Renormalization in self-consistent approximation schemes at finite temperature: Theory

Hendrik van Hees and Jörn Knoll
GSI Darmstadt, Planckstrasse 1, D-64291 Darmstadt, Germany
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Within finite temperature field theory, we show that truncated nonperturbative self-consistent Dyson resummation schemes can be renormalized with local counterterms defined at the vacuum level. The requirements are that the underlying theory is renormalizable and that the self-consistent scheme follows Baym's Φ-derivable concept. The scheme generates both the renormalized self-consistent equations of motion and the closed equations for the infinite set of counterterms. At the same time the corresponding two-particle irreducible generating functional and the thermodynamical potential can be renormalized, consistent with the equations of motion. This guarantees that the standard Φ-derivable properties such as thermodynamic consistency and exact conservation laws hold also for the renormalized approximation schemes. The proof uses the techniques of Bogoliubov-Parasiuk-Hepp-Zimmermann renormalization to cope with the explicit and the hidden overlapping vacuum divergences.

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I. INTRODUCTION

In recent years the question of how to appropriately treat particles in a hot and dense medium has gained growing interest in many areas of physics ranging from plasma physics and condensed matter physics to nuclear and particle physics. Within a Green's function formalism a consistent treatment of such phenomena frequently leads one to consider dressed propagators, which follow from nonperturbative Dyson resummation schemes, rather than perturbative ones, in particular, if damping width effects play a significant role.

As early as the 1960s, based on a functional formulation of Luttinger and Ward [1] and Lee and Yang [2], Baym and Kadanoff [3] considered a class of self-consistent Dyson approximations. Baym reformulated this in terms of a variational principle, defining the so-called Φ-derivable approximations [4]. Since in principle the truncation of the diagrammatic series of the functional Φ can be at an arbitrary level it gives rise to a variety of approximations including Hartree and Hartree-Fock schemes as the simplest schemes. The main virtue of this concept is that the resulting equations of motion are conserving and the corresponding equilibrium limit is thermodynamically consistent. This functional treatment constitutes the basis for the two-particle irreducible (2PI) diagram technique, where the functional Φ generates the driving terms for the equations of motion, like the self-energy. Later the concept was extended to the relativistic case and formulated within the path integral approach by Cornwall, Jackiw, and Tomboulis [5]. There is no formal problem in extending the formalism to the Schwinger-Keldysh real-time path method [6,7] applicable to the general case of nonequilibrium many-body theory.

Despite such early conceptual formulations most applications of self-consistent approximations were pursued on the Hartree or Hartree-Fock level, sometimes supplemented by random phase approximation (RPA) resummations (see, e.g., [8,9]) or perturbative estimates of higher order corrections. Thus, essentially mean field corrections to the self-energies were considered. Genuine two-point or even multipoint contributions to the self-consistent self-energy, which give rise to a finite damping width, imply a new level of complexity. Various new conceptual problems, such as leaving the quasiparticle picture or the issue of renormalization, come in with considerable complications for the numerical solutions of such problems. In the pioneering work of Bielajew and Serot [10] for the first time the renormalization of self-consistent two-point self-energy loops was investigated at zero temperature but finite matter density.

In recent years with the special interest in dense hadronic matter problems the Φ-derivable schemes with higher order self-energy terms were used to derive transport equations [11–14] from the corresponding Kadanoff-Baym equations [15] for the consistent and also conserving [16] transport treatment of particles with finite spectral width beyond the quasiparticle approximation. Finite mass width effects on vector mesons were also investigated within a self-consistent Dyson resummation scheme [17]. In most of these cases, however, the question of renormalization was circumvented by taking into account the imaginary part of the self-energy only, while the real part was neglected, or cutoff recipes—mostly symmetry violating—were employed, or the counterterms were even chosen to be temperature dependent. Yet, especially in the study of phase transitions, e.g., within chiral hadronic models of QCD, or for nonperturbative corrections of hard thermal loop approaches to QCD, e.g., within a Φ-derivable scheme [18,19], it is important to consistently take into account both real and imaginary parts of the self-energies.

Therefore in this paper we address the more formal question of renormalizability of such nonperturbative approximations. We essentially concentrate on the thermodynamic equilibrium case and show how to obtain finite self-consistent dynamical quantities like the in-medium equations of motion and the self-energy of the particles and thermodynamic quantities like the pressure and the entropy. For definiteness and clarity of the presentation we use the ϕ^4 theory as the simplest example to study the related questions. The results and techniques can easily be transferred to other theories.

The paper is organized as follows. In Sec. II we briefly summarize Baym's Φ functional using the combined real-

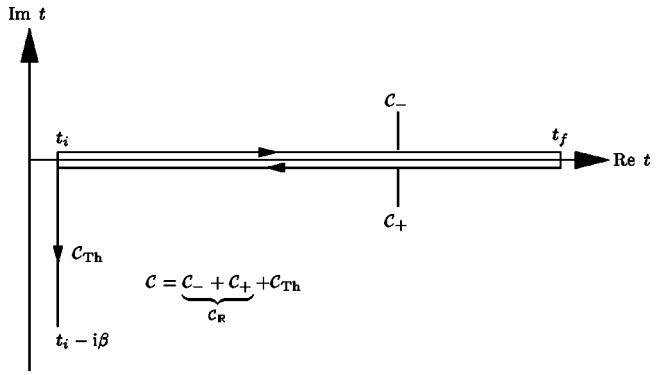


FIG. 1. The Schwinger-Keldysh real-time contour modified for the application to thermal equilibrium of quantum field theory.

and imaginary-time contour appropriate for thermal equilibrium within the path integral formalism [5].

In Sec. III we derive the general formalism for the renormalization of the self-consistent self-energy at finite temperature and the in-matter generating functional Γ. With the help of Weinberg's convergence theorem [20] and the Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) formalism of renormalization theory [21,22] we show that, in close analogy to perturbative renormalization (see, e.g., [23–25]), *any* Φ*-derivable self-consistent approximation scheme can be rendered finite by subtracting pure vacuum counterterms* given by closed recursive equations. Indeed, the main complication arises from the fact that the self-consistent propagator is involved in divergent loops, which gives rise to ''hidden'' divergences that have to be resolved. This leads to a Bethe-Salpeter equation for the divergent vacuum pieces with a kernel compatible with the functional Φ, which needs to be renormalized. The renormalized equations of motion for the self-consistent propagator are shown to be consistent with the renormalized 2PI generating functional, which proves the consistency of counterterms at both levels. The diagrammatical interpretation shows that in strict analogy to perturbative renormalization of thermal quantum field theory this procedure can be interpreted as renormalization of the wave functions, the mass, and the coupling constants *in the vacuum*. We also give a closed expression for the renormalized self-consistent thermodynamical potential.

Numerical solutions for the renormalized self-consistent Dyson equations beyond the standard Hartree approximation up to the self-consistent sunset-diagram level could be achieved; the results are discussed in the second paper of this series [26].

II. Φ-DERIVABLE APPROXIMATIONS

In the case of thermal equilibrium the real- and imaginary-time formalism can be combined by extending the Schwinger-Keldysh contour $\mathcal{C}$, running from t_i to t_f and back to t_i, by appending a vertical part $\mathcal{C}_{\mathrm{Th}}$ running from t_i to $-i\beta$ (see Fig. 1). One uses the fact that the factor $\exp(-\beta\mathbf{H})$ in the canonical density operator can be formally treated as a time evolution in the imaginary-time direction. The functional integral for bosonic fields has to be taken over all fields fulfilling the *periodic boundary condition* $\phi(t_i - i\beta) = \phi(t_i)$, which leads to the Kubo-Martin-Schwinger (KMS) condition for the Green's functions [27]. Since the equilibrium state is invariant in time one can take $t_i \to -\infty$ and $t_f \to +\infty$, which is convenient when formulating the theory in energy-momentum space via a Fourier transformation.

In addition to the usually introduced *one-point auxiliary external source* a *two-point auxiliary external source* is also included. Variations of the latter generate contour-ordered expectation values of the form $\langle \mathcal{T}_{\mathcal{C}} \boldsymbol{\phi}(x_1)\boldsymbol{\phi}(x_2)\rangle$ where $\mathcal{C}$ denotes the extended Schwinger-Keldysh time contour, and $\mathcal{T}_{\mathcal{C}}$ stands for the ordering of the operators due to the ordering of the time arguments along this contour.

The corresponding generating functional is defined within the path integral formalism of quantum field theory as

$$Z[J,B] = N \int D\phi \exp\left[iS[\phi] + i\int_{\mathcal{C}} d(1) J_1 \phi_1 + \frac{i}{2}\int_{\mathcal{C}} d(12) B_{12}\phi_1\phi_2 \right], \tag{1}$$

where N is a normalization constant chosen below. Here and in the following the shorthand notation

$$\int_{\mathcal{C}} d(12\ldots n) f_{12\ldots n} = \int_{\mathcal{C}\times\mathbb{R}^{d-1}} d^d x_1 \cdots d^d x_n f(x_1,\ldots,x_n) \tag{2}$$

for integrations in d-dimensional space is used (in the sense of dimensional regularization[1]). The time integration has to be performed along the time contour introduced above. It is clear that the action functional S also has to be defined as the $\mathcal{C}\times\mathbb{R}^{d-1}$ integral of the Lagrangian.

