# Resonant interaction of photons with gravitational waves

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The interaction of photons with a low-amplitude gravitational wave propagating in a flat space-time is studied by using an exact model of photon dynamics. The existence of nearly resonant interactions between the photons and the gravitational waves, which can take place over large distances, can lead to a strong photon acceleration. Such a resonant mechanism can eventually be useful to build consistent new models of gamma-ray emitters.

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## I. INTRODUCTION

The interaction of photons with the gravitational field is one of the major problems in the theory of gravitation [1]. This is due not only to the historic links with the tests of general relativity, but also to the present-day predictions for gravitational wave detection and to astronomical observations of gravitational lenses.

The most traditional problems associated with photon propagation in a gravitational field are the gravitational redshift of spectral lines emitted by a massive object and the deflection of light passing in the vicinity of a galaxy or a star. One could also add the time delay of radar echoes passing the sun.

In a recent work [2], it was shown that such gravitational effects can be described as particular examples of photon acceleration, a concept successfully developed in plasma physics and in optics [3], and recently considered in the astrophysical context [4,5]. Furthermore, a significant photon blueshift was also predicted as a result of photon interaction with gravitational waves, propagating in a plasma or in the absence of matter. However, the formulation was based on a crude model for the interaction.

Here we reexamine the interaction of photons with a low-amplitude gravitational wave in a flat space-time, by using an exact model of photon dynamics, in order to evaluate in more solid bases the expected amount of photon acceleration. At first sight, this problem is well known in the literature, but the physical consequences of the basic photon equations have not yet been completely understood. In particular, little attention has been given to the study of resonant interactions between photons and gravitational waves. It will be shown here that these interactions can lead to a very strong energy exchange, even for infinitesimal wave amplitudes, because they can act over very large distances and can eventually lead to significant frequency shifts. Such a resonant mechanism of photon acceleration can eventually be useful to build consistent new models of gamma-ray emitters [6].

The contents of this paper are the following. In Sec. II, we state the photon ray trajectories in Hamiltonian form, for photons moving in a generic gravitational field. In Sec. III, we examine the particular case of photon motion in a flat space-time perturbed by a low-amplitude gravitational wave. In Sec. IV, we consider the photon motion along the direction of propagation of the gravitational wave and illustrate our formalism with numerical examples. Finally, in Sec. V, we state our conclusions.

## II. PHOTON EQUATIONS

We know that the gravitational field varies (almost) always on a scale which is much larger than the photon wavelength. This means that the geometric optics approximation for photons is justified. In this work we will use the word "photon" in the classical sense, as equivalent to an electromagnetic wave packet. The electric field associated with an electromagnetic wave, moving in a space- and time-varying gravitational field, can be written as

$$E(x^j) = A \exp i\psi(x^j) \tag{1}$$

with $j=0,1,2,3$. Here $x^j$ represents the four-vector position, $A$ is the amplitude, and $\psi$ is the eikonal function. The wave four-vector $k^j$ can be defined by $k_j = \partial\psi/\partial x^j$, and the dispersion relation for electromagnetic waves is

$$k_i k^i = g^{ij} k_i k_j = 0, \tag{2}$$

where $g^{ij}$ are the components of the metric tensor. If we define the frequency of the electromagnetic wave $\omega$ as $\omega = -c\,\partial\psi/\partial x^0$, we can write Eq. (2) as

$$g^{00}\omega^2 - 2g^{0\alpha}\omega c k_\alpha + g^{\alpha\beta}k_\alpha k_\beta c^2 = 0, \tag{3}$$

where $(\alpha,\beta)=1,2,3$ and $g^{0\alpha}=g^{\alpha 0}$.

Notice that a different definition of frequency can be used, corresponding to the proper frequency $\omega_\tau$, defined as the derivative of the eikonal with respect to the proper time $\tau$: $\omega_\tau = -\partial\psi/\partial\tau = \omega/\sqrt{g_{00}}$. In contrast with $\omega$, which can be seen as a universal frequency, this proper frequency $\omega_\tau$ is the frequency measured by a local observer [7].

