Note on flat foliations of spherically symmetric spacetimes

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It is known that spherically symmetric spacetimes admit flat spacelike foliations. We point out a simple method of seeing this result via the Hamiltonian contraints of general relativity. The method yields explicit formulas for the extrinsic curvatures of the slicings.

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The Painlevé-Gulstrand form [1,2] of the metric of Schwarzschild spacetime,

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + 2\sqrt{2M/r}\,dt\,dr + r^2 d\Omega^2, \qquad (1)$$

demonstrates that this spacetime has a foliation by flat spatial surfaces. This generalizes to all spherically symmetric spacetimes, for which the metric may be written as

$$ds^2 = -f^2(r)dt^2 + 2g(r)\,dt\,dr + r^2 d\Omega^2. \qquad (2)$$

The question of which observers in the spacetime see flat spacelike constant time surfaces was addressed recently in [3]: the surfaces of the foliation are orthogonal to the trajectories of observers freely falling from rest at infinity. (Such flat foliations have also been used to study Hawking radiation [4,5].) A similar result holds for the Reissner-Nordström spacetime [3].

The purpose of this note is to demonstrate a method of showing that all spherically symmetric spacetimes admit flat spacelike foliations via the Hamiltonian equations of general relativity. The approach involves finding solutions, in spherical symmetry, of the Hamiltonian and spatial diffeomorphism constraints of general relativity:

$$\mathcal{H} \equiv \frac{1}{\sqrt{q}} G_{abcd}\tilde{\pi}^{ab}\tilde{\pi}^{cd} - \sqrt{q}\,{}^{(3)}R + \tilde{\rho} = 0, \qquad (3)$$

$$\mathcal{C}^a \equiv \partial_b \tilde{\pi}^{ba} + \tilde{P}^a = 0, \qquad (4)$$

where $(q_{ab}, \tilde{\pi}^{ab})$ are the canonically conjugate Hamiltonian variables, $\tilde{\rho}$ and $\tilde{P}_a$ are the matter energy density and momentum, $G_{abcd} = (g_{ac}g_{bd} + g_{ad}g_{bc} - g_{ab}g_{cd})/2$ is the DeWitt supermetric, D_a is the covariant derivative of q_{ab}, and a tilde denotes densities of weight 1.

The constraints are first class; therefore they continue to be satisfied on all time slices under evolution. Furthermore the lapse and shift can be chosen such that flat slices evolve to flat slices. This is easily done by solving the Hamiltonian equations for the lapse and shift such that the time derivative of the spatial metric is zero, i.e., it remains flat. Thus it is sufficient to find flat slice initial data in spherical symmetry to show that there exist flat slice foliations.[1]

Following [7], we assume that a flat slice foliation exists, and seek solutions of the constraints. Let $q_{ab} = e_{ab}$, the flat Euclidean 3-metric in (global) rectangular coordinates x^a. The general form of the conjugate momentum is given by

$$\tilde{\pi}^{ab} = f(r)n^a n^b + g(r)e^{ab}, \qquad (5)$$

where $f(r)$ and $g(r)$ are functions to be determined, $n^a = x^a/r$ is the unit radial vector and $r^2 = x^a x^b e_{ab}$. That this is the most general form for $\tilde{\pi}^{ab}$ is easy to see by noting that the symmetric $\tilde{\pi}^{ab}$ must be constructed from the only objects at hand given flat slicing and spherical symmetry: n^a and e_{ab}.

Substituting these into the constraints gives

$$(f-3g)(f+g) + \tilde{\rho} = 0, \qquad (6)$$

$$f' + g' + \frac{2f}{r} + \tilde{P}^r = 0, \qquad (7)$$

where $\tilde{P}^r$ is the radial matter momentum. These are the two equations of interest: the unknown functions are $(f(r), g(r))$, with the given matter variables $(\tilde{\rho}, \tilde{P}^r)$. These equations may be rewritten in a form more amenable for general solution by replacing $g(r)$ with the canonical variable $\tilde{\pi}(r) := e_{ab}\tilde{\pi}^{ab} = f + 3g$, and solving the algebraic constraint:

$$f = \pm\tfrac{1}{2}\sqrt{\tilde{\pi}^2 - 3\tilde{\rho}}, \qquad (8)$$

$$f' + \frac{3f}{r} + \frac{\tilde{\pi}'}{2} + \frac{3}{2}\tilde{P}^r = 0. \qquad (9)$$

[1] After this work was submitted for publication, we learned of Ref. [6] where an initial value approach for flat slice foliations is also studied. Our approach stems from the slightly earlier work [7].