The generating functional W for *connected diagrams reads*

$$W[J,B] = -i \ln(NZ[J,B]). \tag{3}$$

The *mean field* and the *connected Green's functions* are defined by

$$\varphi_1 = \frac{\delta W}{\delta J_1}, \quad G_{12} = -\frac{\delta^2 W}{\delta J_1 \delta J_2} \Rightarrow \frac{\delta W}{\delta B_{12}} = \frac{1}{2}(\varphi_1\varphi_2 + iG_{12}). \tag{4}$$

The last formula immediately follows from the definition of the partition sum (1) and (3) using the Feynman-Kac formula

$$\langle \mathcal{T}_{\mathcal{C}} \boldsymbol{\phi}(x_1)\cdots\boldsymbol{\phi}(x_n)\rangle = \int D\phi\, \phi(x_1)\cdots\phi(x_n)\exp(iS[\phi]). \tag{5}$$

Since the real-time part of this contour is closed by itself it can be shown that the functional (1) factorizes into the real-

[1]We use this notation for convenient regularization only, in order to write down sensible nonrenormalized functionals.

time part and the imaginary-time part. Thus the Feynman rules for the calculation of connected Green's functions apply separately to the real and the imaginary parts of the contour [28], since the functional (3) splits into a sum of the contributions from the vertical and the real-time part of the contour, respectively:

$$Z[J,B]=Z[J,B]Z_{\mathrm{Th}}[J,B],$$

$$W[J,B]=W[J,B]+W_{\mathrm{Th}}[J,B]-i\ln N. \tag{6}$$

Within the real-time part the effect of the heat bath is completely taken into account by means of the analytical properties of the Green's function. It is uniquely determined by the KMS condition which itself is a consequence of the above mentioned periodic boundary conditions for bosons within the path integral (see the Appendix for details of the analytic properties of Green's functions and self-energies).

By a functional Legendre transformation on φ and G one obtains the *effective quantum action*:

$$\Gamma[\varphi,G]=W[J,B]-\int_{\mathcal{C}} d(1)\,\varphi_1 J_1 - \frac{1}{2}\int_{\mathcal{C}} d(12)(\varphi_1\varphi_2+iG_{12})B_{12}-i\ln N. \tag{7}$$

Now, as is well known from the usual functional formalism of quantum field theory, a formal saddle point expansion of the effective quantum action is an expansion in orders of $\hbar$ around the classical solution, where G is considered as an independent quantity, which gives

$$\Gamma[\varphi,G]=S[\varphi]+\frac{i}{2}\mathrm{Tr}\ln(M^2G^{-1}) + \frac{i}{2}\int_{\mathcal{C}} d(12)D_{12}^{-1}(G_{12}-D_{12})+\Phi[\varphi,G]. \tag{8}$$

Herein the free propagator in the presence of a mean field is given by

$$D_{12}^{-1}=\frac{\delta^2 S[\varphi]}{\delta\varphi_1\delta\varphi_2}. \tag{9}$$

The arbitrary constant M^2 accounts for the overall normalization and cancels out for any physical quantity as we shall see below.

In the case of an ideal gas it is sufficient to subtract the pure vacuum part to render this functional finite, which leads to the well known result. At $T=0$ this subtraction corresponds to the renormalization of the total ground-state energy to zero.

As we shall discuss below in the case of interacting particles this description is not sufficient to render the effective action and thus the pressure finite, since we need additional subtractions of vacuum subdivergences to renormalize it.

In the above sense the functional Φ in Eq. (8) contains the parts of order $O(\hbar^2)$ and higher. Since simple power counting shows the $\hbar$ order of diagrams to be identical with the number of loops, Φ as a functional of G consists of all closed diagrams with at least two loops. The lines within the diagrams stand for dressed Green's functions G while the vertices are the bare vertices of the classical action with the presence of the background field φ, which can be immediately read off from $S_I[\varphi+\phi']$ around $\phi'=0$ beginning at order ϕ'^3.

The equations of motion are now given by the fact that we wish to study the theory with vanishing auxiliary sources j and B. From Eqs. (4) and (7) we obtain

$$\frac{\delta\Gamma}{\delta\varphi_1}=-j_1\overset{!}{=}0, \quad \frac{\delta\Gamma}{\delta G_{12}}=-\frac{i}{2}B_{12}\overset{!}{=}0. \tag{10}$$

Using Eq. (8) the equations of motion read

$$\frac{\delta S}{\delta\varphi_1}=-\frac{i}{2}\int_{\mathcal{C}} d(1'2')\frac{\delta D^{-1}_{1'2'}}{\delta\varphi_1}G_{1'2'}-\frac{\delta\Phi}{\delta\varphi_1},$$

$$\Sigma_{12}:=D_{12}^{-1}-G_{12}^{-1}=2i\frac{\delta\Phi}{\delta G_{12}}. \tag{11}$$

The first line is of the form of a Klein-Gordon equation with the quantum corrections to the classical theory on the right hand side. The second equation is the *Dyson equation* and shows that the variation of the Φ functional with respect to G is the self-energy defined with respect to the classical field dependent propagator (9). This shows that the Φ functional must be *two-particle irreducible*. No propagator line must contain a self-energy insertion. In other words the closed diagrams representing contributions to Φ must not split into disconnected pieces when cutting two lines. Diagrammatically the derivative of a functional with respect to G corresponds to opening one line of the diagrams representing it. In that sense Φ is the generating functional for *skeleton diagrams for the self-energy* where the lines represent fully dressed propagators. Thus the functional formalism avoids double counting in a natural way by omitting all nonskeleton diagrams from the Dyson-resummed equations of motion. Altogether the Φ-functional formalism provides a closed system of exact equations of motion for the full two-point function and the full mean field. Solving these equations would be equivalent to finding the full propagator of the quantum field theory, which of course is impossible in practice.

One obtains approximations by truncating the series for Φ at a certain vertex or loop order (which corresponds to the respective order in the coupling λ or $\hbar$), while preserving the forms (10) and (11) of the self-consistent equations of motion. Approximations of this kind respect the conservation laws for the expectation values of Noether currents for symmetries that are linearly operating on the field operators (including space-time symmetries and the corresponding conserved quantities such as energy, momentum, and angular momentum) [4,11].

In the case of thermal equilibrium, setting the mean field and the propagator to the solution of the self-consistent equations the effective action gives the grand canonical potential $\Omega=-T\Gamma[\varphi,G]_{J,K=0}=-T\ln Z(\beta)$ [27]. Since the real-time

part of the contour in Fig. 1 is closed it vanishes for the solution of the equations of motion. A short summary of the analytic continuation from the real- to the imaginary-time formalism is given in the Appendix.

Thus the formalism leads to a well defined treatment for bulk thermodynamical quantities of the system (like energy, pressure, entropy, etc.).

All these quantities can be calculated either with real-time Green's functions or with the corresponding imaginary-time functions, because as summarized in the Appendix real- and imaginary-time propagators are connected by the analytic properties of the Green's functions originating from the KMS condition. For our purpose the real-time formalism is preferred, because of its simplicity with respect to the analytic structure of Green's functions, which easily permits one to deal with the mixture of finite temperature and vacuum pieces occurring in the subtraction scheme. This also avoids the necessity of performing an analytic continuation from imaginary-time to real-time Green's functions, which is complicated to obtain for numerical results.

In order to exemplify the method we apply the formalism to ϕ^4 theory with the Lagrangian

$$\mathscr{L}=\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)-\frac{m^2}{2}\phi^2-\frac{\lambda}{4!}\phi^4. \tag{12}$$

III. RENORMALIZABILITY OF Φ-DERIVABLE APPROXIMATIONS

In this section we show that, in close analogy to the renormalization of perturbative diagrams, any **Φ**-derivable self-consistent approximation can also be renormalized with the help of local temperature-independent counterterms.

The proof uses the same line of argument as in the perturbative case: The reason is that the renormalization theory rests completely on Weinberg's power counting theorem [20], which is formulated for a general class of Green's functions with a given asymptotic behavior. It does not depend on the special form of the propagators.

The first step is a simple topological argument leading to the superficial degree of divergence for a given diagram γ, which for the ϕ^4 theory simply follows from the number E of external lines [29]:

$$\delta(\gamma)=4-E. \tag{13}$$

Due to field reflection symmetry only diagrams with an even number of external lines are different from 0; the only divergent functions are those represented by diagrams where the number of external legs is 0 (i.e., contributions to the total mean energy and the thermodynamical potential), 2 [self-energy (Green's function)], and 4 (four-point vertex functions).

The second step is an expansion of the regularized unrenormalized self-consistent self-energy around the self-consistent vacuum propagator, which shows that the asymptotic behavior of the diagrams and subdiagrams is ruled by their pure vacuum parts.

Then an equation of motion for the temperature-dependent ''infinite part'' of the regularized self-energy is derived and it is shown that it can be renormalized by a temperature-independent subtraction procedure.

A. BPHZ scheme for the vacuum

We first apply the BPHZ renormalization theorem [21,22,30] for the vacuum. The only difference from the perturbative case is that we apply it to diagrams with self-consistent propagator lines. This is justified since Weinberg's power counting theorem is independent of the special form of the propagators but only needs their asymptotic behavior stated above.

We summarize the BPHZ scheme as follows. A subdiagram γ of a diagram Γ is defined as any set of lines and vertices contained in Γ which itself builds a proper vertex diagram: $\gamma\subseteq\Gamma$. A subdiagram γ is called a *renormalization part* if its superficial degree of divergence (or its dimension) is greater than or equal to 0. In our case this means it has at most four external legs. Two subdiagrams γ_1 and γ_2 are called *nested* $\gamma_1\subseteq\gamma_2$ if γ_1 is a subdiagram of γ_2. If they have no line or vertex in common, $\gamma_1\cap\gamma_2=\varnothing$, they are called *disjoined*. If they are neither nested nor disjoined they are called *overlapping*: $\gamma_1\bigcirc\gamma_2$.

