Spherically symmetric braneworld solutions with an $^{(4)}R$ term in the bulk

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An analysis of a spherically symmetric braneworld configuration is performed when the intrinsic curvature scalar is included in the bulk action; the vanishing of the electric part of the Weyl tensor is used as the boundary condition for the embedding of the brane in the bulk. All the solutions outside a static localized matter distribution are found; some of them are of the Schwarzschild-(A)dS$_4$ form. Two modified Oppenheimer-Volkoff interior solutions are also found; one is matched to a Schwarzschild-(A)dS$_4$ exterior, while the other is not. A nonuniversal gravitational constant arises, depending on the density of the considered object; however, the conventional limits of Newton's constant are recovered. An upper bound of the order of TeV for the energy string scale is extracted from the known solar system measurements (experiments). On the contrary, in the usual brane dynamics, this string scale is calculated to be larger than TeV.

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I. INTRODUCTION

The desire to explore physics beyond the standard model has led us to explore the ideas that spacetime is of a dimension larger than four, and that we are essentially confined to a four-dimensional hypersurface. String theories provide a framework for exploring such ideas, but nevertheless we are still far away from having a viable low-energy realization of these theories. Braneworld models are relevant world realizations in which some underlying features are often minimized. Replacing, for example, a whole field with a constant (solitonic solution) may probably oversimplify the reality but at the same time make it possible to obtain a more concrete picture, with the hope that any new behavior appearing will still be present in the more complete theory. Not only at the cosmological level, but also at a local one—concerning stars, galaxies, clusters of galaxies—has a brane solution to be consistent with the various astrophysical observations, which are often more reliable than the cosmological ones.

Attempts at obtaining braneworld solutions are cast into two categories. First, the bulk space assumes a given geometry, a coordinate system is adopted, and the influence on the brane geometry is somehow extracted. It seems a disadvantage of this approach that the bulk is prefixed and also that the brane embedding obtained is not gauge invariant (independent of the coordinate system chosen). Second, do not specify the exact bulk geometry, adopt a coordinate system adapted to the brane (Gauss normal coordinates or some relevant one), and deduce a brane dynamics, containing imprints from the bulk. Assumptions about the brane geometry are often sufficient to obtain an exactly closed brane dynamics. This approach allows for a brane dynamically interacting with a bulk, though this situation is not necessarily considered. A disadvantage of this method is that finding a bulk geometry in which the brane is the boundary may be a very difficult task. A probable advantage would be the extraction of common braneworld characteristics holding for a broad class of bulk backgrounds. In both approaches, if the codimension is one, Israel matching conditions are necessarily used. In the present paper we shall elaborate on the second approach.

The effective brane equations have been obtained [1] when the effective low-energy theory in the bulk is higher-dimensional gravity. However, a more fundamental description of the physics that produces the brane could include [2] higher-order terms in a derivative expansion of the effective action, such as a term for the scalar curvature of the brane, and higher powers of curvature tensors on the brane. A brane action that contains powers of the brane curvature tensors has also been used in the context of the AdS conformal field theory (CFT) correspondence (e.g., [3]) to regularize the action of a bulk AdS space which diverges when the radius of the AdS space becomes infinite. If the dynamics is governed not only by the ordinary five-dimensional Einstein-Hilbert action, but also by the four-dimensional Ricci scalar term induced on the brane, new phenomena appear. In [4,5] it was observed that the localized matter fields on the brane (which couple to bulk gravitons) can generate via quantum loops a localized four-dimensional worldvolume kinetic term for gravitons (see also [6–9]). That is to say, four-dimensional gravity is induced from the bulk gravity to the brane worldvolume by the matter fields confined to the brane. It was also shown that an observer on the brane will see correct Newtonian gravity at distances shorter than a certain crossover scale, despite the fact that gravity propagates in extra space which was assumed there to be flat with infinite extent; at larger distances, the force becomes higher dimensional. The first realization of the induced gravity scenario in string theory was presented in [10]. Furthermore, new closed string couplings on Dp-branes for the bosonic string were found in

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[11]. These couplings are quadratic in derivatives and therefore take the form of induced kinetic terms on the brane. For the graviton in particular these are the induced Einstein-Hilbert term as well as terms quadratic in the second fundamental tensor. Considering the intrinsic curvature scalar in the bulk action, the effective brane equations have been obtained in [12]. Results concerning cosmology have been discussed in [13–17].

The original Randall-Sundrum models [18], based on a Minkowski brane and a specific relation between the bulk cosmological constant and the brane tension, have drawn much attention because they might be realizable in supergravity and superstring compactifications [19–22]. However, any Ricci-flat four-dimensional metric can be embedded (with the common warped embedding) in $(\mathrm{A})\mathrm{dS}_5$ (e.g., [23,24]). In this way, a black-string solution [23,25–27] can easily be constructed. Furthermore, it is known that any four-dimensional Einstein spaces can foliate an $(\mathrm{A})\mathrm{dS}_5$ bulk [28–33]. Thus, asymptotically nonflat black holes [Schwarzschild-$(\mathrm{A})\mathrm{dS}_4$] can be obtained as slices of the above precise bulks. Almost all treatments on spherically symmetric braneworld solutions, such as those previously mentioned, representing, for example, the exterior of a star, do not take account of the finite extension of the object. Until now, there is no known exact five-dimensional solution for astrophysical brane black holes. Furthermore, looking for bulks having some interior star solution as part of their boundaries is even harder. In [34–36], some interior and exterior solutions were found, without including the ${}^{(4)}R$ term.

In the present paper, we discuss the gravitational field of an uncharged, nonrotating spherically symmetric rigid object when there is a contribution in the dynamics from the brane intrinsic curvature invariant. In Sec. II, we find all the possible exterior solutions containing one undetermined parameter, which is the parameter of the Newtonian term. Some of these solutions are of the Schwarzschild-$(\mathrm{A})\mathrm{dS}_4$ form. In two cases, we can also solve the interior problem which reduces to a generalization of the Oppenheimer-Volkoff solution, and thus determine the unknown parameter. This is found to be different from the conventional value of a localized spherically symmetric distribution within the framework of four-dimensional general relativity. Hence, a nonuniversal Newton's constant, depending on the density of the object, naturally arises. In Sec. III, taking account of the classical experiments of gravity in the solar system, we can set an upper bound for the five-dimensional Planck mass of the order of TeV. The revival of the conventional results is discussed, and also a comparison with the more standard brane dynamics is presented. Finally, in Sec. IV are our conclusions.