In the presence of a plasma there will be a lower cutoff frequency $\omega_p$, which is merely but the electron plasma frequency, such that propagation of transverse electromagnetic waves will not be allowed, for $\omega_\tau \leqslant \omega_p$. This means that the zero on the right-hand side of Eq. (3) will be replaced by $\omega_p^2$ [8]. The plasma frequency is, in general, a function of the four coordinates $x^i$. This means that we can write $\omega$ as a function of $k_\alpha$ and $x^i$:

$$\omega=\omega(k_\alpha,x^i). \tag{4}$$

The explicit expression for $\omega$ is obtained by solving Eq. (3). We get

$$\omega=y^{0\alpha}k_\alpha c+\sqrt{\nu^2+(y^{0\alpha}k_\alpha)^2c^2-y^{\alpha\beta}k_\alpha k_\beta} \tag{5}$$

with $y^{0\alpha}=g^{0\alpha}/g^{00}$, $\nu^2=\omega_p^2/g^{00}$, and $y^{\alpha\beta}=g^{\alpha\beta}/g^{00}$.

As a simple example, we can consider a locally Euclidean metric, such that $g^{\alpha\beta}=-\delta^{\alpha\beta}$ and $g^{00}=1/g_{00}$, where $\delta^{\alpha\beta}$ is the Krönecker delta symbol. Replacing this in the above photon dispersion relation, we obtain

$$\omega^2=\nu^2+k^2c^2g_{00}. \tag{6}$$

This states the analogy between a gravitational field and a dielectric medium. This simple model, with a space- and time-varying value for $g_{00}$, was explored in our previous paper [2].

We also know that the photon trajectories verify the variational principle $\delta\int k_i dx^i=0$, from which we get the photon equations of motion:

$$\frac{dx^\alpha}{dt}=\frac{\partial\omega}{\partial k_\alpha},\quad \frac{dk_\alpha}{dt}=-\frac{\partial\omega}{\partial x^\alpha}, \tag{7}$$

where $\alpha=1,2,3$ and $\omega=\omega(k_\alpha,x^\alpha,t)$ is determined by Eq. (5).

### III. PERTURBED FLAT SPACE-TIME

Let us assume a flat space-time perturbed by a moving gravitational field perturbation. Two possible examples of perturbation can be imagined: (a) a gravitational wave propagating far away from the emitter; (b) the gravitational field associated with an ultraintense laser pulse [9]. This simple but physically meaningful model is also compatible with the existence of a very low density plasma background. For instance, a typical plasma frequency for the intergalactic gas is $\omega_p\simeq 200\ \mathrm{s}^{-1}$ [1]. However, its influence on photon trajectories such that $\omega\gg\omega_p$ is negligible and allows us to take $\omega_p=\nu=0$ in the following. The metric tensor associated with a perturbed flat space-time is determined by

$$g^{ij}=\eta^{ij}+h^{ij} \tag{8}$$

with $|h^{ij}|\ll 1$. Here $\eta^{ij}$ are the components of the Minkowski metric tensor: $\eta^{\alpha\beta}=-\delta^{\alpha\beta}$, $\eta^{00}=1$. In this case, we have

$$y^{0\alpha}=\frac{\eta^{0\alpha}+h^{0\alpha}}{\eta^{00}+h^{00}}\simeq h^{0\alpha},$$

$$y^{\alpha\beta}=\frac{\eta^{\alpha\beta}+h^{\alpha\beta}}{\eta^{00}+h^{00}}\simeq-\delta^{\alpha\beta}+(h^{00}\delta^{\alpha\beta}+h^{\alpha\beta}). \tag{9}$$

Substituting $\omega_p=0$ in Eq. (5), we get

$$\frac{\omega}{c}=h^{0\alpha}k_\alpha+\sqrt{k^2(1-h^{00})+(h^{0\alpha}k_\alpha)^2-h^{\alpha\beta}k_\alpha k_\beta}, \tag{10}$$

where we have used $k^2=\delta^{\alpha\beta}k_\alpha k_\beta$. To the lowest order in $h^{ij}$, we can then write

$$\omega\simeq kc[1+f(h^{ij})] \tag{11}$$

with

$$f(h^{ij})=h^{0\alpha}\frac{k_\alpha}{k}-\frac{1}{2}\left(h^{00}+h^{\alpha\beta}\frac{k_\alpha k_\beta}{k^2}\right). \tag{12}$$

This can also be written as

$$f(h^{ij})=\Omega_{ij}h^{ij} \tag{13}$$

with

$$\Omega_{ij}=\delta_{i0}\delta_j^\alpha\frac{k_\alpha}{k}-\frac{1}{2}\left(\delta_{i0}\delta_{j0}+\delta_i^\alpha\delta_j^\beta\frac{k_\alpha k_\beta}{k^2}\right). \tag{14}$$