For any diagram Γ we denote the integrand following from the Feynman rules by I_Γ. For a set of pairwise disjoined subdiagrams $\gamma_1,\gamma_2,\ldots,\gamma_n$ we write the integrand in terms of the integrands of the subdiagrams I_{γ_j} and the remainder of the integrand denoted by $I_{\Gamma\setminus\{\gamma_1,\ldots,\gamma_n\}}$, usually called the *reduced diagram*:

$$I_\Gamma=I_{\Gamma\setminus\{\gamma_1,\ldots,\gamma_n\}}\prod_{j=1}^{n}I_{\gamma_j}. \tag{14}$$

The original scheme by Bogoliubov and Parasiuk [21] defines the integrand R_Γ of the renormalized diagram recursively. If a diagram does not contain any renormalization part but is itself divergent it is called *primitively* divergent. In that case the renormalized integrand is defined by $R_\Gamma=(1-t_\Gamma)I_\Gamma$. Here t_Γ is the operator of the Taylor expansion with respect to the external momenta around 0 up to the order of the dimension $\delta(\gamma)$ of the divergent diagram, which is in our case $4-E$:

$$t_\gamma I_\gamma(p_1,\ldots,p_k):=\begin{cases}\displaystyle\sum_{j=0}^{\delta(\gamma)}\frac{1}{j!}\sum_{\substack{\mu_1,\ldots,\mu_k\geqslant 0\\ \mu_1+\ldots+\mu_k=j}}\left.\frac{\partial^j I_\gamma(p_1,\ldots,p_k)}{\partial p_1^{\mu_1}\cdots\partial p_k^{\mu_k}}\right|_{p_1=\ldots=p_k=0}p_1^{\mu_1}\cdots p_k^{\mu_k},\\ 0\quad\text{for}\quad\delta(\gamma)<0.\end{cases} \tag{15}$$

If the diagram is convergent the integrand is unchanged under renormalization.

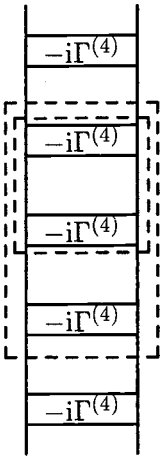
If the diagram is not only primitively divergent but contains divergent subdiagrams the integrand for the diagram with all subdivergences subtracted is called $\overline{R}_\Gamma$ and the renormalized integrand is defined by

$$R_\Gamma = \begin{cases} \overline{R}_\Gamma & \text{if } \delta(\Gamma) < 0, \\ (1 - t_\Gamma)\overline{R}_\Gamma & \text{if } \delta(\Gamma) \geqslant 0. \end{cases} \tag{16}$$

From Weinberg's power counting theorem it follows that after this recursive procedure the integral over the internal momenta of R_Γ is finite. The definition of the *counterterms* by the Taylor operator t_γ for any renormalization part γ of the diagram shows that these are polynomials in the external momenta to the order $\delta(\gamma)$ and thus they can be interpreted as counterterms to the corresponding wave function normalization factors, masses, and coupling constants in the original Lagrangian.

Zimmermann solved Bogoliubov's and Parasiuk's recursion with his *forest* formula. A forest is defined as any set of subdiagrams (including the empty set and the whole diagram itself) that are *pairwise nonoverlapping*. One can depict these sets by drawing boxes around the subdiagrams; in a forest these boxes are not allowed to overlap but they can be nested. A forest is restricted if each of its boxes contains only renormalization parts. To each restricted forest $\mathfrak{F}$ one associates again an integrand, namely,

$$\Omega_{\mathfrak{F}} = \tilde{\prod}_{\gamma \in \mathfrak{F}} (-t_\gamma) I_\Gamma = \tag{17}$$



The diagram above shows an example for a typical ladder diagram, which we shall consider in the following section. For this case $\delta(\gamma) = 0$ for all subdiagrams and the diagram itself such that for each box only the subdiagram value at vanishing external momenta is to be subtracted. The tilde over the product sign in Eq. (17) stands for the fact that in the case of nested diagrams within the forest one has to apply the Taylor operators from the innermost to the outermost diagrams while for disjoined subdiagrams the expressions are naturally independent of the order of Taylor operators, since then

$$I_\Gamma = I_{\Gamma \setminus \{\gamma_1, \ldots, \gamma_n\}} \prod_{k=1}^{n} I_{\gamma_k}. \tag{18}$$

The forest formula then says that the integrand of the renormalized diagram is given by the sum over all restricted forests:

$$R_\Gamma = \sum_{\mathfrak{F} \in \mathcal{F}_R(\Gamma)} \Omega_{\mathfrak{F}}. \tag{19}$$

It is understood that the empty set stands for the diagram itself, i.e., without any box around a subdiagram.

The BPHZ scheme described chooses the renormalization point for the divergent diagrams at external momenta set to 0. It is clear that by another finite renormalization we can switch to any renormalization scheme appropriate for the application considered. In our case of ϕ^4 theory we chose the on-shell scheme. We have to define the coupling constant, the mass, and the wave function normalization. This can be formulated in terms of the proper self-energy and the proper four-point vertex (the three-point vertex can be set identically to 0 because of symmetry under the field reflection $\phi \to -\phi$ without destroying the renormalizability of the theory, so that we do not have to consider terms linear or cubic in the fields within the Lagrangian):

$$\Gamma^{(4,\mathrm{vac})}(s,t,u=0) = \frac{\lambda}{2}, \quad \Sigma^{(\mathrm{vac})}(p^2 = m^2) = 0,$$

$$\partial_{p^2} \Sigma^{(\mathrm{vac})}(p^2 = m^2) = 0. \tag{20}$$

Here s, t, and u are the usual Mandelstam variables for two-particle scattering, p is the external momentum of the self-energy, and m^2 is the renormalized mass of the particles. The first condition defines the coupling constant at vanishing momentum transfer for the two-particle scattering to be given by λ, the second condition chooses m to be the physical mass of the particles, while the third condition ensures that the residuum of the propagator at $p^2 = m^2$ is 1 and thus the on-shell wave function is normalized to 1, as it should be.

B. The finite temperature self-energy at the regularized level

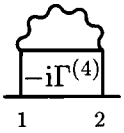
In this section we want to isolate those vacuum subparts inherent in the pure temperature part of the self-energy that need to be renormalized at physical space-time dimension $d = 4$. For this purpose we assume a regularization scheme, e.g., dimensional regularization, and extract those vacuum parts from the self-energy that diverge in the limit $d \to 4$. For the sake of clarification we mark all equations with an asterisk that diverge in the limit $d \to 4$ and need special renormalization treatment. All other equations are generally valid, even if all expressions are replaced by their renormalized quantities.

In order to extract the divergent vacuum pieces we take the self-energy as a functional of the self-consistent propagator and expand it around the vacuum value

$$\Sigma_{12}=\Sigma_{12}^{(\mathrm{vac})}+\underbrace{\Sigma_{12}^{(0)}+\Sigma_{12}^{(r)}}_{\Sigma_{12}^{(\mathrm{matter})}} \tag{21}$$

Here $\Sigma_{12}^{(\mathrm{vac})}$ is the vacuum ($T=0$) self-energy. Its renormalization poses no particular problem and can be done according to standard rules. For the examples discussed in our second paper [26] we show how to do this in practical cases for numerical solutions to the self-consistent equations of motion. The second and third terms in Eq. (21) contain the in-matter or finite temperature components of the self-energy. Thereby

$$-\mathrm{i}\Sigma_{12}^{(0)}=\int_{\mathcal{C}}\mathrm{d}(1'2')\left(-\frac{\delta \mathrm{i}\Sigma_{12}}{\delta G_{1'2'}}\bigg|_{T=0}G_{1'2'}^{(\mathrm{matter})}\right)= \tag{22*}$$



contains the parts of Σ linear in the matter (temperature) part $G^{(\mathrm{matter})}$ of the full propagator

$$\mathrm{i}G = \mathrm{i}G^{(\mathrm{vac})} + \mathrm{i}G^{(\mathrm{matter})} \tag{23}$$

At this level it is important to recognize that all loops involving vertices from both sides of the real-time contour (cf. Fig. 1) are UV convergent due to the analytical properties of the $\{-+\}$ and $\{+-\}$ propagators (A16) to (A18). Such loops contain at least one thermal weight factor $n(p_0)$ which decays exponentially at large $|p_0|$, since the Θ-function parts completely drop out for large loop momenta. Therefore all mixed components like G^{-+} or Σ^{+-} have to be excluded from the subtraction scheme. Thus, the expansion point $G^{(\mathrm{vac})}$ in Eq. (23) is defined as the contour-diagonal part of the propagator in the vacuum limit ($T\to 0$), i.e., with vanishing $G^{-+(\mathrm{vac})}=G^{+-(\mathrm{vac})}:=0$. Likewise all vacuum structures like $\Sigma^{(\mathrm{vac})}$, and the four-point functions $\Gamma^{(4,\mathrm{vac})}$ and $\Lambda^{(\mathrm{vac})}$ defined below, are "diagonal" in the real-time contour placement.

The remaining self-energy piece $\Sigma^{(r)}$ in Eq. (21) contains at least two $G^{(\mathrm{matter})}$ lines which therefore are never involved in any divergent loops due to the 2PI property of the Φ functional.[2] Thus, there are no hidden subdivergences in $\Sigma^{(r)}$, and possible divergent vacuum substructures can be directly renormalized using the BPHZ rules given above.

On the other hand the diagrams of $\Sigma^{(0)}$ deserve special attention, since there the single $G^{(\mathrm{matter})}$ line is involved in logarithmically divergent loops, if all vertices of $\Gamma^{(4)}_{12,1'2'}$ are placed on the same side of the contour. As mentioned the terms with mixed vertices are finite.

