\frac{1}{2}m^2\phi^2\right], \tag{3.24}$$

$$\bar{P}\simeq\frac{l}{2}\left[\left(\frac{m^2}{4\lambda_0^2H^2}+\frac{1}{2}\right)\dot{\phi}^2-\frac{1}{2}m^2\phi^2\right]. \tag{3.25}$$

When $|m^2|\overline{l^2}\ll|m^2|/H^2\ll1$, we have $H^2\overline{l^2}\ll1$. In this case, using the eigenvalue λ_0^2 given by Eq. (3.16), we obtain

$$\bar{\rho}\simeq\frac{l}{2}\left(\dot{\phi}^2+\frac{1}{2}m^2\phi^2\right)=\frac{1}{2}\dot{\Phi}^2+\frac{1}{2}m_{\rm eff}^2\Phi^2,$$

$$\bar{P}\simeq\frac{l}{2}\left(\dot{\phi}^2-\frac{1}{2}m^2\phi^2\right)=\frac{1}{2}\dot{\Phi}^2-\frac{1}{2}m_{\rm eff}^2\Phi^2, \tag{3.26}$$

where we have introduced the rescaled four-dimensional scalar field Φ defined by

$$\Phi:=\sqrt{l}\,\phi. \tag{3.27}$$

Since $H^2\overline{l^2}\ll1$ in this case, the extra dimension is sufficiently compact in comparison with the Hubble radius of the brane. Therefore the dynamics on the brane on super Hubble horizon scales is expected to be dominated by the zero-mode solution and to be essentially the same as the conventional four-dimensional theory. The above result indeed supports this expectation.

In contrast, when $|m^2|\overline{l^2}\gg1$, if the effective four-dimensional mass of the scalar field should be still given by that of the zero-mode solution, the effective energy density and pressure on the brane are given by

$$\bar{\rho}\simeq\frac{l}{2}\left(\frac{3}{4}\dot{\phi}^2+\frac{1}{2}m^2\phi^2\right),$$

$$\bar{P}\simeq\frac{l}{2}\left(\frac{11}{12}\dot{\phi}^2-\frac{1}{2}m^2\phi^2\right). \tag{3.28}$$

Apparently these do not agree with those given by the zero-mode solution. If we introduce the rescaled four-dimensional field for this case as

$$\Phi:=\sqrt{\frac{5l}{6}}\phi, \tag{3.29}$$

which minimizes the discrepancy, we have

$$\bar{\rho}=\frac{1}{2}\dot{\Phi}^2+\frac{1}{2}m_{\rm eff}^2\Phi^2-\frac{1}{20}\dot{\Phi}^2,$$

$$\bar{P}=\frac{1}{2}\dot{\Phi}^2-\frac{1}{2}m_{\rm eff}^2\Phi^2+\frac{1}{20}\dot{\Phi}^2. \tag{3.30}$$

It is worth noting that the discrepancy is small even in the limit $|m^2|\overline{l^2}\to\infty$, provided Φ is slowly rolling on the brane. Nevertheless, this implies that the dynamics on the brane cannot be described by the zero-mode solution alone even at the classical level, which in turn suggests that the quantum fluctuations on the brane may be affected significantly.

Before closing this section, we comment on the possible generality of our brane-inflation scenario. Since the restriction on the sign of m^2 may be relaxed as discussed above, it may be said that inflation on the brane is a natural consequence of the Randall-Sundrum type brane-world scenario once bulk (gravitational) scalar fields are taken into account. The actual issue is not if inflation can occur but how inflation can end, that is, it is the cosmological constant problem which should be solved in the brane-world scenario.

IV. QUANTUM FLUCTUATIONS

In this section, we calculate quantum fluctuations of the bulk scalar field. Kobayashi, Koyama, and Soda recently discussed the effect of quantum fluctuations of a massless bulk scalar field on the de Sitter brane [33]. Here we consider the case of arbitrary mass.

On the bulk background given by the metric (3.4), the fluctuation of the bulk scalar field, $\delta\phi$, can be expanded as

$$\delta\phi(r,t,\boldsymbol{x})=\int d\lambda\, u_\lambda(r)\,\phi_\lambda(t,\boldsymbol{x}), \tag{4.1}$$

where ϕ_λ satisfies the four-dimensional field equation,

$$[-{}^{(4)}\Box+\lambda^2H^2]\phi_\lambda=0, \tag{4.2}$$

and u_λ satisfies the eigenvalue equation,

$$\left[\frac{d^2}{dr^2}+\frac{4}{\bar{l}}\coth\left(\frac{r}{\bar{l}}\right)\frac{d}{dr}-\left(m^2-\frac{\lambda^2}{\overline{l^2}\sinh^2(r/\bar{l})}\right)\right]u_\lambda=0. \tag{4.3}$$

The d'Alembertian ${}^{(4)}\Box$ is the one with respect to the de Sitter metric (3.6). Note that Eq. (4.3) is the same as Eq. (3.11).