Let the metric perturbation $h^{ij}$ be associated with a weak gravitational wave. This can be described by the following metric solution [10]:

$$ds^2=cdt^2-dx^2-(1-a)dy^2-(1+a)dz^2+2b\,dy\,dz, \tag{15}$$

where we define

$$a=A\sin(k_0x^0+k_1x^1+\phi)=A\sin(qx-\Omega t+\phi),$$
$$b=B\sin(qx-\Omega t+\phi'). \tag{16}$$

Here $A$ and $B$ are the wave amplitudes, and $\phi$ and $\phi'$ are constant phases, corresponding to the two orthogonal polarization states for gravitational waves propagating along the axis $x=x^1$, with wave number $q=k_1$ and frequency $\Omega=qc$. From here we get

$$h_{22}=-h_{33}=a,\quad h_{23}=b, \tag{17}$$

and all the other components of $h_{ij}$ are equal to zero.

In order to be specific, let us focus on a well-defined polarization state, for instance $A\neq 0$ and $B=0$. Equation (12) will then be reduced to

$$f(h^{ij})=f(a)=\frac{a}{2}\left[\left(\frac{k_y}{k}\right)^2-\left(\frac{k_z}{k}\right)^2\right], \tag{18}$$

where $k_y=k_2$ and $k_z=k_3$.

The photon-dispersion relation becomes $\omega=kc[1+f(a)]$ and the canonical equations of motion can now be written as

$$\frac{dx}{dt}=c\frac{k_x}{k},\quad \frac{dy}{dt}=c(1+a)\frac{k_y}{k},$$

$$\frac{dz}{dt} = c(1-a)\frac{k_z}{k}, \tag{19}$$

and

$$\frac{dk_x}{dt} = -\frac{kc}{2}\left[\left(\frac{k_y}{k}\right)^2 - \left(\frac{k_z}{k}\right)^2\right]\frac{\partial a}{\partial x}, \tag{20}$$

$$\frac{dk_y}{dt} = -\frac{\partial \omega}{\partial y} = 0, \quad \frac{dk_z}{dt} = -\frac{\partial \omega}{\partial y} = 0. \tag{21}$$

We can conclude from here that the photon momentum components perpendicular to the direction of propagation of the gravitational wave, $k_y$ and $k_z$, are constants of motion. We can therefore consider the photon motion in the plane $xy$ and assume $k_z = 0$. The motion along the $y$ direction is trivially determined by

$$k_y = \text{const}, \quad y(t) = ck_y \int^t \frac{1+a(t')}{k(t')} dt'. \tag{22}$$

The problem is then reduced to solving the parallel photon motion:

$$\frac{dx}{dt} = c\frac{k_x}{k}, \quad \frac{dk_x}{dt} = -\frac{1}{2}\left(\frac{k_y}{k}\right)^2 \frac{\partial a}{\partial x}. \tag{23}$$

These two equations have to be complemented by

$$\frac{d\omega}{dt} = \frac{\partial \omega}{\partial t} = +\frac{1}{2}\left(\frac{k_y}{k}\right)^2 \frac{\partial a}{\partial t}. \tag{24}$$

Notice that photon acceleration can only take place for oblique propagation, $k_y \neq 0$, in agreement with our previous work [2]. It should also be noticed, using Eq. (16), that

$$\frac{\partial a}{\partial t} = -\frac{\Omega}{q}\frac{\partial a}{\partial x}. \tag{25}$$

Comparing Eqs. (23) and (24), we conclude that

$$\frac{d\omega}{dt} = \frac{\Omega}{q}\frac{dk_x}{dt} = c\frac{dk_x}{dt}. \tag{26}$$

Or, equivalently,

$$I = \omega - ck_x = \text{const}. \tag{27}$$

The existence of this constant of motion is well known in the theory of photon acceleration [3].

## IV. PARALLEL PHOTON MOTION

In order to discuss the solution of the parallel photon motion, as described by Eq. (23), it is convenient to use the relative parallel momentum

$$p = \frac{k_x}{k} = \frac{k_x}{\sqrt{k_x^2 + k_y^2}}. \tag{28}$$

From here, we can also derive

$$k_x = k_y \frac{p}{\sqrt{1-p^2}}, \quad k^2 = k_y^2 \frac{1}{1-p^2}. \tag{29}$$