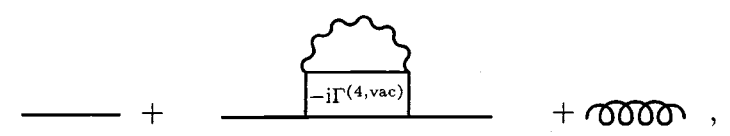
The divergences result from the fact that the functional variation of Σ with respect to G at $T=0$ defines a vacuum vertex function

$$-i\Gamma^{(4)}_{12,1'2'}=-\frac{\delta\Sigma_{12}}{\delta G_{1'2'}}\bigg|_{T=0}=-2i\,\frac{\delta^2\Phi}{\delta G_{12}\delta G_{1'2'}}\bigg|_{T=0} \tag{24}$$

with four external legs. Its diagonal part (all vertices on one contour side) defines $\Gamma^{(4,\mathrm{vac})}_{12,1'2'}$, which is of divergence degree 0. Assuming $G^{(\mathrm{matter})}$ of divergence degree -4, it is involved in a logarithmically divergent loop. Thus, this part of $\Sigma^{(0)}$, called $\Sigma^{(0,\mathrm{div})}$, accounts for all terms of divergence degree 0 and consequently $\Sigma^{(r)}$ is of divergence degree -2.

In order to trace all subdivergences hidden in $\Sigma^{(0)}$, the vacuum structure inherent in $G^{(\mathrm{matter})}$ has to be resolved. For this purpose the diagonal parts of the full propagator (i.e., G^{--} and G^{++}) need to be further decomposed

$$\mathrm{i}G_{12}=\mathrm{i}G_{12}^{(\mathrm{vac})}+\underbrace{\mathrm{i}\int_{\mathcal{C}}\mathrm{d}(1'2')\,G_{11'}^{(\mathrm{vac})}\Sigma_{1'2'}^{(0,\mathrm{div})}G_{2'2}^{(\mathrm{vac})}+\mathrm{i}G_{12}^{(\mathrm{r})}}_{\mathrm{i}G^{(\mathrm{matter})}} \tag{25}$$



(26*)

[2]Any $G^{(\mathrm{matter})}$ line in $\Sigma^{(r)}$ is either involved in loops with further $G^{(\mathrm{matter})}$ lines which are finite or attached to a pure vacuum piece. Due to the 2PI property of Φ this vacuum piece has more than four external legs also leading to finite loops for this $G^{(\mathrm{matter})}$ line.

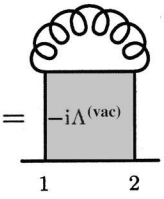
where here and below all vertices 1, 2, $1'$, and $2'$ are placed on the same side of the contour, in this way defining the divergent piece of $\Sigma^{(0)}$. The remaining part $G^{(\mathrm{r})}$ is of divergence degree -6.

Since $\Gamma^{(4,\mathrm{vac})}$ is a pure vacuum four-point function, Eqs. (21) and (25) also show $\Sigma^{(0,\mathrm{div})}$ as a functional linear in $G^{(\mathrm{matter})}$:

$$\Sigma_{12}^{(0,\mathrm{div})}=\int_{\mathcal{C}} d(1'2')\Gamma_{12,1'2'}^{(4,\mathrm{vac})}\left(\int_{\mathcal{C}} d(1''2'')G_{1'1''}^{(\mathrm{vac})}\Sigma_{1''2''}^{(0,\mathrm{div})}G_{2''2'}^{(\mathrm{vac})}+G_{1'2'}^{(\mathrm{r})}\right) \tag{27*}$$

and thus also linear in $G_{1'2'}^{(\mathrm{r})}$. A simple iteration argument, starting with $\Sigma_{12}^{(0,\mathrm{div})}=0$, shows that Eq. (27*) is solved by the ansatz

$$-\mathrm{i}\Sigma_{12}^{(0,\mathrm{div})}=\int_{\mathcal{C}} \mathrm{d}(1'2')\,\Lambda_{12,1'2'}^{(\mathrm{vac})}G_{1'2'}^{(\mathrm{r})}=$$

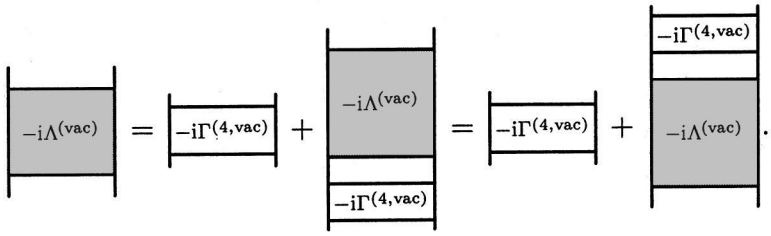


(28)

leading to the pure vacuum equation of motion

$$\Lambda_{12,1'2'}^{(\mathrm{vac})}=\Gamma_{12,1'2'}^{(4,\mathrm{vac})}+i\int_{\mathcal{C}} d(3456)\Gamma_{12,34}^{(4,\mathrm{vac})}G_{35}^{(\mathrm{vac})}G_{46}^{(\mathrm{vac})}\Lambda_{56,1'2'}^{(\mathrm{vac})} \tag{29*}$$

for the vacuum four-point function $\Lambda^{(\mathrm{vac})}$. The diagrammatic interpretation shows that this has the form of an inhomogeneous Bethe-Salpeter (BS) ladder equation



(30*)

From the construction it is clear that this is a very particular BS equation, namely, the one that complies with the self-consistent Dyson resummation scheme defined through Eq. (11). Thus these ladders, which are of *s*-channel type (forward scattering), are implicitly contained in the self-consistent vacuum self-energy. Since Weinberg's power counting theorem shows that $\Lambda^{(\mathrm{vac})}$ is of divergence degree 0, like any four-point function, Eq. (28) implies that $\Sigma^{(0,\mathrm{div})}$ is indeed also of divergence degree 0 as assumed above. Then $G^{(\mathrm{r})}$ is of divergence degree -6 and the loops in Eq. (28), which close $\Lambda^{(\mathrm{vac})}$ with a $G^{(\mathrm{r})}$ line, contain no further divergences. Thus, once $\Lambda^{(\mathrm{vac})}$ is renormalized, the self-energy is renormalized too, within a temperature-independent subtraction scheme.

Since Φ is 2PI, the variational relation (24) defines the BS kernel $\Gamma_{12,1'2'}^{(4,\mathrm{vac})}$ as a proper skeleton diagram, i.e., it contains no self-energy insertions, and cutting the diagram such that the pairs of space time points (12) and $(1'2')$ are separated cuts more than two lines. Thus the BS kernel has the appropriate irreducibility properties for the resummation to the complete four-point function $\Lambda^{(\mathrm{vac})}$, again showing the virtue of the Φ-functional formalism to avoid double counting.

From Eq. (28) it is obvious that on switching to the momentum space representation both $\Gamma^{(4,\mathrm{vac})}$ and $\Lambda^{(\mathrm{vac})}$ are not needed in their full momentum dependence but rather only as a function of the two momenta given by the Fourier transformation with respect to the space-time point pairs (12) and $(1'2')$. Because of Eq. (24) and through Eq. (29*) both $\Gamma^{(4,\mathrm{vac})}$ and $\Lambda^{(\mathrm{vac})}$ obey the symmetry relations

$$\Gamma_{12,1'2'}^{(4,\mathrm{vac})}=\Gamma_{1'2',12}^{(4,\mathrm{vac})}\quad\text{or}\quad\Gamma^{(4,\mathrm{vac})}(p,q)=\Gamma^{(4,\mathrm{vac})}(q,p),$$

$$\Lambda_{12,1'2'}^{(\mathrm{vac})}=\Lambda_{1'2',12}^{(\mathrm{vac})}\quad\text{or}\quad\Lambda^{(\mathrm{vac})}(p,q)=\Lambda^{(\mathrm{vac})}(q,p). \tag{31}$$

C. Renormalization of the vacuum Bethe-Salpeter equation

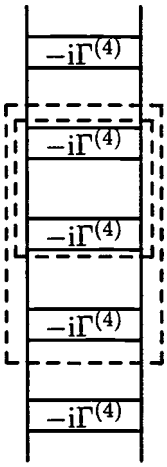
In energy-momentum representation the regularized BS equation (29*) and its equivalent ''adjoint'' version become

$$\Lambda^{(\mathrm{vac})}(p,q)=\Gamma^{(4,\mathrm{vac})}(p,q)+i\int\frac{d^dl}{(2\pi)^d}\Gamma^{(4,\mathrm{vac})}(p,l)\times[G^{(\mathrm{vac})}(l)]^2\Lambda^{(\mathrm{vac})}(l,q) \tag{32*}$$

$$=\Gamma^{(4,\mathrm{vac})}(p,q)+i\int\frac{d^dl}{(2\pi)^d}\Lambda^{(\mathrm{vac})}(p,l)\times[G^{(\mathrm{vac})}(l)]^2\Gamma^{(4,\mathrm{vac})}(l,q). \tag{33*}$$

The renormalization of the BS equation (29*) is not straightforward. First the BS kernel $\Gamma^{(4,\mathrm{vac})}(p,q)$ has to be renormalized following the BPHZ rules outlined in Sec. III. Representing $\Lambda(p,q)$ as a sum of ladder diagrams, this BS kernel forms the rungs in each ladder. The complication arises from overlapping subdivergences: in the general case each internal rung is part of various diverging subdiagrams through the loops involving two or more rungs. However, two observations help to settle this renormalization issue.

(i) The rungs given by the BS kernel are 2PI with respect to cuts separating the top from the bottom extremities of the rung. This implies that there are no divergent subdiagrams that cut into the inner structure of any rung, since this would involve a cut of more than two lines and the resulting subdiagram would have more than four external lines. Thus divergent subdiagrams always have complete rungs as subdiagrams, as shown in the valid counterterm below.



(ii) If one takes the difference of two Λ functions which differ only in one of the momenta, e.g., $\Lambda(p,q)-\Lambda(p',q)$, all those counterterms cancel out that contain boxes that cut the outer p or p' lines, respectively, since for these counterterms the argument p or p' is replaced by zero. Thus, the only boxes left are those that exclude the outermost rung attached to the p lines. These boxes, however, just define the renormalized result of the subdiagram complementary to this outermost rung, which is again a ladder diagram.