II. FOUR-DIMENSIONAL SPHERICALLY SYMMETRIC SOLUTIONS

We consider a three-dimensional brane Σ (with normal vector field n^A) embedded in a five-dimensional spacetime M. Capital latin letters $A,B,\ldots=0,1,\ldots,4$ denote full spacetime, lower-case greek $\mu,\nu,\ldots=0,1,\ldots,3$ run over the brane worldvolume, while lower-case latin letters span some three-dimensional spacelike surface foliating the brane, i.e., $i,j,\ldots=1,\ldots,3$. For convenience, we can quite generally choose a coordinate y such that the hypersurface $y=0$ coincides with the brane. The total action for the system is taken to be

$$S=\frac{1}{2\kappa_5^2}\int_M\sqrt{-{}^{(5)}g}({}^{(5)}R-2\Lambda_5)d^5x+\frac{1}{2\kappa_4^2}\int_\Sigma\sqrt{-{}^{(4)}g}({}^{(4)}R-2\Lambda_4)d^4x+\int_M\sqrt{-{}^{(5)}g}L_5^{mat}d^5x+\int_\Sigma\sqrt{-{}^{(4)}g}L_4^{mat}d^4x. \tag{1}$$

For clarity, we have separated the cosmological constants Λ_5,Λ_4 from the rest of the matter content L_5^{mat},L_4^{mat} of the bulk and the brane, respectively. Λ_4/κ_4^2 can be interpreted as the brane tension of the standard Dirac-Nambu-Goto action, or as the sum of a brane worldvolume cosmological constant and a brane tension. We are basically concerned with the case with no fields in the bulk, i.e., ${}^{(5)}T_{AB}=0$.

From the dimensionful constants κ_5^2,κ_4^2 the Planck masses M_5,M_4 are defined as

$$\kappa_5^2=8\pi G_{(5)}=M_5^{-3},\quad \kappa_4^2=8\pi G_{(4)}=M_4^{-2}, \tag{2}$$

with M_5,M_4 having dimensions of $(\text{length})^{-1}$. Then, a distance scale r_c is defined as

$$r_c\equiv\frac{\kappa_5^2}{\kappa_4^2}=\frac{M_4^2}{M_5^3}. \tag{3}$$

Varying Eq. (1) with respect to the bulk metric g_{AB}, we obtain the equations

$${}^{(5)}G_{AB}=-\Lambda_5 g_{AB}+\kappa_5^2[{}^{(5)}T_{AB}+{}^{(loc)}T_{AB}\delta(y)], \tag{4}$$

where

$${}^{(loc)}T_{AB}\equiv-\frac{1}{\kappa_4^2}\sqrt{\frac{-{}^{(4)}g}{-{}^{(5)}g}}({}^{(4)}G_{AB}-\kappa_4^2\,{}^{(4)}T_{AB}+\Lambda_4 h_{AB}) \tag{5}$$

is the localized energy-momentum tensor of the brane. ${}^{(5)}G_{AB},{}^{(4)}G_{AB}$ denote the Einstein tensors constructed from the bulk and the brane metrics, respectively. Clearly, ${}^{(4)}G_{AB}$ acts as an additional source term for the brane through ${}^{(loc)}T_{AB}$. The tensor $h_{AB}=g_{AB}-n_An_B$ is the induced metric on the hypersurface y=constant, with n^A the normal vector on these.

The way the y coordinate has been defined allows us to write, at least in the neighborhood of the brane, the five-line element in the block diagonal form

$$ds_{(5)}^2=-N^2dt^2+g_{ij}dx^idx^j+dy^2, \tag{6}$$

where N,g_{ij} are generally functions of t,x^i,y. The distributional character of the brane matter content makes necessary for the compatibility of the bulk equations (4) the following modified (due to ${}^{(4)}G^{\mu}_{\nu}$) Israel-Darmois-Lanczos-Sen conditions [37–40]:

$$[K^{\mu}_{\nu}]=-\kappa_5^2\left({}^{(loc)}T^{\mu}_{\nu}-\frac{{}^{(loc)}T}{3}\,\delta^{\mu}_{\nu}\right), \qquad (7)$$

where the square brackets mean discontinuity of the extrinsic curvature $K_{\mu\nu}=(1/2)\partial_y g_{\mu\nu}$ across $y=0$. A $\mathbf{Z}_2$ symmetry on reflection around the brane is considered throughout.

One can derive from Eqs. (4),(7) the induced brane gravitational dynamics [12], which consists of a four-dimensional Einstein gravity, coupled to a well-defined modified matter content. More explicitly, one gets

$${}^{(4)}G^{\mu}_{\nu}=\kappa_4^2\,{}^{(4)}T^{\mu}_{\nu}-\left(\Lambda_4+\frac{3}{2}\alpha^2\right)\delta^{\mu}_{\nu}+\alpha\left(L^{\mu}_{\nu}+\frac{L}{2}\,\delta^{\mu}_{\nu}\right), \qquad (8)$$

where $\alpha\equiv 2/r_c$, while the quantities L^{μ}_{ν} are related to the matter content of the theory through the equation

$$L^{\mu}_{\lambda}L^{\lambda}_{\nu}-\frac{L^2}{4}\,\delta^{\mu}_{\nu}=\mathcal{T}^{\mu}_{\nu}-\frac{1}{4}(3\alpha^2+2\mathcal{T}^{\lambda}_{\lambda})\,\delta^{\mu}_{\nu}, \qquad (9)$$

and $L\equiv L^{\mu}_{\mu}$. The quantities $\mathcal{T}^{\mu}_{\nu}$ are given by the expression

$$\mathcal{T}^{\mu}_{\nu}=\left(\Lambda_4-\frac{1}{2}\Lambda_5\right)\delta^{\mu}_{\nu}-\kappa_4^2\,{}^{(4)}T^{\mu}_{\nu}+\frac{2}{3}\kappa_5^2\left[{}^{(5)}\overline{T}^{\mu}_{\nu}+\left({}^{(5)}\overline{T}^{y}_{y}-\frac{{}^{(5)}\overline{T}}{4}\right)\delta^{\mu}_{\nu}\right]-\overline{\mathsf{E}}^{\mu}_{\nu}, \qquad (10)$$