Finally, the substitution $\tilde{\pi}(r)=\sqrt{3\tilde{\rho}(r)}\cosh z(r)$ gives $f(r)=\pm\sqrt{3\tilde{\rho}}\sinh z(r)/2$, and Eq. (9) becomes

$$\pm z'+\frac{3}{2r}(1-e^{\mp 2z})+e^{\mp z}\frac{\sqrt{3}\tilde{P}^r}{\sqrt{\tilde{\rho}}}+\frac{\tilde{\rho}'}{2\tilde{\rho}}=0. \qquad (10)$$

Thus the problem is reduced to solving Eq. (10). This nonlinear ordinary differential equation can be solved at least numerically for given matter energy-momentum. In particular, for $\tilde{P}^r=0$ this gives a rather explicit relationship between $\tilde{\rho}$ and z, and hence $\tilde{\pi}$.

It is worth mentioning that a problem can arise since $\tilde{\rho}$ is a function of r, so that the expression inside the square root in Eq. (8) can become negative. This occurs, for example, in the foliation of the Reissner-Nordström geometry. It was dealt with in [3] by computing the hypersurface in the domain where the expression remains non-negative and then continuing numerically across the boundary (where it becomes negative) by using a Taylor expansion. This procedure may be adopted for other spacetimes where this problem occurs, including cases where there is more than one such boundary. In the following we discuss some special cases of interest, where the solutions are *explicit*.

(i) Schwarzschild spacetime ($\tilde{\rho}=\tilde{P}^r=0$). This case is most easily solved using Eq. (7). The solution is $f=3g$ with $f=Cr^{-3/2}$ where C is an integration constant. This constant is related to the Schwarzschild mass M by $C\sim\sqrt{M}$. Note that $f=-g$ also solves the Hamiltonian constraint, but this gives zero extrinsic curvature which is the Minkowski solution.

(ii) Reissner-Nordström spacetime ($\tilde{\rho}=Q^2/r^4$, and $\tilde{P}^r=\tilde{E}^r\partial_r A_r=0$ since the spatial part of the vector potential A_a vanishes for this metric). Equation (10) reduces to

$$z'\mp\frac{1}{2r}(3e^{\mp 2z}+1)=0. \qquad (11)$$

This has the general solution

$$z(r)=\pm\frac{1}{2}\ln(Cr-3). \qquad (12)$$

Thus

$$\tilde{\pi}=\frac{\sqrt{3}Q}{r^2}\cosh z=\frac{\sqrt{3}Q}{2r^2}\frac{Cr-2}{\sqrt{Cr-3}}. \qquad (13)$$

With the integration constant C written as $C=6M/Q$, the solutions are

$$f_{\pm}=\pm\frac{\sqrt{\tilde{\pi}^2-3Q^2/r^4}}{2}, \quad g_{\pm}=\frac{\tilde{\pi}}{3}\pm\frac{\sqrt{\tilde{\pi}^2-3Q^2/r^4}}{2}. \qquad (14)$$

This result agrees with the forms given in [3] obtained by constructing the flat slice foliation of the Reissner-Nordström metric via the use of freely falling frames.

(iii) de Sitter spacetime ($\tilde{\rho}=\sqrt{q}\Lambda>0$, $\tilde{P}^r=0$). Equation (10) is

$$z'\mp\frac{3}{2r}(e^{\mp 2z}-1)=0, \qquad (15)$$

which has the general solution

$$z=\pm\ln(\sqrt{1-K/r^3}) \quad (K>0). \qquad (16)$$

Thus

$$\tilde{\pi}=\pm\frac{1}{2}\sqrt{3\Lambda r^3/(r^3-K)}. \qquad (17)$$

(iv) Anti–de Sitter spacetime ($\tilde{\rho}=-\sqrt{q}/l^2$, $\tilde{P}^r=0$). Equation (10) may be rewritten as

$$(1\pm\sin z)z'\pm 3\frac{\cos z}{r}=0. \qquad (18)$$

This has the solution

$$z=\arctan\mp(1-Kr^3) \quad (K>0), \qquad (19)$$

which gives

$$\tilde{\pi}=\mp\frac{\sqrt{3}}{l}(1-Kr^3). \qquad (20)$$

In summary, we have given a simple method for explicitly finding flat slice initial data for arbitrary spherically symmetric spacetimes, with the general case summarized in Eq. (10). The method also gives a way to verify that all spherically symmetric spacetimes have flat slice foliations, via the existence of solutions of Eq. (10).

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