This permits one to establish recursive relations for the renormalized expressions of the two possible differences:

$$\Lambda^{(\mathrm{ren})}(p,q)-\Lambda^{(\mathrm{ren})}(p',q)=\Gamma^{(4,\mathrm{vac})}(p,q)-\Gamma^{(4,\mathrm{vac})}(p',q)+i\int\frac{d^4l}{(2\pi)^4}[\Gamma^{(4,\mathrm{vac})}(p,l)-\Gamma^{(4,\mathrm{vac})}(p',l)]\times[G^{(\mathrm{vac})}(l)]^2\Lambda^{(\mathrm{ren})}(l,q), \tag{34}$$

$$\Lambda^{(\mathrm{ren})}(p',q)-\Lambda^{(\mathrm{ren})}(p',q')=\Gamma^{(4,\mathrm{vac})}(p',q)-\Gamma^{(4,\mathrm{vac})}\times(p',q')+i\int\frac{d^4l}{(2\pi)^4}\Lambda^{(\mathrm{ren})}(p',l)[G^{(\mathrm{vac})}(l)]^2\times[\Gamma^{(4,\mathrm{vac})}(l,q)-\Gamma^{(4,\mathrm{vac})}(l,q')], \tag{35}$$

where now $\Gamma^{(4,\mathrm{vac})}$ stands for the renormalized Bethe-Salpeter kernel. Since the renormalized function $\Gamma^{(4,\mathrm{vac})}(p,l)-\Gamma^{(4,\mathrm{vac})}(0,l)$ is of divergence degree less than zero the integrals are finite. This set of renormalized equations can be used to construct the renormalized Λ function using

$$\Lambda^{(\mathrm{ren})}(0,0):=\pm\frac{\lambda}{2} \tag{36}$$

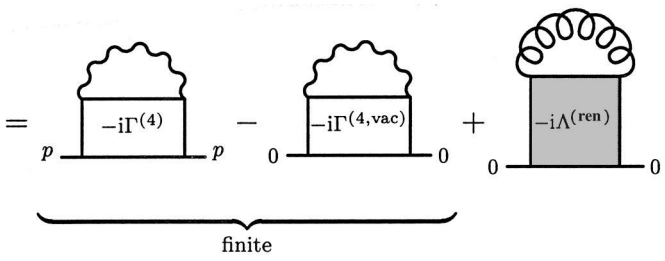
on the two real-time branches $\mathcal{C}_{\mp}$ due to our renormalization condition (20). In a kind of sweep-up sweep-down scheme, first the "half sided" $\Lambda^{(\mathrm{ren})}(0,q)$ function can be constructed by solving Eq. (35) for $p'=0$. Using this half sided function as the input for Eq. (34), the full momentum dependence of $\Lambda^{(\mathrm{ren})}(p,q)$ can be obtained. This scheme fully complies with the BPHZ renormalization prescription. It has the remarkable feature that, although it is explicitly asymmetric in p and q, it constructs a completely symmetric renormalized four-point function which can be combined to give the complete result

$$\Lambda^{(\mathrm{ren})}(p,q)=\Gamma^{(4,\mathrm{vac})}(p,q)+i\int\frac{d^4l}{(2\pi)^4}[\Gamma^{(4,\mathrm{vac})}(p,l)-\Gamma^{(4,\mathrm{vac})}(0,l)][G^{(\mathrm{vac})}(l)]^2\Lambda^{(\mathrm{vac})}(l,q)+i\int\frac{d^4l}{(2\pi)^4}\Lambda^{(\mathrm{ren})}(0,l)[G^{(\mathrm{vac})}(l)]^2\times[\Gamma^{(4,\mathrm{vac})}(l,q)-\Gamma^{(4,\mathrm{vac})}(l,0)]. \tag{37}$$

For numerical applications it is important to realize that only the half sided $\Lambda^{(\mathrm{ren})}(0,l)$, and not the full momentum dependence $\Lambda^{(\mathrm{ren})}(p,q)$, is explicitly needed. Since the half side Λ has essentially two-point function properties it can be numerically constructed using similar techniques as for self-energies.

Indeed one can express the complete renormalized self-energy part linear in $G^{(\mathrm{matter})}$ in the form

$$\Sigma^{(0)}(p)=\Sigma^{(0)}(p)-\Sigma^{(0)}(0)+\Sigma^{(0)}(0)=\int\frac{d^4l}{(2\pi)^4}[\Gamma^{(4)}(p,l)-\Gamma^{(4,\mathrm{vac})}(0,l)]G^{(\mathrm{matter})}(l)+\int\frac{d^4l}{(2\pi)^4}\Lambda^{(\mathrm{ren})}(0,l)G^{(\mathrm{r})}(l) \tag{38}$$



with $G^{(\mathrm{matter})}$ and $G^{(\mathrm{r})}$ from Eqs. (23) and (25). Here $\Gamma^{(4)}$ is the full contour valued kernel (24) including mixed contour vertices, while $\Gamma^{(4,\mathrm{vac})}$ and $\Lambda^{(\mathrm{ren})}$ are diagonal in the contour vertices. Because of the 2PI properties of $\Gamma^{(4)}$ the difference $\Gamma^{(4)}(l,p)-\Gamma^{(4,\mathrm{vac})}(l,0)$ is of divergence degree less than 0. Therefore the first integral represented by the difference of the first two diagrams is finite, since $G^{(\mathrm{matter})}$ is of divergence degree -4. This difference represents the most naive subtraction, which by itself, however, would be false, since it contains temperature-dependent counterterms. The heart of the above derivation is that these false *T*-dependent counterterms are precisely compensated by the last term. The fact that the counterterm structures never mix the two real-time contour branches also removes the problem of pinch singularities which otherwise could arise due to the vanishing external momentum.

This completes the proof that the self-energies can be renormalized with *T*-independent counterterms.

D. Renormalization of the real-time Γ functional

In this section we derive the renormalized real-time generating functional for the in-matter equations of motion. Thus, we restrict the contour integrations to the real-time contour $\mathcal{C}_{\mathbb{R}}$ and the corresponding traces to the real-time traces $\mathrm{Tr}_{\mathbb{R}}$. For functions in momentum representation the corresponding contour matrix algebra in the $\{-+\}$ notation (cf. the Appendix) is implied.

For the renormalization procedure we use the ansatz (25) for the full propagator *G* together with the form (28) for the logarithmically divergent part of the self-energy, where the renormalized four-point function Λ resolves the subdivergences hidden in both the propagator and the self-energy. For this purpose we decompose the generating functional Γ into its vacuum part, which is solely a functional of $G^{(\mathrm{vac})}$, and the in-matter part

$$\Gamma=\Gamma^{(\mathrm{vac})}[G^{(\mathrm{vac})}]+\Gamma^{(\mathrm{matter})}[G^{(\mathrm{vac})},G^{(\mathrm{matter})}]. \tag{39}$$

Thereby it is implied that the vacuum problem is already solved through its equation of motion resulting from the functional variation of $\Gamma^{(\mathrm{vac})}$. Given $G^{(\mathrm{vac})}$, the equations of motion in matter result from the functional variation of $\Gamma^{(\mathrm{matter})}$ with respect to $G^{(\mathrm{matter})}$.

Compared to the two-point self-energy the Γ and Φ functionals have no external points and essentially result from the diagrams of the self-energies by closing the extremities. Therefore one has to explicitly expand the corresponding expressions up to second order in $G^{(\mathrm{matter})}$ before one comes to the situation where the remaining pieces are void of hidden subdivergences. Thus we write

$$\begin{aligned}\Phi_{\mathbb{R}}&=\Phi_{\mathbb{R}}^{(\mathrm{vac})}+\mathrm{Tr}_{\mathbb{R}}\frac{\delta\Phi^{(\mathrm{vac})}}{\delta G^{(\mathrm{vac})}}G^{(\mathrm{matter})}\\&\quad+\frac{1}{2!}G^{(\mathrm{matter})}\frac{\delta^2\Phi^{(\mathrm{vac})}}{\delta G^{(\mathrm{vac})^2}}G^{(\mathrm{matter})}+\Phi_{\mathbb{R}}^{(\mathrm{r})}\\&=\Phi_{\mathbb{R}}^{(\mathrm{vac})}-\frac{i}{2}\mathrm{Tr}_{\mathbb{R}}\,G^{(\mathrm{matter})}\Sigma^{(\mathrm{vac})}\\&\quad+\frac{1}{4}G^{(\mathrm{matter})}\Gamma^{(4)}G^{(\mathrm{matter})}+\Phi_{\mathbb{R}}^{(\mathrm{r})}.\end{aligned} \tag{40}$$

Here we have used Eqs. (11) and (24) for the vacuum parts defined through the variation of Φ with respect to *G*. At the same time we introduced the real-time trace in momentum space,

$$\mathrm{Tr}\,A\cdots B=\int\frac{d^dl}{(2\pi)^d}A(l)\cdots B(l), \tag{41}$$

and the functional tensor contraction for four-point functions with propagators,

$$G_1\Gamma^{(4)}G_2=\int\frac{d^dl_1}{(2\pi)^d}\int\frac{d^dl_2}{(2\pi)^d}G_1(l_1)\Gamma^{(4)}(l_1,l_2)G_2(l_2). \tag{42}$$

In all expressions the functions are contour matrix functions which imply the corresponding contour matrix algebra and the contour trace; cf. Eqs. (A19) and (A20).

Applying the arguments given for $\Sigma^{(\mathrm{r})}$ for $\Phi^{(\mathrm{r})}$ also, no $G^{(\mathrm{matter})}$ line is involved in divergent loops such that after renormalization of possible vacuum subdivergences the en-

tire diagram is finite. Thus only terms with at most two $G^{(\mathrm{matter})}$ lines need further care.