with ${}^{(5)}\overline{T}={}^{(5)}\overline{T}^{A}_{A}$, ${}^{(5)}\overline{T}^{A}_{B}=g^{AC}\,{}^{(5)}\overline{T}_{CB}$. The overbars on ${}^{(5)}T^{A}_{B}$ and the electric part $\mathsf{E}^{\mu}_{\nu}=C^{\mu}_{A\nu B}n^A n^B$ of the five-dimensional Weyl tensor C^{A}_{BCD} mean that the quantities are evaluated at $y=0$. $\overline{\mathsf{E}}^{\mu}_{\nu}$ carries the influence of nonlocal gravitational degrees of freedom in the bulk onto the brane [1] and makes the brane equations (8) not closed in general. This means that there are bulk degrees of freedom which cannot be predicted from data available on the brane. One needs to solve the field equations in the bulk in order to determine E^{μ}_{ν} on the brane. In the present paper, to make Eq. (8) closed, we shall set $\overline{\mathsf{E}}^{\mu}_{\nu}=0$ as a boundary condition of the propagation equations in the bulk space. This is somewhat simplified from the viewpoint of geometric complexity, but it is the first step for investigating the characteristics carried by the brane curvature invariant on the local brane dynamics we are interested in. Treatments and solutions without this assumption, in the context of the usual brane dynamics, have been given in [26,41,42,43,34–36,44]. Because of the block-diagonal form of the metric (6) the solution of the algebraic system (9), whenever

$$\mathcal{T}^{i}_{j}=\tau\delta^{i}_{j}, \qquad (11)$$

is

$$L^{0}_{0}=\pm\frac{1}{2\mathsf{B}}[(7-4n_+n_-)\mathcal{T}^{0}_{0}-(3-4n_+n_-)\tau+3\alpha^2], \qquad (12)$$

$$L^{i}_{j}=SE^{i}_{j}, \qquad (13)$$

$$L^{0}_{i}=L^{i}_{0}=0, \qquad (14)$$

where

$$S=\frac{1}{2\mathsf{B}}|\mathcal{T}^{0}_{0}+3(\tau+\alpha^2)|, \qquad (15)$$

$$\mathsf{B}=[-6(n_+-1)(n_--1)\mathcal{T}^{0}_{0}+2n_+n_-(1-n_+n_-)\tau+3(3-2n_+n_-)\alpha^2]^{1/2}, \qquad (16)$$

while the matrix E^{i}_{j} is either diag$(+1,+1,+1)$ (with $n_+=3,n_-=0$) or diag$(+1,+1,-1)$ (with $n_+=2,n_-=1$).

Inspecting Eq. (8), we see that the inclusion of the term ${}^{(4)}R$ has brought a convenient decomposition of the matter terms. First, the standard energy-momentum tensor enters without having made any choice for the brane tension Λ_4 in terms of M_4,M_5 (in [45,1] it has to be $\Lambda_4=3\alpha^2/2$). Note that if ${}^{(4)}R$ is not included in the action, for $\Lambda_4=0$, ordinary energy-momentum terms cannot arise. Furthermore, in that case, Λ_4 has to be positive in order for κ_4^2 to be positive. Second, the additional matter terms (which appear here as square roots instead of squares of the four-dimensional energy-momentum tensor) all contain the factor α of the energy string scale. Thus, conventional four-dimensional general relativity is revived on some region of a four-spacetime, whenever these extra terms remain suppressed relative to the conventional ones; the specific value of α determines the regional validity of general relativity.

From now on, we are interested in static (noncosmological) local braneworld solutions arising from the action (1). More specifically we consider a spherically symmetric line element

$$ds^2_{(4)}=-B(r)dt^2+A(r)dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2). \qquad (17)$$

The matter content of the three-universe is a localized spherically symmetric untilted perfect fluid (e.g., a star) ${}^{(4)}T_{\mu\nu}=(\rho+p)u_{\mu}u_{\nu}+pg_{\mu\nu}$ with $\rho=p=0$ for $r>R$, plus the cosmological constant Λ_4. The matter content of the bulk is a cosmological constant Λ_5. This matter content enters $\mathcal{T}^{\mu}_{\nu}$ in Eq. (10) and thus determines L^{μ}_{ν} on the right hand side of our dynamical equations (8). The result is

$$L^{0}_{0}=\pm\frac{1}{2\mathsf{B}}\{|4\Lambda_4-2\Lambda_5+3\alpha^2|+\kappa_4^2[(7-3n_+n_-)\rho+(n_++3n_-)p]\}, \qquad (18)$$

$$S=\frac{1}{2\mathsf{B}}|4\Lambda_4-2\Lambda_5+3\alpha^2+\kappa_4^2(\rho-3p)|, \qquad (19)$$

$$\mathsf{B}=\{(3-4n_-)(4\Lambda_4-2\Lambda_5+3\alpha^2)-4\kappa_4^2[3(n_--1)\rho+(n_+-3)p]\}^{1/2}, \qquad (20)$$

with the only restriction imposed by the square root appearing in B. Thus, necessarily, $4\Lambda_4-2\Lambda_5+3\alpha^2$ is nonnegative (nonpositive) for $E^i_j=\delta^i_j$ (for the other choice of E^i_j).

For the metric (17), one evaluates the Ricci tensor ${}^{(4)}R_{\mu\nu}$ and then constructs the field equations (8). The combination ${}^{(4)}R_{rr}/2A+{}^{(4)}R_{\theta\theta}/r^2+{}^{(4)}R_{00}/2B$ provides the following differential equation for $A(r)$:

$$\left(\frac{r}{A}\right)'=1-\kappa_4^2\rho(r)r^2-\left(\Lambda_4+\frac{3}{2}\alpha^2\right)r^2+\frac{\alpha}{2}[3L_0^0+(n_+-n_-)S]r^2, \tag{21}$$

$('\equiv d/dr)$. Eliminating A'/A from Eq. (21) in the $(\theta\theta)$ component of Eq. (8), we get an equation for B'/B, from which we obtain

$$\frac{(AB)'}{AB}=Ar\{\kappa_4^2(\rho+p)-\alpha[L_0^0+(2-n_++n_-)S]\}. \tag{22}$$