Using Eqs. (16) and (23), with $\phi = -\pi/2$, we can then write the equations describing the parallel motion in the form

$$\frac{dx}{dt} = cp, \tag{30}$$

$$\frac{dp}{dt} = -\frac{c}{2}Aq(1-p^2)^2 \sin(qx - \Omega t). \tag{31}$$

It is also useful to write the photon-dispersion relation in terms of the relative photon momentum $p$, as

$$\omega = kc[1+f(a)] = kc\left[1 + \frac{a}{2}(1-p^2)\right]. \tag{32}$$

Replacing this in the expression for the photon invariant $I$, we obtain

$$I = \omega - kcp = \omega \frac{1 - p + a(1-p^2)/2}{1 + a(1-p^2)/2}. \tag{33}$$

It is also convenient to introduce new space and time variables, such that

$$\eta = qx - \Omega t, \quad \tau = \frac{A}{2}ct. \tag{34}$$

Replacing $\eta$ and $\tau$ in the parallel photon equations of motion (31), we finally get

$$\begin{aligned} d\eta/d\tau &= \epsilon(p-1), \\ dp/d\tau &= -(1-p^2)^2 \sin \eta \end{aligned} \tag{35}$$

with $\epsilon = 2q/A$.

These equations resemble those of an asymmetric pendulum [11]. Their analytical solutions are not obvious, but they can be easily numerically integrated, which is suitable for our purposes here. Given an initial value $p(0)$ of the relative parallel momentum of the photon, we can calculate the future values $p(\tau)$. Using these results in the expression for the photon invariant $I$, as given by Eq. (33), we can also obtain the frequency of the photon $\omega(\tau)$. For gravitational waves with a very small amplitude, $A \ll 1$, we can get from Eq. (33)

$$\omega(\tau) = \omega(0)\frac{1-p(0)}{1-p(\tau)}. \tag{36}$$

These frequency shifts are not simply a mathematical result of our choice of the space-time coordinates. They can take a physical reality for any local observer (or interacting particle, such as an atom or a dust grain) which will actually see a local photon frequency $\omega_\tau$ nearly equal to $\omega(\tau)$, because $g_{00} \simeq 1$.

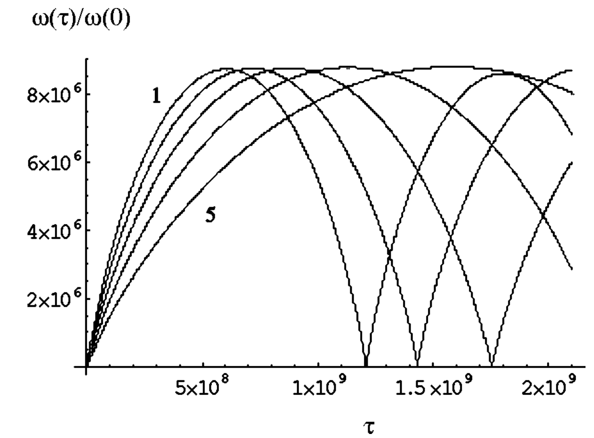


FIG. 1. Photon frequency shift obtained by numerical integration of Eqs. (45) and (46), for $\epsilon = 0.5$ and $p(0) = 0.99$.

For trajectories such that $p(\tau)$ increases with time and approaches 1, this expression shows that the photon frequency can be significantly upshifted. Numerical examples, obtained by solving Eqs. (35) and (36), show that frequency upshifts of nearly $10^7$ can be obtained for $\epsilon = 0.5$ (see Fig. 1). The maximum frequency shift is weakly dependent on the initial parallel momentum $p(0)$. However, it is very sensitive to the value of the parameter $\epsilon$, as shown in Fig. 2.

These examples show that an infrared photon can be accelerated up to the gamma-ray energy range of the electromagnetic spectrum by a low-amplitude gravitational wave. In order to see their relevance in the astrophysical context, we notice that, for gravitational waves with a typical frequency of $\Omega = 10^4\ \mathrm{s}^{-1}$, which corresponds to a frequency of nearly 1 kHz, we obtain for the gravitational wave amplitude the value of $A = 2q/\epsilon = 4\Omega/c \simeq 10^{-4}$, over a short distance of $d = 2\tau/A \sim 1$ a.u. The same photon frequency upshifts could also be attained with lower amplitudes, by assuming lower gravitational wave frequencies. For instance, for a gravitational wave frequency of $\Omega = 10^{-4}\ \mathrm{s}^{-1}$, we would only need an amplitude of $A \simeq 10^{-12}$, over a scale of a few hundred parsecs. This shows that such an effect could occur away (but not too far away) from a radiating object, and that it could give credibility to a model for gamma-ray bursters based on gravitational wave emitters.