Since Φ by itself is not an observable we directly step to the definition of the Γ functional that relates to the thermodynamic potential in the equilibrium case. Exploiting the stationarity condition of Γ at the vacuum level, i.e., using the vacuum equations of motion for $G^{(\mathrm{vac})}$, all terms linear in $G^{(\mathrm{matter})}$ drop out and we find for the functional with all proper vacuum subdivergences subtracted

$$\overline{\Gamma}_{\mathbb{R}}^{(\mathrm{matter})}[G^{(\mathrm{matter})}]=\frac{i}{2}\mathrm{Tr}_{\mathbb{R}}\Bigg(G^{(\mathrm{matter})}\Sigma^{(\mathrm{matter})}-\sum_{k=2}^{\infty}\frac{(G^{(\mathrm{vac})}\Sigma^{(\mathrm{matter})})^k}{k}\Bigg)-\frac{i}{4}\mathrm{Tr}_{\mathbb{R}}\,G^{(\mathrm{matter})}\Sigma^{(0)}+\overline{\Phi}_{\mathbb{R}}^{(\mathrm{r})}. \tag{43*}$$

Renormalized to zero at the vacuum level this expression is a functional of the in-matter part of the propagator. The remaining divergent parts of $\overline{\Gamma}^{(\mathrm{matter})}$ arise from terms quadratic in $G^{(\mathrm{matter})}$, i.e.,

$$\overline{\Gamma}_{\mathbb{R}}^{(\mathrm{matter,div})}=\frac{i}{4}\mathrm{Tr}_{\mathbb{R}}\,G^{(0)}\Sigma^{(0)}+\frac{1}{4}G^{(0)}\Gamma^{(4,\mathrm{vac})}G^{(0)}+\frac{1}{2}G^{(0)}\Gamma^{(4,\mathrm{var})}G^{(\mathrm{r})} \tag{44*}$$

with

$$G^{(0)}=G^{(\mathrm{vac})}\Sigma^{(0,\mathrm{vac})}G^{(\mathrm{vac})}, \tag{45}$$

where again in both relations above all expressions are contour diagonal. Both $\Sigma^{(0,\mathrm{div})}$ and $G^{(0)}$ are linear in $G^{(\mathrm{matter})}$. Using the equations of motion for $\Sigma^{(0,\mathrm{div})}$ and $\Lambda^{(\mathrm{vac})}$ one arrives at an expression for the divergent part of $\overline{\Gamma}^{(\mathrm{matter})}$ which contains only quantities that were already renormalized in the previous subsection:

$$\Gamma^{(\mathrm{matter,div,ren})}=\frac{1}{4}(G^{(\mathrm{r})}\Lambda^{(\mathrm{ren})}G^{(\mathrm{r})}-G^{(\mathrm{r})}\Gamma^{(4,\mathrm{vac})}G^{(\mathrm{r})}). \tag{46}$$

Substituting this for the divergent part we obtain after some algebraic simplifications

$$\Gamma_{\mathbb{R}}^{(\mathrm{matter,ren})}[G^{(\mathrm{matter})}]=\frac{i}{2}\overline{\mathrm{Tr}}_{\mathbb{R}}\Bigg(G^{(\mathrm{r})}\Sigma^{(\mathrm{matter})}-\frac{1}{2}G^{(\mathrm{vac})}\Sigma^{(\mathrm{r})}G^{(\mathrm{vac})}\Sigma^{(\mathrm{r})}-\sum_{k=3}^{\infty}\frac{(G^{(\mathrm{vac})}\Sigma^{(\mathrm{matter})})^k}{k}\Bigg)+\frac{1}{4}G^{(\mathrm{r})}\Lambda^{(\mathrm{vac})}G^{(\mathrm{r})}+\Phi_{\mathbb{R}}^{(\mathrm{r})}$$
$$+\text{mixed contour } \mathrm{Tr}_{\mathbb{R}} \text{ terms from Eq. (43*)}, \tag{47}$$

where $\overline{\mathrm{Tr}}$ includes only the contour diagonal parts. This expression, which now can be considered as a functional of $G^{(\mathrm{matter})}$, or through Eqs. (25) and (28) of $G^{(\mathrm{r})}$, is void of any hidden subdivergences, since all matter-or T-dependent parts of the propagator like $G^{(\mathrm{matter})}$ or $G^{(\mathrm{r})}$ are involved in convergent loops.

Now it remains to be proven that this renormalization procedure for the Γ functional is consistent with the renormalization of the self-energy given in the previous section. In other words, we wish to show that the vanishing functional variation of $\Gamma^{(\mathrm{ren})}[G^{(\mathrm{matter})}]$ complies with the Dyson equation of motion and the renormalized self-energy. From the BPHZ formalism we expect this to hold true, because it ensures that subdivergences can be renormalized first and then the remaining divergences which come into play by closing the diagrams: The renormalized result is independent of the order of counterterm subtractions.

It is sufficient to show this for the contour diagonal parts of Eq. (47). We write the functional variation of $\Gamma^{(\mathrm{ren})}$ as

$$\delta\Gamma_{\mathbb{R}}^{(\mathrm{matter,ren})}=\frac{\delta\Gamma_{\mathbb{R}}^{(\mathrm{matter,ren})}}{\delta\Sigma^{(\mathrm{r})}}\delta\Sigma^{(\mathrm{r})}+\frac{\delta\Gamma_{\mathbb{R}}^{(\mathrm{matter,ren})}}{\delta G^{(\mathrm{r})}}\delta G^{(\mathrm{r})}, \tag{48}$$

where $\Sigma^{(\mathrm{r})}$ is supposed to be a functional of $G^{(\mathrm{r})}$. Both terms in Eq. (48) independently vanish. The first term drops out by virtue of the Dyson equation, which together with Eq. (25) ensures that

$$\delta\Bigg(\sum_{k=3}^{\infty}\frac{(G^{(\mathrm{vac})}\Sigma^{(\mathrm{matter})})^k}{k}\Bigg)=(G^{(\mathrm{r})}-G^{(\mathrm{vac})}\Sigma^{(\mathrm{r})}G^{(\mathrm{vac})})\,\delta\Sigma^{(\mathrm{matter})}. \tag{49}$$

The second term shrinks to

$$\delta\Gamma_{\mathbb{R}}^{(\mathrm{matter,ren})}=\frac{i}{2}\mathrm{Tr}_{\mathbb{R}}(\underbrace{\Sigma^{(\mathrm{r})}\delta G^{(\mathrm{r})}+G^{(\mathrm{vac})}\Sigma^{(\mathrm{r})}G^{(\mathrm{vac})}\delta\Sigma^{(0)}}_{\Sigma^{(\mathrm{r})}\delta G^{(\mathrm{matter})}})+\delta\Phi_{\mathbb{R}}^{(\mathrm{r})}\overset{!}{=}0, \tag{50}$$

which indeed implies

$$\Sigma^{(\mathrm{r})}=2i\frac{\delta\Phi_{\mathbb{R}}^{(\mathrm{r})}}{\delta G^{(\mathrm{matter})}}, \tag{51}$$

compatible with the definition of $\Sigma^{(\mathrm{r})}$. It is important to note that through the functional variation (48) only convergent loops are opened, such that none of the counterterms are af-

fected by this variation. This explicitly demonstrates the consistency of the BPHZ renormalization scheme for the self-consistent approximations: The operations of variation with respect to G and renormalization are commutative, i.e., one can construct the renormalized self-energy in two equivalent ways. The first uses the *unrenormalized* Φ functional and defines the renormalized self-energy by applying the BPHZ renormalization theorem to its diagrams, which are defined by opening any line of the unrenormalized Φ functional. In this way we have defined the renormalized self-energy in the previous section. Subsequently we renormalized the Γ functional by substituting the renormalized functions thereby defined for the divergent vacuum subdiagrams with two and four external legs. After subtracting the pure vacuum contribution this leads to the finite renormalized functional (47).

As we have now seen, the second way to define the renormalized self-energy is to renormalize the Γ functional first. Then the variation with respect to the *renormalized* propagator leads to the *renormalized* equations of motion and thus directly to the *renormalized* self-energy which shows the consistency of the local vacuum counterterms including all combinatorial factors for both the generating functional and the equation of motion.

We wish to clarify that the equations derived in this section for the in-matter parts of the propagator are all valid also in the general nonequilibrium case of quantum field theory provided the density operator at time t_0 amends a Wick decomposition. This is valid for statistical operators at initial time of the form [31]

$$\mathbf{R}=\frac{1}{Z}\exp\left(-\sum_k \alpha_k \mathbf{A}_k\right) \quad \text{with} \quad Z=\mathrm{Tr}\exp\left(-\sum_k \alpha_k \mathbf{A}_k\right), \tag{52}$$

where the $\mathbf{A}_k$ are one-particle operators. Our arguments for the renormalizability with α_k-independent (i.e., state-independent) local counterterms should hold, since the statistical operator is normalized, $\mathrm{Tr}\,\mathbf{R}=1$. Thus, any in-matter part of the propagator leads to similar reductions of the degree of divergence as for the Bose-Einstein distribution functions used here. This ensures that the power counting arguments for the nonvacuum parts are still valid. Note in particular that the real-time functional Γ has only functional meaning, namely, as a tool to derive the equations of motion, since its value at the physical solution vanishes.