Since ϕ_λ can be regarded as a four-dimensional scalar field with mass λH, it can be quantized following the standard procedure as

$$\phi_\lambda(t,\boldsymbol{x})=\int d^3\boldsymbol{k}[a_{k\lambda}\psi_{k\lambda}(t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}+(\text{H.c.})], \tag{4.4}$$

where $a_{\boldsymbol{k}\lambda}$ and $a^{\dagger}_{\boldsymbol{k}\lambda}$ are the annihilation and creation operators which satisfy

$$[a_{\boldsymbol{k}\lambda},a^{\dagger}_{\boldsymbol{k}'\lambda'}]=\delta(\boldsymbol{k}-\boldsymbol{k}')\,\delta(\lambda-\lambda'),\tag{4.5}$$

and $\psi_{k\lambda}$ is the positive frequency function satisfying

$$\left[\frac{d^2}{dt^2}+3H\frac{d}{dt}+(k^2e^{-2Ht}+\lambda^2)H^2\right]\psi_{k\lambda}=0;$$

$$\psi_{k\lambda}\dot{\psi}^{*}_{k\lambda}-\dot{\psi}_{k\lambda}\psi^{*}_{k\lambda}=iH^3e^{-3Ht}.\tag{4.6}$$

Choosing the de Sitter–invariant Euclidean vacuum as the state to be annihilated by $a_{\boldsymbol{k}\lambda}$, we have

$$\psi_{k\lambda}(t)=\frac{\sqrt{\pi}}{2}e^{i\pi\mu/2}H(-\eta)^{3/2}\mathrm{H}^{(1)}_{\mu}(-k\eta),\tag{4.7}$$

where $\eta=-e^{-Ht}$ is the conformal time and $\mu=\sqrt{9/4-\lambda^2}$.

The eigenvalue equation (4.3) for $u_\lambda(r)$ is solved under the boundary condition,

$$\left.\frac{d}{dr}u_\lambda\right|_{r=r_0}=0,\tag{4.8}$$

together with the normalization condition,

$$2(\bar{l}H)^2\int_0^{r_0}dr\,\sinh^2(r/\bar{l})u_\lambda u^{*}_{\lambda'}=\delta(\lambda-\lambda').\tag{4.9}$$

This normalization condition ensures the correct canonical quantization of $\delta\phi$ in the bulk.

Solving Eqs. (4.3) under these conditions, we find there are one bound state and an infinite number of continuous modes. The bound state has the smallest eigenvalue λ_0^2 and is the same as the background solution given by Eq. (3.14) apart from the normalization. It is given by

$$u_{\lambda_0}(r)=\frac{1}{BH\bar{l}\sinh^2(r/\bar{l})}\mathrm{Q}^{\nu}_{\lambda_0-1/2}[\coth(r/\bar{l})],\tag{4.10}$$

$$B^2=2\int_0^{r_0}dr\left(\frac{\mathrm{Q}^{\nu}_{\mu_0-1/2}[\coth(r/\bar{l})]}{\sinh(r/\bar{l})}\right)^2,\tag{4.11}$$

where $\nu=-\sqrt{m^2\bar{l}^2+4}$, and $\mu_0=\sqrt{9/4-\lambda_0^2}$ is determined by

$$\left(\mu-\nu+\frac{1}{2}\right)\mathrm{Q}^{\nu}_{\mu+1/2}[\coth(r_0/\bar{l})]=\left(\mu-\frac{3}{2}\right)\coth(r_0/\bar{l})\mathrm{Q}^{\nu}_{\mu-1/2}[\coth(r_0/\bar{l})],\tag{4.12}$$

which is equivalent to Eq. (3.15) [39].