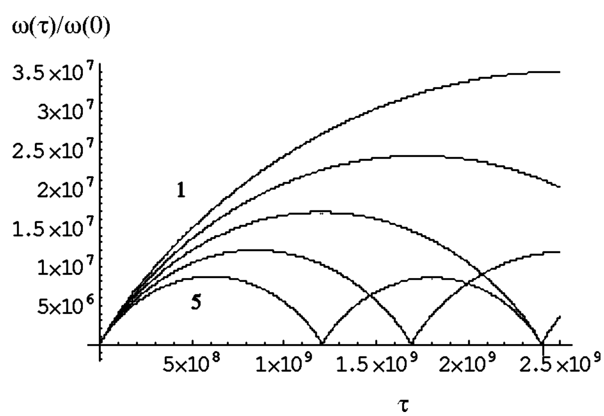


FIG. 2. Photon frequency shift obtained by numerical integration of Eqs. (45) and (46) for $p(0) = 0.987$ and $\epsilon = 0.51 - 0.01n$ with $n = 1 - 5$.

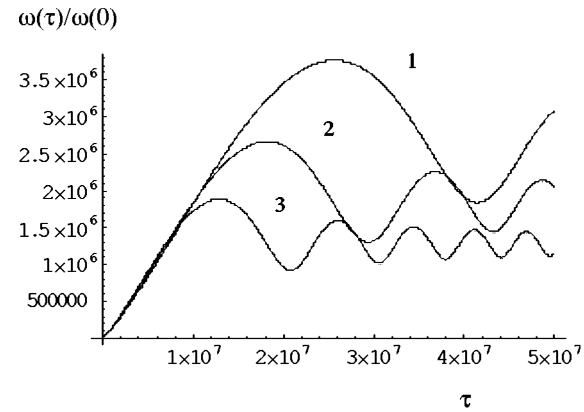


FIG. 3. Photon frequency shift obtained by numerical integration of Eqs. (45) and (46) for $\epsilon = 0.5\sqrt{1 + n\alpha^2\tau^2}$ with $\alpha = 3 \times 10^{-5}$ and $n = 1,2,4$.

Notice that different frequency upshifts, and also downshift, could be obtained by varying the initial photon parameters, $p(0)$ and $\eta(0)$, thus leading to a very broad spectrum. Notice also that the parallel photon momentum $p$, and consequently its frequency, varies periodically in the field of the gravitational wave, which means that the effect of photon acceleration is reversible. The same occurs in the laboratory for probe photons moving in the wakefield of a strong laser pulse. This means that photon acceleration can only become irreversible if the interaction of the photons with the gravitational wave is interrupted by some cause, related to scattering, to the finite width of the gravitational wave packet, or the fading away of its amplitude. This is illustrated in Fig. 3, where a $1/R$ decrease in the gravitational wave amplitude is considered. These questions of irreversibility are well understood in laboratory plasmas.

## V. CONCLUSIONS

Using a ray-tracing formulation of the photon dynamics in a gravitational field, formally analogous to that used for photons moving in nonstationary dielectric media [3], we have shown that photons can strongly interact in a flat space-time with low-amplitude gravitational waves. Such an interaction can take a nearly resonant character, because both the photons and the gravitational waves move with the same velocity, which can lead to extremely high-frequency upshifts over distances that can be considered short on the astronomical scale.

The numerical examples, taken from our simple but exact model, suggest that it is possible to accelerate infrared photons up to the gamma-ray domain with low-amplitude gravitational waves. Such an effect can eventually take place in empty regions not far away from gravitational wave emitters, and could provide a natural and convincing explanation for the observed gamma-ray bursts [6]. Three distinct aspects of our model are very interesting for that purpose: (i) the interactions take place in empty space, far away from any astronomical object (even if one can imagine that some emitter of gravitational waves should exist); (ii) the interactions quite naturally lead to a very large frequency spectrum, because the final frequency upshift strongly depends on the phase with respect to the gravitational wave, and to a lesser degree on the angle of photon propagation with respect to the direction of the gravitational wave; (iii) the photons are deflected by the acceleration process from their initial direction, thus losing the information of their source when observed at large distances. An other possible application of our results could be the production of ultrahigh energy photons, because the frequency shifts are independent of the initial photon frequency, due to the nondispersive character of the flat spacetime.

The existence of such photon acceleration processes shows that a significant transfer of energy can take place between the gravitational wave and the nearly resonant photons. But an estimate of the value of such a transfer cannot be based on the single photon dynamics studied used, and can only be elucidated with a global statistical description of the photon gas. This will be left to a future work.

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