E. Renormalization of the thermodynamical potential

Using the thermodynamic part, i.e., the vertical branch, of the contour given in Fig. 1 the Γ functional provides a finite value which indeed relates to the thermodynamic potential Ω. For the evaluation one uses the relationship between the Matsubara functions and the real-time functions given in the Appendix, Eq. (A21), for the thermodynamic trace $\mathrm{Tr}_{\mathrm{Th}}$, which takes due account of the thermodynamic weights in the partition sum:

$$\begin{aligned}\mathrm{Tr}_{\mathrm{Th}}\{h(p)\}&=\beta V\int\frac{d^dp}{(2\pi)^d}(h^{-+}+h^{+-})\\&=-2i\beta V\int\frac{d^dp}{(2\pi)^d}\,\mathrm{sgn}(p_0)\\&\quad\times\left(n(p_0)+\frac{1}{2}\right)\mathrm{Im}\,h_R(p),\end{aligned} \tag{53}$$

where h^{-+} and h^{+-} are the Wightman functions and h_R is the retarded function of h. Furthermore,

$$n(p_0)=\frac{1}{\exp(\beta|p_0|)-1} \tag{54}$$

is the thermal Bose-Einstein factor resulting from the summation over Matsubara frequencies expressed in terms of complex contour integrals; cf. Eq. (A21). For this thermal contour $\mathcal{C}_{\mathrm{Th}}$ closed diagrams such as those of Φ and Γ also attain a finite value. In this case the rule is first to omit one of the momentum integrations, which in this way defines a two-point function. Its renormalized retarded value can be calculated according to the real-time contour rules used above. Subsequently one applies Eq. (53) for the final integration. We show now that after renormalization of the two-point function we need only subtract the overall vacuum divergence inherent in this final integral.

To obtain this result we have to go back to the regularized expression for the *unrenormalized* effective potential (8) which relates to the thermodynamical potential via

$$\Omega^{(\mathrm{reg})}(T)=-T\Gamma^{(\mathrm{reg})}_{\mathrm{Th}} \tag{55}$$

where

$$\Gamma^{(\mathrm{reg})}_{\mathrm{Th}}(T)=\Gamma^{(\mathrm{vac,reg})}_{\mathrm{Th}}(T)+\Gamma^{(\mathrm{matter,reg})}_{\mathrm{Th}}(T) \quad \text{with} \quad \Gamma^{(\mathrm{vac,reg})}_{\mathrm{Th}}(T)=\mathrm{Tr}_{\mathrm{Th}}\left[\frac{i}{2}\ln\left(-\frac{M^2}{G^{\mathrm{vac}}}\right)+\frac{i}{2}\Sigma^{\mathrm{vac}}G^{\mathrm{vac}}\right]+\Phi^{\mathrm{vac}}_{\mathrm{Th}}(T),$$

$$\Gamma^{(\mathrm{matter,reg})}_{\mathrm{Th}}(T)=\frac{i}{2}\mathrm{Tr}_{\mathrm{Th}}\left[G^{(\mathrm{r})}\Sigma^{(\mathrm{matter})}-\frac{1}{2}G^{(\mathrm{vac})}\Sigma^{(\mathrm{r})}G^{(\mathrm{vac})}\Sigma^{(\mathrm{r})}-\sum_{k=3}^{\infty}\frac{(G^{(\mathrm{vac})}\Sigma^{(\mathrm{matter})})^k}{k}\right]+\frac{1}{4}G^{(\mathrm{r})}\Lambda^{(\mathrm{vac})}G^{(\mathrm{r})}+\Phi^{(\mathrm{r})}_{\mathrm{Th}}(T). \tag{56}$$

Here the subscript ''Th'' specifies the quantities resulting from the thermal trace. The matter part results from the form (47). Subsequently one replaces all quantities by their renormalized ones (denoted by a bar over the functions) and cancels the overall divergence by subtracting the $T \to +0$ value,

$$\begin{aligned}\Omega^{(\mathrm{ren})}(T) &= -T(\overline{\Gamma}^{(\mathrm{reg})}_{\mathrm{Th}}(T) - \overline{\Gamma}^{(\mathrm{reg})}_{\mathrm{Th}}(+0)) \\ &= -T(\overline{\Gamma}^{(\mathrm{vac,reg})}_{\mathrm{Th}}(T) - \overline{\Gamma}^{(\mathrm{vac,reg})}_{\mathrm{Th}}(+0) \\ &\quad + \overline{\Gamma}^{(\mathrm{matter,reg})}_{\mathrm{Th}}(T)). \end{aligned} \tag{57}$$

This procedure is legitimate as long as the new loops due to the final thermodynamical trace (53) do not induce new subdivergences, but only overall divergences. It is obvious that for the matter part the final trace loop involving the factor $n+1/2$ is completely convergent, since all loops are regular once Λ is renormalized. For the two vacuum terms the components proportional to the factor 1/2 in the thermal trace (53) cancel out, such that all terms are proportional to $n(p_0)$, which cuts off the p_0 integration, while the momentum integrations are also limited due to the vacuum thresholds: the imaginary parts of the vacuum functions are zero for $p^2 < m^2$, where m is the mass of the stable vacuum particle. Thus these final loop integrals are also finite, defining a finite thermodynamical potential.

The vacuum part essentially determines the kinetic energy part of Ω as can be seen from the most simple example of an ideal gas. Here of course all self-energies and Φ are vanishing, the retarded propagator at finite temperature is $D_R(p) = [p^2 - m^2 + i\eta\sigma(p_0)]^{-1}$, and the renormalization is done by subtracting the pure vacuum part. Thus, the free thermodynamical potential becomes

$$\Omega^{(\mathrm{id.\ gas})} = -V \int \frac{d^4 l}{(2\pi)^4} n(l_0)\pi\Theta(l^2 - m^2), \tag{58}$$

which can be brought to a more familiar form by an integration by parts:

$$\Omega^{(\mathrm{id.\ gas})} = -pV = V \int \frac{d^3\vec{l}}{(2\pi)^3} \ln[1 - \exp(-\beta\sqrt{\vec{l}^2 + m^2})]. \tag{59}$$

IV. CONCLUSIONS AND OUTLOOK

For the example of ϕ^4 theory we have shown that self-consistent Dyson resummations based on a Φ-derivable scheme can be renormalized with local counterterms defined on the self-consistently determined vacuum level. This result was obtained with the help of Weinberg's power counting theorem and using the BPHZ renormalization scheme with the usual modifications for finite temperature diagram rules, which can be summarized in the simple rule that the ''contraction boxes'' defining the counterterms have to exclude subdiagrams that contain any temperature line.

The hidden subdivergence structure of the self-consistent scheme has been resolved. This leads to a Bethe-Salpeter equation for the vacuum four-point function compatible with the chosen Φ approximation, which we have renormalized. The method is free of pinch singularities. Closed equations could be formulated that resum the nonperturbative structure of both the equations of motion, i.e., the self-energies, and also the nonperturbative counterterm structure. The complexity of these equations is comparable to standard Dyson resummation schemes and therefore in principle does not imply new techniques. The first numerical applications, which include the construction of the *BS* kernel, the solution of the half sided four-point function, and thus the renormalized self-energies up to the self-consistent sunset self-energy, are presented in a second paper [26]. The renormalization of the generating functional Γ (cf. Sec. III D) shows that the derivation and thus the renormalized in-matter equations of motion apply equally to the general nonequilibrium case.

This also proves that there is no arbitrariness in studying the in-medium modifications of model parameters like the mass and the coupling constants within this class of approximation scheme: It is sufficient to adjust them in the vacuum, for instance by fitting them to scattering data, in order to predict without ambiguity how they change in the dense and hot medium. *The in-medium modifications are ruled completely by the model and its vacuum parameters alone and no further assumptions need to be made.*

Although demonstrated for the ϕ^4 theory, the method is in principle general, since the derivation relies only on the analytic and asymptotic form of the propagators. In particular, the renormalization of hidden overlapping divergences in the logarithmically divergent Bethe-Salpeter equations is general. Still, there are a number of restrictions of the self-consistent Dyson resummation within the Φ-derivable scheme, which concern global and local symmetries and the corresponding conservation laws and Ward-Takahashi identities.

The Φ-functional formalism only ensures the conservation laws for the expectation values of charges associated with the symmetry by Noether's theorem. However, in general the Ward-Takahashi identities are violated for the self-energy and higher vertex functions. Heuristically the problem can be traced back to the violation of crossing symmetry by the self-consistent scheme: Our derivation shows that the self-consistent solution of the self-energy involves Bethe-Salpeter ladder resummations of the four-point function, but only in the *s* channel. The crossing symmetric *t*- and *u*-channel contributions to the four-point function are not included.

The symmetry properties of the $\Gamma[\varphi, G]$ functional have already been investigated by us with the help of the path-integral method applied here [32,33]. We show that it is always possible to define a *nonperturbative* approximation to the effective action $\Gamma_{\mathrm{eff}}[\varphi]$ which respects linearly realized symmetries of the classical action provided the symmetry is not anomalously broken. The self-energy and higher vertex functions defined from this improved approximation action formalism then fulfill the Ward-Takahashi identities of the underlying symmetry. As a result the effective action Γ_{eff} enforces that additional *t*- and *u*-channel Bethe-Salpeter resummations are needed to restore the crossing symmetry together with the Ward-Takahashi identities for the self-energy

and the vertex functions. However, these vertex functions are not self-consistently calculated and thus some problems also remain within this approximation: For instance, in the case of the linear sigma model the O(N) symmetry is restored for the vertex functions and the Goldstone modes become massless. Yet the phase transition from the Nambu-Goldstone phase at low temperatures to the Wigner-Weyl phase at high temperatures is found to be of first order rather than second order [9].

In the case of a local gauge symmetry the problems become even more intricate. Self-consistent schemes beyond the classical field level for the gauge fields generally violate local gauge symmetries for the same reasons as for global symmetries. However, this immediately causes the excitation of spurious modes of the gauge fields, which leads to violation of the unitarity of the S matrix, the positive definiteness of the statistical operator, and the causality structure of the Green's functions. Nevertheless, a gauge invariant effective action $\Gamma_{\rm eff}$ within a background-field approach can be formulated that provides gauge covariant polarization functions [17].