Rewriting Eq. (4.3) in the standard Schrödinger form [35], we find the continuous mode spectrum for $\lambda>3/2$. It should be noted that the continuous mass spectrum is independent of the five-dimensional mass of the field. These continuous modes are called the Kaluza-Klein modes and their existence is the main signature of the brane world. The Kaluza-Klein mode solutions are given by

$$u_\lambda(r)=\frac{1}{N\sqrt{\bar{l}}}\left(\frac{\lambda^2}{\lambda^2-9/4}\right)^{1/4}\frac{1}{H\bar{l}\sinh^2(r/\bar{l})}\{\mathrm{P}^{\nu}_{\mu-1/2}[\coth(r/\bar{l})]-\alpha_\lambda(z_0)\mathrm{Q}^{\nu}_{\mu-1/2}[\coth(r/\bar{l})]\},\tag{4.13}$$

where

$$\mu=\sqrt{9/4-\lambda^2}=i\sqrt{\lambda^2-9/4},\quad \nu=-\sqrt{m^2\bar{l}^2+4},\tag{4.14}$$

and

$$N^2=\left|\frac{\Gamma(\mu)}{\Gamma(\mu-\nu+\frac{1}{2})}\right|^2+\left|\frac{\Gamma(-\mu)}{\Gamma(-\mu-\nu+\frac{1}{2})}-\pi\alpha_\lambda(z_0)e^{\nu\pi i}\frac{\Gamma(\mu+\nu+\frac{1}{2})}{\Gamma(\mu+1)}\right|^2,$$

$$\alpha_\lambda(z_0)=\frac{(\mu-\nu+1/2)\mathrm{P}^{\nu}_{\mu+1/2}(z_0)+(3/2-\mu)z_0\mathrm{P}^{\nu}_{\mu-1/2}(z_0)}{(\mu-\nu+1/2)\mathrm{Q}^{\nu}_{\mu+1/2}(z_0)+(3/2-\mu)z_0\mathrm{Q}^{\nu}_{\mu-1/2}(z_0)};$$

$$z_0\equiv\coth(r_0/\bar{l}).\tag{4.15}$$

Assuming the state to be the Euclidean vacuum, the vacuum expectation value $\langle\delta\phi^2\rangle$ on the brane is given by

$$\langle\delta\phi^2(\eta)\rangle|_{r=r_0}=\int d^3k\left[P_0(k;\eta)+\int_{3/2}^{\infty}d\lambda\,P(\lambda,k;\eta)\right],\tag{4.16}$$

where $P_0(k;\eta)$ and $P(\lambda,k;\eta)$ are the power spectra of the zero mode and the Kaluza-Klein mode,

$$P_0(k;\eta)=|\psi_{k\lambda_0}(\eta)|^2|u_{\lambda_0}(r_0)|^2,\tag{4.17}$$

$$P(\lambda,k;\eta)=|\psi_{k\lambda}(\eta)|^2|u_\lambda(r_0)|^2.\tag{4.18}$$

As already noted, the Kaluza-Klein fields ϕ_λ are independent of the five-dimensional mass m. Thus the difference between the massless and massive bulk scalar fields appears only through the difference in the amplitudes of u_λ on the brane. To see the contributions of the Kaluza-Klein modes relative to the zero mode, we plot $|u_\lambda(r_0)|^2/|u_{\lambda_0}(r_0)|^2$ for various values of $m^2\bar{l}^2$ in Fig. 1. There we have set $|m^2|/H^2=0.3$ for all the massive models. We find the continuous mode spectrum is rather insensitive to the value of $m^2\bar{l}^2$ as long as $|m^2|/H^2\ll 1$.

The bare expectation value (4.16) is ultraviolet divergent, and the divergence is cubic now in contrast to the quadratic divergence in four-dimensional theory. As far as the four-dimensional divergence is concerned, it may be regularized by cutting off the contributions of the physical wave numbers larger than H, $-k\eta>1$, as in conventional models of inflation. Then the regularized value may be estimated by