There are various different cases (namely, eight) according to the choice of E^i_j and the alternative signs of L_0^0,S. However, in the outside region, there are only four different cases, according to n_+,n_- and the $\pm$ sign in Eq. (18). In all these cases, we can integrate Eq. (21) in the outside region, obtain the solution $A_>(r)$, and from Eq. (22) get the solution $B_>(r)$. The result is

$$\frac{1}{A_>(r)}=1-\frac{\gamma}{r}-\beta r^2,\quad r\geqslant R, \tag{23}$$

$$B_>(r)=\frac{1}{A_>(r)}F_{n_+,n_-}(r),\quad r\geqslant R, \tag{24}$$

with

$$\beta=\frac{1}{3}\Lambda_4+\frac{1}{2}\alpha^2-\alpha\frac{n_+-n_-\pm 3}{12\sqrt{|3-4n_-|}}\sqrt{|4\Lambda_4-2\Lambda_5+3\alpha^2|}, \tag{25}$$

$$F_{n_+,n_-}(r)=1+[f(r)^{\alpha r_1\{[n_+-(2+3\sqrt{3})n_-]/6\sqrt{3}\beta(3\gamma-2r_1)\}\sqrt{|4\Lambda_4-2\Lambda_5+3\alpha^2|}}-1]\delta_{n_+\mp 1,4-3n_-}, \tag{26}$$

$$f(r)=(r-r_1)\left(\frac{r}{A_>}\right)^{1-2\gamma/r_1}g(r)^{\sqrt{|r_1-\gamma|/(r_1+3\gamma)}}, \tag{27}$$

where r_1 is the minimum horizon distance and $g(r)$ is equal to

$$|[r+r_1/2+\sqrt{(r_1+3\gamma)/4\beta r_1}]/[r+r_1/2-\sqrt{(r_1+3\gamma)/4\beta r_1}]|$$

for $\beta>0$, or $e^{2\arctan\sqrt{4|\beta|r_1/(r_1+3\gamma)}(r+r_1/2)}$ for $\beta<0$. For $\beta=0$, $g(r)^{\sqrt{|r_1-\gamma|}/\beta}$ is replaced by $(r-\gamma)^{4\gamma^{5/2}}e^{2\sqrt{\gamma}(r+3\gamma)(r-\gamma)}$. The $\pm$, $\mp$ signs appearing in Eqs. (25), (26) correspond to the $\pm$ sign of Eq. (18). A multiplicative constant of integration for $B_>$ has been absorbed into a redefinition of time and γ is a constant of integration. Note that for $\beta\leqslant 0$ there is only one horizon $r_1<\gamma$, while for $\beta>0$ (and $27\beta\gamma^2<4$ to have well defined horizons) there are two horizons $\gamma<r_1<3\gamma$ and $1/\sqrt{3\beta}<r_2<1/\sqrt{\beta}$.

The solutions (23),(24) are not yet completely defined unless the parameter γ is determined, i.e., the interior solution is found. In the case $E^i_j=\delta^i_j$, we can find two situations where Eq. (21) does not contain p and so we can integrate it in the interior region [we give these solutions below, equation (31)].

As it is seen from Eqs. (24) and (26), all the exterior solutions are either of the form where $A_>,B_>$ are inverse to each other, or of the form where the product $A_>B_>$ is equal to $f(r)$ to a power appearing in Eq. (26). The first class of these solutions is of the Schwarzschild-(A)dS$_4$ form, while the second is not. For zero β (we can interpret β as the effective brane cosmological constant) the first class of these solutions reduces to Schwarzschild-like, while the second does not. Non-Schwarzschild-like exterior solutions were also obtained in [34,41,36,44], but this fact was attributed to the nonvanishing $\bar{\mathsf{E}}^{\mu}_{\nu}$. Such irregular behavior also appears here, due to the intrinsic curvature invariant, without involving nonlocal bulk effects on the brane. There is one case of our non-Schwarzschild–(A)dS$_4$ solutions with $\beta>0$, $\gamma/r_1=2/3$, where at large distances—given that the second horizon is actually at cosmological distances—$A_>B_>$ is almost one, i.e., the solution asymptotes to the Schwarzschild-dS$_4$ solution.

If we take the covariant derivative (denoted by $_|$) with respect to the induced brane metric $h_{AB}=g_{AB}-n_An_B$ of Eqs. (7), and make use of Codacci's equations, and of the bulk equations (4), we arrive at the equations ${}^{(4)}T^A_{B|A}=-[{}^{(5)}T_{CD}]n^Ch^D_B$. When the matter content of the bulk space is only a cosmological constant, then the common conservation law of our world is obtained. For the static case we are discussing, this law is equivalent to the equation

$$\frac{B'}{B}=-\frac{2p'}{\rho+p}. \tag{28}$$

Thus, for $r\leqslant R$ we get the equation for $p(r)$:

$$\frac{p'}{\rho+p}=\frac{1-A}{2r}-\frac{Ar}{2}\left[\kappa_4^2p-\left(\Lambda_4+\frac{3}{2}\alpha^2\right)+\frac{\alpha}{2}L_0^0-\frac{3}{2}\alpha\left(\frac{4}{3}-n_++n_-\right)S\right]. \tag{29}$$

We assume a uniform distribution $\rho(r)=\rho_o=3M/4\pi R^3$ for $r\leqslant R$. Then, the immediate integration of Eq. (28) gives

$$B_<(r)=\frac{(1-\gamma/R-\beta R^2)F_{n_+,n_-}(R)}{[1+(4\pi R^3/3M)p(r)]^2},\quad r\leqslant R, \tag{30}$$

in which, the continuity of $B(r)$ at $r=R$ and the condition $p(R)=0$ have been used. The vanishing of the pressure at the surface, which is certainly physically reasonable, is a consequence of the application of the Israel matching conditions at the stellar surface [46,47]. The pressure $p(r)$ in Eq. (30) is found from Eq. (29).

Now, we proceed, as we said before, with the two cases where we can solve the system of equations (21),(29). Both have $E^i_j=\delta^i_j$. The first case corresponds to the upper sign of the $\pm$ sign in Eq. (18), and the quantity inside the absolute value of Eq. (19) in the interior of the stars being positive. The second case corresponds to the lower $\pm$ sign in Eq. (18) and negative quantity in Eq. (19). In these cases, integration of Eq. (21) gives

$$\frac{1}{A_<(r)}=1-\left(\beta+\frac{\gamma}{R^3}\right)r^2,\quad r\leqslant R. \tag{31}$$

The parameters γ and β [from Eq. (25)] are given in terms of $M,\alpha,\Lambda_5,\Lambda_4$, by:

First solution

$$\frac{\gamma}{R^3}=\frac{\kappa_4^2 M}{4\pi R^3}+\frac{\alpha}{2\sqrt{3}}\sqrt{4\Lambda_4-2\Lambda_5+3\alpha^2}-\frac{\alpha}{2\sqrt{3}}\sqrt{4\Lambda_4-2\Lambda_5+3\alpha^2+\frac{3\kappa_4^2 M}{\pi R^3}}, \tag{32}$$

$$\beta=\frac{1}{3}\Lambda_4+\frac{1}{2}\alpha^2-\frac{\alpha}{2\sqrt{3}}\sqrt{4\Lambda_4-2\Lambda_5+3\alpha^2}. \tag{33}$$

Second solution

$$\frac{\gamma}{R^3}=\frac{\kappa_4^2 M}{4\pi R^3}+\frac{\alpha}{2\sqrt{3}}\sqrt{4\Lambda_4-2\Lambda_5+3\alpha^2+\frac{3\kappa_4^2 M}{\pi R^3}}, \tag{34}$$

$$\beta=\frac{1}{3}\Lambda_4+\frac{1}{2}\alpha^2. \tag{35}$$

The first solution, as it is seen from Eq. (26) and Eq. (24), is matched to a Schwarzschild-(A)dS$_4$ exterior solution, while the second solution is matched to a non–Schwarzschild-(A)dS$_4$ exterior solution. Note that no additional constant of integration enters the above solution since we have required that the metric is nondegenerate at $r=0$. In the special case with $4\Lambda_4-2\Lambda_5+3\alpha^2=0$, Eq. (34) is matched to an exterior Schwarzschild-(A)dS$_4$ solution.

