PHYSICAL REVIEW D **66**, 106003 (2002)

Stability of the nonextremal enhançon solution: Perturbation equations

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(Received 23 August 2002; published 18 November 2002)

We consider the stability of the two branches of nonextremal enhançon solutions. We argue that one would expect a transition between the two branches at some value of the nonextremality, which should manifest itself in some instability. We study small perturbations of these solutions, constructing a sufficiently general ansatz for linearized perturbations of the nonextremal solutions, and show that the linearized equations are consistent. We show that the simplest kind of perturbation does not lead to any instability. We reduce the problem of studying the more general spherically symmetric perturbation to solving a set of three coupled second-order differential equations.

DOI: 10.1103/PhysRevD.66.106003 PACS number(s): 11.25.Mj, 04.70.Bw

I. INTRODUCTION

A key issue in string theory is the role and physical interpretation of singularities in supergravity solutions. Some singular solutions, such as negative mass Schwarzschild solutions, are genuinely unphysical [1], and are simply excluded from consideration; no corresponding source exists. String theory provides resolutions of many other singularities through various mechanisms. Recently, new singularity resolution mechanisms have played an important part in the understanding of field theories with partially broken supersymmetry in the anti–de Sitter—conformal field theory (AdS-CFT) correspondence [2–7]. A simple example of this new class of mechanisms is the resolution of the repulson singularity of [8,9] by the enhançon mechanism [10].

Generally, one of the simplest questions to consider from the bulk spacetime side of the AdS-CFT correspondence is the finite-temperature behavior of the theory. One would expect that the theories with reduced supersymmetry should have interesting phase structures. At high temperatures, one would expect to find that the partition function is dominated by a black hole solution, and there may be some symmetry-breaking phase transitions as the temperature decreases. Attempts to investigate these issues by studying black hole solutions on the AdS side were made in [11–16]. Considerable progress was made on obtaining suitable black hole solutions. However, because of the complexity of the setup, no exact closed-form solutions are available.

In this paper, we will begin an investigation of the phase structure associated with nonextremal black hole versions of the enhançon solution of [10], using the simple explicit solutions generalizing the enhançon found in [10,17]. We will focus on studying whether the solutions have classical instabilities which could provide the mechanism for transitions between them. We analyze the linearized perturbation equations around the nonextremal enhançon background, generalizing the analysis of [18] in the extreme case. Although the enhançon is somewhat different from the asymptotically AdS cases, the underlying physics should be similar. It would be interesting to extend our work to consider the stability of the nonextremal fractional brane solutions of [19], which are more closely related to asymptotically AdS cases.

In Sec. II, we review the extremal and nonextremal enhançon solutions. There are two branches of nonextremal solutions, arising from an ambiguity of a choice of sign in the solution of the supergravity equations. One branch joins on to the extremal enhançon solution studied previously, and always has a shell of branes outside the horizon. (The proportion of the energy carried by the shell and by the black hole inside the shell in this solution was not determined at this supergravity level; a better understanding of the internal dynamics of the shell is required to obtain a unique solution for given asymptotic charges.) The other branch appears at a finite value of the nonextremality parameter. Above this critical value of the nonextremality, both types of solution are possible. At large energies, the effects of the D-brane charges should be negligible, so the solution with a horizon, which for large mass is approximately the usual uncharged black hole solution, has the right physical behavior. On the other hand, if we begin slowly adding energy to the extremal enhançon, we will obtain a solution on the branch with a shell. We would expect that there is some nontrivial transition between these two families of solutions as we vary the energy.[1]

We are going to focus on the linearized stability analysis, but we will begin by discussing the thermodynamic aspects.[2] In Sec. III we will compare the entropies of the two solu-

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[1]Since the enhançon is like a monopole solution, we expect the physics to be similar to that of the Einstein–Yang-Mills-Higgs system (see [20] and references therein). For any given value of the asymptotic charges, only one of the two solutions should be stable. However, to see this physics, it may be necessary to include the effects of the non-Abelian gauge fields, as in [21], which we do not do.

[2]One interesting suggestion in [11] was that in some cases, black hole solutions should exist only for temperatures greater than a critical value. We will see that for the nonextremal enhançon, solutions with a regular event horizon exist only for values of the nonextremality parameter greater than a critical value—that is, for sufficiently large energies. There also appears to be a maximum temperature for these solutions, but no minimum.

tions, and see that the horizon branch has larger entropy at large mass, as we would expect. We can also calculate the specific heat for the horizon branch; for the branch with a shell, the ambiguity in the division of energy between the shell and the black hole prevent us from obtaining a well-defined answer for the specific heat.