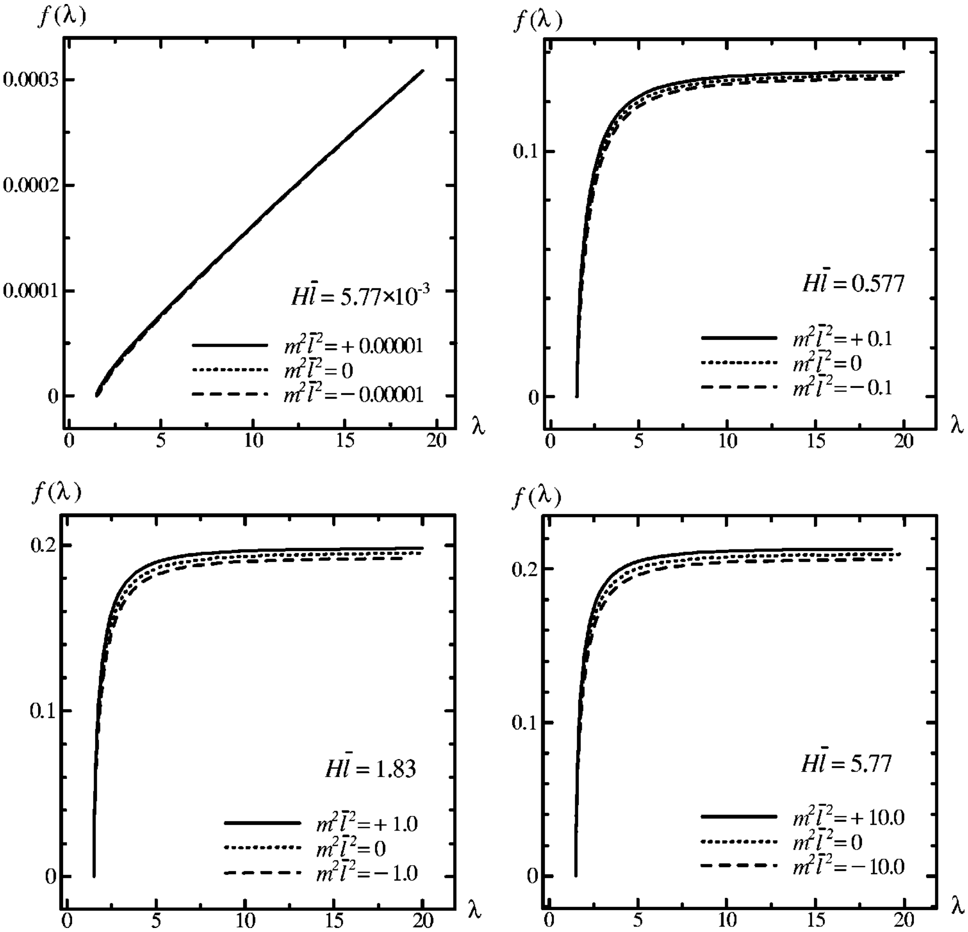

FIG. 1. $f(\lambda) \equiv |u_\lambda(r_0)|^2 / |u_{\lambda_0}(r_0)|^2$ for various values of $|m^2|\bar{l}^2$. We set $|m^2|/H^2 = 0.3$ except for massless cases.

simply evaluating $P_0(k;\eta)$ and $P(\lambda,k;\eta)$ at $-k\eta = 1$. However, the expectation value would still diverge because of the infinite contribution of the Kaluza-Klein modes. In the limit $\lambda \to \infty$,

$$P(\lambda,k;\eta) \propto \frac{1}{\lambda}, \tag{4.19}$$

so the Kaluza-Klein correction diverges logarithmically.

Lacking the relation of $\langle \delta\phi^2 \rangle$ to physical observables, we have no principle for how it should be regularized. Furthermore, the divergencies of the bulk five-dimensional theory would require not only local counterterms in the bulk but also surface counterterms on the brane, and the latter will generally be nonlocal functionals of the brane metric and matter fields that may not be absorbed by the renormalization of four-dimensional observables. It is hoped that there exist physically significant observables that are calculable without fully renormalizing the theory.

Discussions on these issues are, however, out of scope of this paper. We therefore adopt a simple cutoff regularization by introducing a cutoff parameter λ_c and define the regularized power spectrum,

$$P_{\lambda_c}(k) := P_0(k)\left[1 + \int_{3/2}^{\lambda_c} d\lambda \frac{P(\lambda,k)}{P_0(k)}\right], \tag{4.20}$$

where $P_0(k)$ and $P(\lambda,k)$ are $P_0(k;\eta)$ and $P(\lambda,k;\eta)$ evaluated at $-\eta = 1/k$, respectively. Then, in the limit $\lambda_c \to \infty$, we have

$$\Delta_{\lambda_c} := \frac{P_{\lambda_c}(k)}{P_0(k)} - 1 \to \alpha + \beta \ln \lambda_c. \tag{4.21}$$

The parameters α and β are independent of λ_c, and they not only describe the asymptotic behavior of the divergence but also reflect the cutoff-independent, physical properties of the regularized spectrum. Table I gives the parameters α and β, and the Kaluza-Klein correction Δ_{λ_c} for several choices of the cutoff parameter λ_c, for various cases of the five-dimensional mass but with the same value of $|m^2|/H^2 = 0.3$. Since the Hubble scale H is the only natural scale we have on the brane, the value of λ_c is taken to be not too greater than H.

From Table I, we see that both α and β decrease as $|m^2|\bar{l}^2$ becomes small. Hence we deduce that the Kaluza-Klein correction vanishes in the limit $|m^2|\bar{l}^2 \to 0$. This is consistent with our intuition that the five-dimensional theory reduces to the conventional four-dimensional theory for $\bar{l} \ll 1/H$. The Kaluza-Klein correction becomes large as $|m^2|\bar{l}^2$ increases. But instead of growing indefinitely, it seems to be saturated at a small finite value. This is the quantum version of the fact we found for the classical background in Sec. III,

TABLE I. The Kaluza-Klein correction to the power spectrum of $\langle \delta\phi^2 \rangle$ on the brane for various values of $m^2\overline{l^2}$. The parameter $|m^2|/H^2$ is set to 0.3 for all the entries. See Eq. (4.21) for the definitions of α, β, and Δ_{λ_c}.