From Eq. (29), $p(r)$ for our two solutions is found to be

$$p(r)=-\rho_0\frac{\sqrt{1-(\beta+\gamma/R^3)r^2}\ominus\sqrt{1-(\beta+\gamma/R^3)R^2}}{\sqrt{1-(\beta+\gamma/R^3)r^2}\ominus\omega\sqrt{1-(\beta+\gamma/R^3)R^2}}, \tag{36}$$

where

$$\omega^{-1}=1-\frac{2}{\kappa_4^2\rho_0}\left(\beta+\frac{\gamma}{R^3}\right)\times\left(1\mp\frac{\sqrt{3}\alpha}{\sqrt{4\Lambda_4-2\Lambda_5+3\alpha^2+4\kappa_4^2\rho_0}}\right)^{-1}. \tag{37}$$

The symbol $\ominus$ means $-$, except from the (rather irregular) case with $\omega<0, p>\rho_0/|\omega|$, where it becomes $+$. In the limit $\alpha,\Lambda_4\to 0$ both solutions for $A_<(r),B_<(r),p(r)$ reduce to the known Oppenheimer-Volkoff solution. Also, in the limit $\alpha\to 0$, the exterior solutions corresponding to Eqs. (32),(33) and Eqs. (34),(35) reduce to the Kottler [48,49] solution of four-dimensional general relativity.

It is of some importance to notice the following. Although three unrelated parameters $\alpha,\Lambda_4,\Lambda_5$ (which are supposed to be fundamental) enter our problem, the final exterior solutions contain only two combinations of them, namely, the parameters γ,β. Thus, from exterior experimental data only two constraints on $\alpha,\Lambda_4,\Lambda_5$ can be extracted. However, the interior solutions contain, furthermore, the parameter ω, which means that a third combination of $\alpha,\Lambda_4,\Lambda_5$ could be obtained from possible astrophysical information. Thus, $\alpha,\Lambda_4,\Lambda_5$ can be uniquely determined from local measurements. Of course, as is seen from Eq. (37), if the parameters $\alpha,\Lambda_4,\Lambda_5$ are extremely small (as will be seen in the next section), the influence of the bulk effects onto the interior solution is also small.

It can be seen from Eq. (32) that for a given set of parameters $\alpha,\Lambda_4,\Lambda_5$, the relative change $(\gamma/2G_{(4)}M)-1$ in the parameter of the Newtonian term is negative, and it is an increasing function of ρ_0. This deviation from the common situation can be interpreted as an object-dependent gravitational constant, while M remains unchanged, i.e., $\gamma=2G_{(4)}(\rho_0)M$, where

$$G_{(4)}(\rho_0)/G_{(4)}=1+2\left(1+\frac{4\Lambda_4-2\Lambda_5}{3\alpha^2}\right)^{-1/2}\times\frac{1}{s\rho_0}(1-\sqrt{1+s\rho_0})$$

and

$$s=32\pi\frac{G_{(4)}}{3\alpha^2}\left(1+\frac{4\Lambda_4-2\Lambda_5}{3\alpha^2}\right)^{-1}.$$

Then, $G_{(4)}(\rho_0)$ starts from the value $G_{(4)}\{1-[1+(4\Lambda_4-2\Lambda_5)/3\alpha^2]^{-1/2}\}$ when $\rho_0\to 0$, and asymptotically tends to $G_{(4)}$ for $\rho_0\to\infty$. In this picture, G_N, the measured Newton's constant, is not a universal quantity, but simply corresponds to $G_{(4)}(\rho_{0,everyday})$, where $\rho_{0,everyday}$ is the density of com-

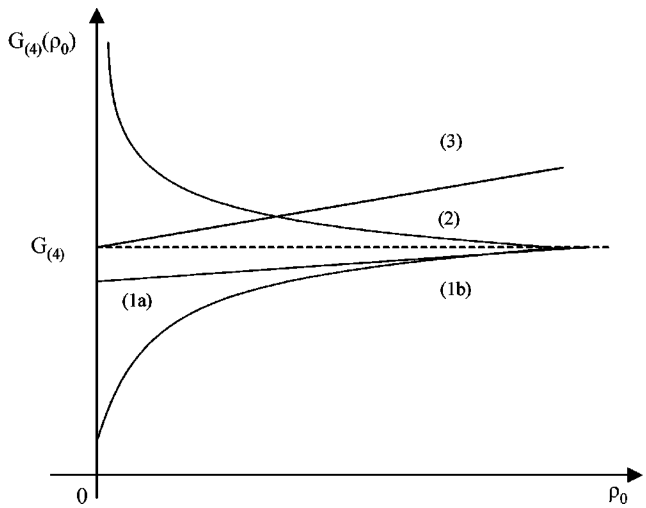

FIG. 1. The ρ_0 dependence of Newton's constant in various models.

mon matter $\sim$g/cm^3. There is a characteristic value of energy, which can be associated with these densities, namely, $\alpha_e = \sqrt{G_N \rho_{0,everyday}} \sim 10^{-14}$ cm^{-1}. If $4\Lambda_4 - 2\Lambda_5 \gg 3\alpha^2$ (plot 1a in Fig. 1), $G_{(4)}(\rho_0)$ is always almost equal to $G_N \simeq G_{(4)}$, and no significant deviations from Newton's constant universality exist. Otherwise (plot 1b in Fig. 1), significant deviations from G_N can arise. Thus, there exist only two situations which do not contradict the everyday experience of no deviation from Newton's constant universality. These are $\alpha \ll \alpha_e$ or $\alpha \gg \alpha_e$. In the first case, $G_N \simeq G_{(4)}$ and significant deviations from G_N appear at extremely low densities $\rho_0 \ll \alpha^2/G_N$. In the second case, $G_N \simeq G_{(4)}\{1-[1+(4\Lambda_4 - 2\Lambda_5)/3\alpha^2]^{-1/2}\}$ and significant deviations appear for extremely dense objects. In the next section, we will set upper bounds on α, similar to $\alpha \ll \alpha_e$, from solar system experiments, and thus the second case is excluded. If this is really the situation, the possibility for the parameters to have $4\Lambda_4 - 2\Lambda_5 < 0$, which leads to repulsive gravity on very low density objects, is possible. Similar behavior to that described above, but with extra attraction, appears in solution (34) (plot 2 in Fig. 1).