Our main focus is to look for dynamical instabilities that could take us from one branch to the other. We particularly want to see whether there is an instability at some value of the energy which could take us from the branch with a shell to the horizon branch, which we think should be the physical solution at large energies.

In Sec. IV, we set up an appropriate ansatz for the perturbations. We consider only perturbations which are spherically symmetric in the transverse space and translationally invariant along the branes, as we are looking for a transition between two solutions which preserve these symmetries. We consider the most general ansatz consistent with the assumed symmetries. This ansatz is slightly more general than the ansatz for perturbations of the extreme enhançon considered in [18]; we find that our more general ansatz is necessary to obtain nontrivial solutions of the full set of field equations. We use the remaining diffeomorphism freedom to reduce the linearized equations of motion to four second-order equations for four functions characterizing the perturbation. One of these equations is decoupled from the others.

In Sec. V, we consider the stability to this decoupled mode. This equation is in fact identical to the free scalar wave equation in this background. Since the mode is not coupled to the shell, it satisfies simple continuous matching conditions there. We reduce the equation to a one-dimensional bound state problem, and find that the potential is negative in a region near the shell, so one might expect that there could be bound states (and hence an instability). Nevertheless, we present a general argument that there can never be an instability associated with this mode. The idea is that since the equation is just the free wave equation, a constant function is a solution. This implies that the bound state problem has a zero-energy solution with no nodes, and as a consequence, there can be no bound states.

In this paper, we will not consider the solution of the other three coupled equations. The boundary conditions at the shell will be more complicated for these modes, and we will need to solve the equations numerically to determine if there is any instability. This analysis will be continued in a forthcoming paper [22].

II. THE ENHANÇON SOLUTIONS

The original repulson geometry [8,9] is constructed by wrapping N D$(p+4)$-branes on a K3 manifold of volume V. We will also consider including M Dp-branes parallel to the noncompact directions of the D$(p+4)$-branes. This leaves an unwrapped $(p+1)$-dimensional world volume in the six noncompact dimensions. There are $5-p$ noncompact spatial dimensions transverse to the branes. We will consider the case $p=2$, so we have coordinates r,θ,ϕ in the transverse directions, and x^μ, $\mu=0,1,2$ in the noncompact directions along the branes. The Einstein frame metric and fields are

$$\begin{aligned} ds^2 &= Z_2^{-5/8} Z_6^{-1/8} \eta_{\mu\nu} dx^\mu dx^\nu + Z_2^{3/8} Z_6^{7/8} (dr^2 + r^2 d\Omega) \\ &\quad + V^{1/2} Z_2^{3/8} Z_6^{-1/8} ds_{\mathrm{K3}}^2, \\ e^{2\Phi} &= g_s^2 Z_2{}^{1/2} Z_6{}^{-3/2}, \\ C_{(3)} &= (Z_2 g_s)^{-1} dx^0 \wedge dx^1 \wedge dx^2, \\ C_{(7)} &= (Z_6 g_s)^{-1} dx^0 \wedge dx^1 \wedge dx^2 \wedge V \varepsilon_{\mathrm{K3}}, \end{aligned} \tag{1}$$

where the harmonic functions are

$$Z_6 = 1 + \frac{r_6}{r}, \quad Z_2 = 1 - \frac{r_2}{r}, \tag{2}$$

the parameters are related by

$$r_6 = \frac{g_s N \alpha'^{1/2}}{2}, \quad r_2 = \frac{V_*}{V} r_6 \left(1 - \frac{M}{N}\right), \tag{3}$$

and $d\Omega$ denotes the metric on the unit two-sphere. The running K3 volume is

$$V(r) = V \frac{Z_2(r)}{Z_6(r)}. \tag{4}$$

$V = V_* = (2\pi\sqrt{\alpha'})^4$ at the enhançon radius,

$$r_{\mathrm{e}} = \frac{2V_*}{V - V_*} r_6 \left(1 - \frac{M}{2N}\right). \tag{5}$$

The repulson singularity would occur at $r = r_2 < r_{\mathrm{e}}$.