$m^2\overline{l^2}$	α	β	$\Delta_{\lambda_c=2}$	$\Delta_{\lambda_c=5}$	$\Delta_{\lambda_c=10}$
-10.0	-0.058	0.096	0.016	0.097	0.16
-1.0	-0.056	0.089	0.014	0.089	0.16
-0.1	-0.046	0.060	0.0070	0.053	0.093
-10^{-5}	-0.003	7.5×10^{-4}	1.5×10^{-6}	2.1×10^{-5}	5.9×10^{-5}
$+10^{-5}$	-0.003	8.5×10^{-4}	2.1×10^{-6}	2.5×10^{-5}	6.8×10^{-5}
$+0.1$	-0.051	0.071	0.0093	0.065	0.11
$+1.0$	-0.062	0.11	0.019	0.11	0.18
$+10.0$	-0.064	0.12	0.021	0.12	0.20

that is, the zero-mode dominance of the classical solution even in the limit $|m^2|\overline{l^2}\to\infty$. Nevertheless, in contrast to the case of the classical background, the quantum Kaluza-Klein correction is non-negligible, of the order of about 10%.

Before concluding this section, it may be worth mentioning the following observation. In the case of $m^2<0$, one might expect instability of some modes to appear if $|m^2|\overline{l^2}\gg 1$, that is, when the bulk curvature is negligible on the Compton length scale of the field. But our result that the Kaluza-Klein correction always remains small implies the absence of such instability. This may be understood as follows. If we transform the bulk metric (3.4) to a static chart [11],

$$ds^2=-\frac{R^2}{\overline{l^2}}dT^2+\frac{\overline{l^2}}{R^2}dR^2+R^2\,d\mathbf{x}^2, \tag{4.22}$$

the location of the brane can be parametrized as $(T,R)=[T(\tau),R(\tau)]$ where

$$R(\tau)=\frac{1}{H}e^{H\tau}, \quad T(\tau)=\overline{l}\sqrt{1+(H\overline{l})^2}e^{-H\tau}. \tag{4.23}$$

Since the growth rate of instability would be $O(|m|)$, while the brane radius increases exponentially at the rate H, the slow-roll condition $|m^2|/H^2\ll 1$ implies that the field in the bulk is stretched so fast that the instability has no time to grow.

V. SUMMARY

Based on a brane-inflation model of a Randall-Sundrum-type brane world [4,5] proposed by Himemoto and Sasaki [35], we have investigated the quantum fluctuations of a massive bulk scalar field ϕ with mass m which drives slow-roll inflation on the brane. In this model, the potential of the scalar field $V(\phi)$ (>0) modifies the curvature radius of the five-dimensional anti–de Sitter space from the Randall-Sundrum value l to $\overline{l}$ $(>l)$, and inflation occurs on the brane with the expansion rate H proportional to $\sqrt{V(\phi)}$.

We have calculated the power spectrum of the expectation value $\langle \delta\phi^2 \rangle$ on the brane. Provided $|m^2|/H^2\ll 1$ which ensures slow-roll inflation of the classical background, we have found that the contribution of the Kaluza-Klein modes, which features the existence of the extra dimension, is much smaller than that of the zero mode if $|m^2|\overline{l^2}\ll 1$, whereas it is small but non-negligible if $|m^2|\overline{l^2}\gg 1$. The Kaluza-Klein contribution in the latter case has been found to be of the order of 10%. Although $\langle \delta\phi^2 \rangle$ does not describe any observable quantity in our model, our result indicates the importance of the Kaluza-Klein contributions to physical observables when $|m^2|\overline{l^2}\gg 1$. In particular, the subsequent cosmological evolution may be substantially affected.

In our analysis, we had to introduce a cutoff parameter to regularize the logarithmic divergence of the power spectrum which arises from integral over the infinite number of the Kaluza-Klein modes, because of the lack of physical significance in $\langle \delta\phi^2 \rangle$ itself as mentioned above. Therefore, in order to verify our conclusion in a more rigorous and quantitative manner, it is necessary to evaluate quantities directly related to observables instead of $\langle \delta\phi^2 \rangle$. One such is the cosmological density perturbations in our brane-inflation model. This issue is left for future study.

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