The solution (32),(33), since it is matched to an exterior Schwarzschild-(A)dS$_4$ solution, will be used in the next section to bound the parameters encountered from experimental data of our solar system (deflection of light coming from distant stars, precession of perihelia, and radar echo delay). Since there are two parameters β, γ in the exterior solution, connected to the three $\alpha, \Lambda_4, \Lambda_5$, it is necessary to drop one of these three by hand, in order to get an estimation of the other two. It is obvious that α cannot be this one, since this is too restrictive and in fact analyses of this case have been performed [50,51]. Also, we do not set $\Lambda_5 = 0$, since then β cannot be negative (β negative implies $\Lambda_5 < 0$). Although β is the same quantity that in cosmology plays the role of the effective cosmological constant [16,12] and it is then positive, in the present work we do not claim any connection with cosmology, so we would prefer to be able to also deal with a negative β. As will be discussed in the next section, this may be of importance for galactic scale phenomena. In the following, we choose $\Lambda_4 = 0$. Then, from Eqs. (32),(33) we find that for $G_{(4)} \simeq G_N$ the values are

$$\alpha^2 = \frac{1}{\gamma R^3}(2G_N M - \gamma)(2G_N M - \gamma - 2\beta R^3), \tag{38}$$

$$\Lambda_5 = 6\beta\left(1 - \frac{\beta\gamma R^3}{(2G_N M - \gamma)(2G_N M - \gamma - 2\beta R^3)}\right), \tag{39}$$

while in the other limiting case

$$\alpha^2 = 2\beta + \frac{2\gamma^2}{R^3}\left[\gamma - 2G_N M - \frac{2\gamma}{\beta R^3}G_N M \pm \sqrt{(2G_N M - \gamma)\left(2G_N M - \gamma + \frac{4\gamma}{\beta R^3}G_N M\right)}\right]^{-1}. \tag{40}$$

Finally, we note that the non-Schwarzschild–(A)dS$_4$ solution (34),(35) could also be used for extracting pheonmenological bounds on the string parameters from the solar-system experiments. We have chosen in this paper the simplest solution for this purpose. However, it is known [52] that the agreement with the solar-system tests of some metric-based relativistic theory requires on kinematical grounds that $AB \simeq 1$ to high accuracy in the vicinity of the sun.

III. CONSTRAINTS FROM CLASSICAL TESTS

A difficulty arising with the calculations of measurable quantities (integrals) comes from the fact that the solution (32),(33) is not asymptotically flat, but diverges at large distances; thus, an expansion in powers of $1/r$, performed in the standard PPN (parametrized post-Newtonian) analysis, does not work here. Hence, one has to make expansions according to parameters of the problem that are sufficiently small, and fortunately such parameters exist.

The motion of a freely falling material particle or photon in a static isotropic gravitational field [Eq. (17)] is described [53] by the equation

$$\left(\frac{d\phi}{dr}\right)^2 = \frac{A_>}{r^4}\left(\frac{1}{J^2 B_>} - \frac{1}{r^2} - \frac{E}{J^2}\right)^{-1}, \tag{41}$$

where J, E are constants of integration ($E > 0$ for material particles and $E = 0$ for photons). At the points of minimum or maximum distance r_0, $dr/d\phi = 0$, and thus

$$J = r_0\left(\frac{1}{B_>(r_0)} - E\right)^{1/2}. \tag{42}$$

We will analyze the three classical solar scale experiments—deflection of electromagnetic waves coming from distant stars by the sun, precession of the perihelia of planets, and time delay of radio waves.

(1) *Deflection of light*. Although the metric is not asymptotically flat, the photon, as can be seen from Eq. (41), has $d\phi/dr \to 0$ as $r \to \infty$, and thus, it moves in a "straight" line

of the background geometry in that region. The deviation from this line is measured by the total deflection angle $\Delta\phi_d=2|\phi(r_0)-\phi(\infty)|-\pi$, where r_0 is the minimum distance of the orbit to the sun (when a ray grazes the sun $r_0=R_\odot$). As is expected and will be shown below, $|\beta|$ has an extremely small value; thus, for $\beta>0$ the horizon r_2 is of cosmological scale and scattering of light can be practically defined even in this case. The angular momentum J is related to the "impact parameter" b through the relation $J=\mathrm{b}(1-\beta\mathrm{b}^2)^{-1/2}$. Integrating Eq. (41) we arrive at

$$\phi(r)-\phi(\infty)=\sqrt{\frac{r_0^3}{r_0-\gamma}}\int_r^\infty\left[r(r-r_0)\times\left(r^2+r_0r-\frac{\gamma r_0^2}{r_0-\gamma}\right)\right]^{-1/2}dr. \quad (43)$$

For the above expression to be well defined, we must have $r_0\geqslant\frac{3}{2}\gamma$ (which is always the case for common stars such as our sun). Note that the parameter β has disappeared in the expression (43), i.e. the deflection phenomenon is the same as if it had occurred in a Schwarzschild field of parameter γ. The expression (43) leads to an elliptic integral. Since γ is almost $2G_NM_\odot$ and r_0 is of the order of $R_\odot$, γ/r_0 is of the order of 10^{-6}. Hence, we can expand the integrand of Eq. (43) to first order in this parameter before integration [54]. It is convenient, simultaneously, to set $\sin u=r_0/r$, and the result is

$$\Delta\phi_d=\frac{2\gamma}{r_0}. \quad (44)$$

The best measurements of the deflection of light from the sun were obtained using radio-interferometric methods [55] and the result found (for $r_0=R_\odot$) was $\Delta\phi_d=1.761\pm0.016$ arc sec. Then, from Eq. (44),

$$29.440\times10^4\ \mathrm{cm}<\gamma<29.979\times10^4\ \mathrm{cm}, \quad (45)$$

which is around the conventional value $2G_NM_\odot=29.539\times10^4$ cm.