From a practical point of view the problem remains of calculating the self-consistent propagators needed for the symmetry-restoring Bethe-Salpeter resummation, which presently can be solved only in simple cases (RPA bubble resummation). In [17] we presented a workaround in terms of a suitably chosen projection method onto the physical (transverse) degrees of freedom of the gauge-field polarization tensor. This procedure, of course, does not lead to a full restoration of local gauge theory but to causal Green's functions and current conservation within the self-energies of matter fields. Alternative methods are restricted to the approximate solution of the self-consistent equations of motion, e.g., in the sense of a Hard thermal loop approximation [19], or to a systematic expansion in terms of the coupling constant or $\hbar$ [18].

The proof of the renormalizability of Φ-derivable approximations opens a broad range of perspectives for effective field theory model applications describing the nonperturbative in-medium properties of particles in dense or finite temperature matter with model parameters fixed at the vacuum level. Further applications point toward the appropriate renormalization of nonequilibrium transport equations [16], where in particular the drift terms, which determine the equation of state, involve the real parts of the self-energies, which generally need renormalization.

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APPENDIX: ANALYTICAL PROPERTIES OF GREEN'S FUNCTIONS

In this appendix we summarize briefly the analytic properties of Green's functions of neutral bosons needed in the main part of the paper. This is most easily done by switching to the operator formalism in the Heisenberg picture. By definition we have for a Hermitian scalar field operator

$$iG(x)=\frac{1}{Z}\mathrm{Tr}\exp(-\beta\mathbf{H})\phi(x)\phi(0):=\langle\mathcal{T}_C\phi(x)\phi(0)\rangle_\beta. \tag{A1}$$

For x^0 on the vertical part, i.e., $x^0=-i\tau$ with $0\leqslant\tau\leqslant\beta$, we obtain the Matsubara Green's function

$$G_M(\tau,\vec{x})=G^{+-}(-i\tau,\vec{x}), \tag{A2}$$

where one has to understand the analytic continuation of the real-time Wightman function G^{+-} on the right hand side. It is important to keep in mind that Eq. (A2) is valid only when the first time argument x^0 in Eq. (A1) is on the vertical part of the contour while the second one is at $x^0=0$. If both arguments of the fields are located on the vertical part according to Eq. (A1) one has to use time ordering along the imaginary time axis.

Since the order of the operators under the trace in Eq. (A1) can be changed cyclically the real-time Wightman functions G^{+-} and G^{-+} are related through

$$G^{+-}(x^0-i\beta,\vec{x})=G^{-+}(x^0,\vec{x}), \tag{A3}$$

where x^0 is a real-time argument on the contour and on the right hand side one has to understand the analytic continuation. One should keep in mind that only the Wightman functions are analytically continuable, not the time ordered or anti time ordered functions of the upper or lower real-time branch, since they contain step functions from the time ordering operator $\mathcal{T}_C$. The same holds true for the imaginary-time ordered Matsubara Green's function.

As in vacuum quantum field theory, because of translation invariance it is customary to use the energy-momentum representation of Green's functions. For the real-time propagators we have the usual description

$$G^{ij}(x)=\int\frac{d^dp}{(2\pi)^d}\exp(-ipx)G^{ij}(p). \tag{A4}$$

We write down the formalism for arbitrary space-time dimensions since none of the considerations depend on it and we need it to obtain well defined nonrenormalized quantities in the sense of dimensional regularization. The periodic boundary condition (A3) translates into the *Kubo-Martin-Schwinger condition* for the Fourier transformed Green's functions:

$$G^{-+}(p)=\exp(-\beta p_0)G^{+-}(p). \tag{A5}$$

The Matsubara Green's function is defined only for imaginary times $-i\tau$ with $0\leqslant\tau\leqslant\beta$. Thus the momentum representation with respect to the time component is a Fourier series with period β according to Eq. (A3) rather than a Fourier integral:

$$G_M(x)=\frac{1}{\beta}\sum_{n=-\infty}^{\infty}\frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}}\exp(-ipx)G_M(ip_0,\vec{p})|_{p^0=\omega_n}$$

$$\text{with}\quad \omega_n=\frac{2\pi}{\beta}n. \tag{A6}$$

Here $x^0=-i\tau$ with $0\leqslant\tau\leqslant\beta$. Using the inverse Fourier transform Eq. (A4) and the KMS condition Eq. (A5) we find the *spectral representation*

$$G_M(i\omega_n,\vec{p})=i\int\frac{dp_0}{2\pi}\frac{\rho(p_0,\vec{p})}{p_0-i\omega_n}$$

$$\text{with}\quad \rho(p)=i[G^{+-}(p)-G^{-+}(p)]. \tag{A7}$$

This shows that the Matsubara propagator is completely represented by the real-time function ρ. With the help of this we define the analytically continued propagator by

$$G_c(k)=-\int\frac{dp_0}{2\pi}\frac{\rho(p_0,\vec{k})}{p_0-k_0}. \tag{A8}$$

It can contain singularities only on the real axis. Using the Fourier transformation (A4) for the limits to the real axis from above and below, we obtain

$$G_c(p_0\pm i\eta,\vec{p})=G_{R/A}(p),\quad p_0\in\mathbb{R}, \tag{A9}$$

with the *retarded and advanced* Green's functions

$$G_{R/A}(x)=\mp i\Theta(\pm t)\langle[\phi(x),\phi(0)]_-\rangle_\beta. \tag{A10}$$

From this we find immediately

$$G_R(x)=G_A^*(-x)\Leftrightarrow G_R(p)=G_A^*(p),$$
$$\rho(p)=-2\,\mathrm{Im}\,G_R(p)=-\rho(-p), \tag{A11}$$

and from Eq. (A8) and the analyticity of $G_c(p)$ in the upper complex p_0 plane it follows that

$$\sigma(p_0)\rho(p)\geqslant 0. \tag{A12}$$

For later use we note the momentum space properties

$$G^{--}+G^{++}=G^{+-}+G^{-+},\quad G_R=G^{--}-G^{-+},$$
$$G_A=G^{--}-G^{+-},\quad G_M=-iG_c(i\omega_n), \tag{A13}$$

which follow immediately from Eqs. (A1), (A7), and (A10).

We also make use of the expressions for the real-time Green's functions in terms of the retarded Green's function, which follow immediately from Eqs. (A7), (A11), (A18):

$$iG^{--}(p)=iG_R(p)+[\Theta(-p_0)+n(p_0)]\rho(|p_0|,\vec{p}), \tag{A14}$$

$$iG^{++}(p)=[\Theta(p_0)+n(p_0)]\rho(|p_0|,\vec{p})-iG_R(p), \tag{A15}$$

$$iG^{-+}(p)=[\Theta(-p_0)+n(p_0)]\rho(|p_0|,\vec{p}), \tag{A16}$$

$$iG^{+-}(p)=[\Theta(p_0)+n(p_0)]\rho(|p_0|,\vec{p}), \tag{A17}$$

and the *Bose-Einstein distribution* defined as

$$n(p_0)=\frac{1}{\exp(\beta|p_0|)-1}. \tag{A18}$$

All relations given above for the Green's functions G directly apply to any two-point function given by local field operators $ih(x,y)=\langle\mathcal{T}_{\mathcal{C}_\mathbb{R}}\mathbf{H}(x)\mathbf{H}(y)\rangle$, e.g., the self-energy.

Real-time contour integrations and traces of translationally invariant two-point functions

$$C(x,y)=\int_{\mathcal{C}_\mathbb{R}}dz\,A(x,z)B(z,y),\quad x,y,z\in\mathcal{C}_\mathbb{R},$$

$$\mathrm{Tr}_\mathbb{R}\,C=\int_{\mathcal{C}_\mathbb{R}}dx\,C(x,x) \tag{A19}$$

transcribe to

$$C(p)^{ij}=\sum_{kl}A(p)^{ik}\sigma_{kl}B(p)^{lj},\quad \sigma=\mathrm{diag}(1,-1),$$

$$i,j,k,l\in\{-,+\},$$

$$\mathrm{Tr}_\mathbb{R}\,C=\sum_{ij}\int\frac{d^dp}{(2\pi)^d}C(p)^{ij}\sigma_{ij} \tag{A20}$$

in contour momentum space representation.

We close this appendix by citing the formula for summation over the Matsubara frequencies needed when calculating quantities related to the vertical branch of the contour [24]. In this paper we use this to calculate the thermodynamical potential:

$$\mathrm{Tr}_{\mathrm{Th}}\,h(p):=\beta V\frac{1}{i\beta}\sum_{n=-\infty}^{\infty}\int\frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}}h(i\omega_n,\vec{p})$$
$$=-\beta V\int\frac{d^dp}{(2\pi)^d}\left[\frac{1}{2}+n(p_0)\right]\{h[p_0+i\eta\sigma(p_0)]$$
$$-h[p_0-i\eta\sigma(p_0)]\}$$
$$=\beta V\int\frac{d^dp}{(2\pi)^d}(h^{-+}+h^{+-}). \tag{A21}$$

In Eq. (A21) we have assumed that the function h is analytic below and above the real axis and that the trace exists. Usually this is only the case for the regularized or renormalized functional traces. It is also clear that due to the exponential damping from the Bose-Einstein distribution (A18) this part of the integral has a superficial degree of divergence reduced by 1 compared to the first part, which is not damped by an n factor.

Equation (A21) shows that the thermodynamical potential can be calculated from real-time quantities since the analytic continuation of the Matsubara Green's function needed on the right hand side is unique and can be obtained from the retarded Green's function as well; cf. Eq. (A9). For a more detailed analysis of the analytic properties, see also [25] and for the general case of Wigner function representations in the nonequilibrium context [11].

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