(2) Precession of perihelia. Here, there are two values r_+,r_- of maximum and minimum distance satisfying Eq. (42). The two constants of motion J,E are expressed in terms of r_+,r_- and are plugged into Eq. (41). The expression arising is very complicated, but referring to [50,51] we can write the precession per orbit $\Delta\phi_p=2|\phi(r_+)-\phi(r_-)|-\pi$ as

$$\Delta\phi_p=\frac{3\pi\gamma}{L}+\frac{6\pi\beta L^3}{\gamma}, \quad (46)$$

where $L^{-1}=(r_+^{-1}+r_-^{-1})/2$ is the *semilatus rectum* of the orbit. Both [50,51] agree on the result Eq. (46). Actually, they refer to the Gibbons-Hawking metric, but their methods can be immediately applied in our case. They disagree on the next order terms, which are, however, negligible compared to the second term of Eq. (46) for stars with small Schwarzschild radius and for slightly eccentric orbits.

For Mercury, the uncertainty in the quantity $\Delta\phi_p-6\pi G_NM_\odot/L$ is $\pm10^{-9}$ rad/orbit. Then, taking into account the range (45) of γ received from deflection, we obtain

$$-7.908\times10^{-43}\ \mathrm{cm}^{-2}<\beta<2.465\times10^{-43}\ \mathrm{cm}^{-2}. \quad (47)$$

The bounds (45),(47) give, from Eq. (38),

$$\alpha<4.379\times10^{-16}\ \mathrm{cm}^{-1}. \quad (48)$$

Actually, as long as the upper bound of $|\beta|$ remains many orders of magnitude smaller than $G_NM_\odot/R_\odot$, the above result, as can be seen from Eq. (38), is insensitive to the exact value of β. Furthermore, the fact that α has an upper instead of a lower bound is due to the specific functional form of the expression (38) in terms of γ. This means that the crossover scale $r_c>4.567\times10^{15}$ cm, i.e., the lower bound of r_c is a few times the diameter of our planetary system. Thus, the five-dimensional fundamental Planck scale M_5 is less than 0.9 TeV. From Eq. (40), one can see that for $\beta\to0$, $\alpha\to0$ and then, from Eqs. (45),(47), an upper bound of the order of 10^{-22} cm^{-1} is set for α, which is incompatible with $\alpha\gg\alpha_e$. Thus, this case is not acceptable.

From Eq. (39), an upper bound for Λ_5 can be obtained:

$$\Lambda_5<3.804\times10^{-43}\ \mathrm{cm}^{-2}. \quad (49)$$

Uncertainties in the measurement of the precession of perihelion are known to exist, due to the rotation of the sun; thus, it is better to examine the bounds on β from the radar echo delay independently.

(3) Radar echo delay. The time required for a radar signal to go from a point r to the closest point r_0 of its orbit to the sun is

$$t(r,r_0)=\int_{r_0}^{r}\left(\frac{A_>}{B_>}\right)^{1/2}\left(1-\frac{r_0^2}{r^2}\frac{B_>}{B_>(r_0)}\right)^{-1/2}dr. \quad (50)$$

As in the deflection of light, expanding to first order in γ/R we obtain

$$t(r,R)=\frac{1}{\sqrt{|\beta|}}\mathrm{arctan}\,h\left(\sqrt{|\beta|}\sqrt{\frac{r^2-R^2}{1-\beta R^2}}\right)+\gamma\left(\ln\frac{\sqrt{1-\beta R^2}r+\sqrt{r^2-R^2}}{R\sqrt{1-\beta r^2}}+\frac{1}{2\sqrt{1-\beta R^2}}\sqrt{\frac{r-R}{r+R}}\frac{1+\beta rR}{1-\beta r^2}\right). \quad (51)$$

This expression holds for $\beta>0$, while for $\beta<0$ $\mathrm{arctan}\,h[\sqrt{|\beta|}\sqrt{(r^2-R^2)/(1-\beta R^2)}]$ has to be replaced by $\pi/2-\arctan[(1/\sqrt{|\beta|})\sqrt{(1-\beta R^2)/(r^2-R^2)}]$. Whenever $|\beta|r^2\ll1$, the above expression takes the form

$$t(r,R)\simeq\sqrt{r^2-R^2}+\gamma\ln\frac{r+\sqrt{r^2-R^2}}{R}+\frac{\gamma}{2}\sqrt{\frac{r-R}{r+R}}+\frac{\beta}{3}(r^2-R^2)^{3/2}. \qquad (52)$$

We will use this expression to get bounds from solar radar echo experiments. Notice, however, that Eq. (51) may be applicable to some more general cases.

In [56], the time delay on solar system scales was measured to an accuracy of 0.1%. A ray that leaves the Earth, grazes the Sun, reaches Mars, and comes back would have a time delay of $248\pm0.25\,\mu s$ where the 248 μs is the exact prediction of the "Shapiro" time delay and the uncertainty ±0.25 μs can be used to constrain β. At superior conjunction, the radii of the Sun to Earth, r_e, and to Mars, r_m, are much greater than the radius of the sun $R_\odot$, and thus $\frac{2}{3}\beta(r_e^3+r_m^3)=\pm0.25$ μs. This constrains β to the range

$$|\beta|<7.555\times10^{-37}\ \ \mathrm{cm}^{-2}. \qquad (53)$$

It is interesting to compare the bounds on the various parameters of a brane theory with an ${}^{(4)}R$ term, with the bounds on the parameters that result from brane dynamics without the ${}^{(4)}R$ term. In [1], the dynamics on the brane is given, instead of Eq. (8), by the following equation:

$${}^{(4)}G^{\mu}_{\nu}=\frac{\kappa_5^4}{6\kappa_4^2}\Lambda_4{}^{(4)}T^{\mu}_{\nu}-\frac{1}{2}\left(\Lambda_5+\frac{\kappa_5^4}{6\kappa_4^4}\Lambda_4^2\right)\delta^{\mu}_{\nu}-\frac{\kappa_5^4}{24}(6^{(4)}\,T^{\mu}_{\rho}\,{}^{(4)}\,T^{\rho}_{\nu}-2\ {}^{(4)}\,T^{(4)}\,T^{\mu}_{\nu}-3\,{}^{(4)}T^{\rho}_{\sigma}\,{}^{(4)}T^{\sigma}_{\rho}\delta^{\mu}_{\nu}+{}^{(4)}T^2\delta^{\mu}_{\nu})-\bar{\mathsf{E}}^{\mu}_{\nu}. \qquad (54)$$

For $\bar{\mathsf{E}}^{\mu}_{\nu}=0$, following the same steps for solving Eq. (54), as before, we arrive at the unique Schwarzschild-(A)dS$_4$ exterior solution $B_>(r)=1/A_>(r)$, where $A_>(r)$ is given by Eq. (23). The parameters of this solution, denoted by the subscript SMS, are given by

$$\gamma_{SMS}=\frac{\kappa_4^2\Lambda_{4,SMS}}{6\pi\alpha_{SMS}^2}\left(1+\frac{3\kappa_4^2M}{8\pi\Lambda_{4,SMS}R^3}\right)M, \qquad (55)$$

$$\beta_{SMS}=\frac{1}{6}\Lambda_{5,SMS}+\frac{\Lambda_{4,SMS}^2}{9\alpha_{SMS}^2}. \qquad (56)$$

It is obvious that the conventional value $2G_NM$ of the Newtonian term can dominate γ_{SMS} only if $\Lambda_{4,SMS}=3\alpha_{SMS}^2/2$. This is the same value that revives the common four-dimensional energy-momentum terms in the general equation (54). This value is substituted in Eqs. (55),(56) and then, using the bounds (45),(47) from the classical tests, we can set bounds on $\alpha_{SMS},\Lambda_{5,SMS}$. More specifically, since $\Lambda_{5,SMS}$ is not contained in Eq. (55), Eq. (45) is enough to find

$$\alpha_{SMS}>2.425\times10^{-13}\ \ \mathrm{cm}^{-1}, \qquad (57)$$

which means $M_5>7$ TeV. Then, from Eq. (56),

$$\Lambda_{5,SMS}<-8.818\times10^{-26}\ \mathrm{cm}^{-2}, \qquad (58)$$

and only a bulk of negative curvature is allowed in this approach. The above results are exact, since now there are only two unknown parameters $\alpha_{SMS},\Lambda_{5,SMS}$ to be determined from the two γ_{SMS},β_{SMS}. It is seen from Eq. (55) (plot 3 in Fig. 1) that the point particle limit of infinite density cannot be obtained (in contrast to the plots 1a, 1b, and 2 of Fig. 1), since then $G_{(4)}\rightarrow\infty$. Even for different boundary conditions [34] the above limit is sometimes not defined at all.

Finally, we make the following comment. In our second solution (34), obtained $\gamma>2G_NM$. Thus, from Eq. (44), the deflection angle $\Delta\phi_d$ is larger than the corresponding "Einstein" deflection $4G_NM/r_0$. This situation of increased deflection (compared to that caused by luminous matter) has been clearly observed in galaxies or clusters of galaxies, and the above solution might serve as a possible way to provide an explanation. In Weyl gravity [57,58,54], the above increase is associated with some parameter like our β (with the difference of a linear instead of a quadratic term), which has to be positive in order to account for this (see also [52,59]). But, then, a $\beta>0$ cannot account for the additional attractive force needed to explain the galactic rotation curves. In our solution, instead, there is the additional freedom for the parameter β to be negative, which can be used for galactic rotation curve fitting. Notice also that the Gibbons-Hawking solution cannot explain the extra deflection in galaxies in this way. Alternative gravity theories have probably not been very successful in illuminating the missing mass problem, but this does not mean that a new gravity modification should not be tested in the arena of local phenomena; it is certain that the whole topic deserves a more thorough investigation.

IV. CONCLUSIONS

In the present paper, we have investigated the influence of the brane curvature invariant included in the bulk action on local spherically symmetric braneworld solutions. The brane dynamics is made closed by assuming the vanishing of the electric part of the Weyl tensor as a boundary condition for the propagation equations in bulk space. All the exterior solutions for a compact rigid object were obtained. Some of them are of the Schwarzschild-(A)dS$_4$ form. Furthermore, two generalized interior Oppenheimer-Volkoff solutions were found, one of which is matched to a Schwarzschild-(A)dS$_4$ exterior, while the other is not. A remarkable consequence is that the bulk space "sees" the finite region of the body and modifies the parameter of the Newtonian term in the outside region. Imposing no contradiction with everyday Newton's constant universality leads to bounds on the string scale. The known classical solar system tests, which were used in the past to check the validity of general relativity, are here used to put precise bounds on the parameters of our model. More specifically, the crossover scale is found to be beyond our planetary system diameter, which means that the upper bound for the energy string scale is of the order of

TeV. The limit of the idealized infinite density point particle is obtained, and significant deviations from the known Newton's constant might occur in extremely low density matter distributions. In the usual brane dynamics, in contrast to our case, the solar tests impose a lower, instead of upper, bound of the above order on the string scale. Furthermore, in that case, to obtain exterior non-Schwarzschild-(A)dS_4 solutions, one has to consider nonlocal bulk effects.

We have followed a braneworld viewpoint to obtain braneworld solutions, ignoring the exact bulk space. We have not provided a description of the gravitational field in the bulk space, but confined our interest to effects that can be measured by brane observers. However, our formalism assures the existence of a five-dimensional Einstein space as the bulk space. Because of the assumptions made to obtain a closed brane dynamics, there is no guarantee that the brane is embeddable in a regular bulk. This is the case for the Friedmann brane [45], whose symmetries imply that the bulk is Schwarzschild-AdS_5 [60,61]. A Schwarzschild brane can be embedded in a "black-string" bulk metric, but this has singularities [23,62]. Investigation of bulk backgrounds which reduce to Schwarzschild [or Schwarzschild-(A)dS_4] black holes is in progress.

It is clear that a density-dependent gravitational constant generally violates, at the weak field limit, Newton's third law of equal action and reaction. This, furthermore, means violation of the conservation of linear momentum and makes it impossible to define precisely the potential energy for a system of two masses. Since the point particle limit does not meet any problem in our model, and the Newtonian limit also arises in metric-based theories for point particles moving along geodesics, we think that an understanding of the motion of an extended body in general relativity (or more generally) would shed light on the above subtleties. Beyond this, in our everyday phenomena, where very low density distributions do not contribute gravitationally, no such difficulties arise. However, such situations may be relevant to early stages of the universe, before or during structure formation.

As a motivation for further speculation, we mention that it would be quite interesting, even for the formal status of the theory, if the existence of the $\rho_0 \to \infty$ asymptotic behavior of the solutions found here, remains valid whenever the ${}^{(4)}R$ term is present. In addition to this, it is known that in cosmology the ${}^{(4)}R$ term revives the desirable early universe of standard general relativity. However, to conclude, as braneworld solutions are continuously investigated, they have to be confronted with the accumulated cosmological and astrophysical observations, if one wishes to consider the underlying theories as viable generalizations of general relativity.

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