The enhançon mechanism discovered in [10] resolves this repulson singularity. The essence of the mechanism is that the singularity can never be formed. If one tries to assemble the repulson from well-separated branes, the constituent branes will stop behaving as pointlike objects and smear out into extended solitons at a certain distance from the would-be singularity; the sphere at this radius is called the enhançon sphere. This effect is due to the appearance of additional light degrees of freedom, enhancing the gauge symmetry from $U(1)$ to $SU(2)$, when the K3 volume is $V = V_* = (2\pi\sqrt{\alpha'})^4$. The metric outside the enhançon sphere is still the repulson geometry, but the sources are distributed over the sphere, leaving flat space inside and removing the singularity. This is the enhançon geometry.

Although this singularity resolution depends on stringy physics, namely the appearance of additional light degrees of freedom which are not contained in the original supergravity description, it was found in [17] that the appearance of a shell at the enhançon radius can be understood from a purely supergravity argument. If we imagine distributing the sources on a spherically symmetric shell, so that the exterior geometry is the repulson, while the spacetime inside the shell is flat, then the energy density of the shell will be positive only if the shell lies outside the enhançon radius. Thus, the enhançon radius provides a minimum position for the shell.

Thus, the above geometry does not always apply for all r. For $M>N$ there is no repulson singularity, and we can assemble sources to form the geometry in Eq. (1).[3] For $M<N$ however, some of the D6-branes cease to be pointlike before we reach the singularity at $r=r_2$, and will form an enhançon shell. This geometry then applies only outside the shell.

We will mainly be interested in $M<N$. We assume that all M D2-branes coincide at the origin, along with N' D6-branes, where $N'\leqslant M$. The remaining $N-N'$ D6-branes lie on an enhançon shell. The geometry inside the shell is

$$g_s^{1/2}ds^2=H_2^{-5/8}H_6^{-1/8}\eta_{\mu\nu}dx^\mu dx^\nu+H_2^{3/8}H_6^{7/8}(dr^2+r^2d\Omega)+V^{1/2}H_2^{3/8}H_6^{-1/8}ds_{K3}^2 \tag{6}$$

and the nontrivial fields are

$$e^{2\Phi}=g_s^2H_2^{1/2}H_6^{-3/2},$$
$$C_{(3)}=(g_sH_2)^{-1}dx^0\wedge dx^1\wedge dx^2, \tag{7}$$
$$C_{(7)}=(g_sH_6)^{-1}dx^0\wedge dx^1\wedge dx^2\wedge V\varepsilon_{\mathrm{K3}},$$

where

$$H_2=1-\frac{r_2-r_2'}{r_{\mathrm{e}}}-\frac{r_2'}{r},\quad r_2'=r_6\frac{V_*}{V}\frac{N'-M}{N}\ , \tag{8}$$

$$H_6=1+\frac{r_6-r_6'}{r_{\mathrm{e}}}+\frac{r_6'}{r},\quad r_6'=r_6\frac{N'}{N}=\frac{g_sN'\alpha'^{1/2}}{2}. \tag{9}$$

The constant terms in the harmonic functions are chosen to ensure continuity of the solution at the shell.

The supergravity argument can be extended to nonextremal generalizations of the enhançon solution, which are difficult to study from the string theory point of view. A nonextremal solution was first written down in [10]. In [17], it was found that there are two branches of nonextremal solutions, arising from an ambiguity of a choice of sign in the solution of the supergravity equations for the usual ansatz. The nonextremal generalization of the exterior geometry is

$$g_s^{1/2}ds^2=Z_2^{-5/8}Z_6^{-1/8}(-Kdt^2+dx_1^2+dx_2^2)+Z_2^{3/8}Z_6^{7/8}(K^{-1}dr^2+r^2d\Omega_2^2)+V^{1/2}Z_2^{3/8}Z_6^{-1/8}ds_{\mathrm{K3}}^2, \tag{10}$$

the dilaton and Ramond-Ramond (RR) fields are

$$e^{2\Phi}=g_s^2Z_2^{1/2}Z_6^{-3/2},$$
$$C_{(3)}=(g_s\alpha_2Z_2)^{-1}dt\wedge dx^1\wedge dx^2, \tag{11}$$
$$C_{(7)}=(g_s\alpha_6Z_6)^{-1}dt\wedge dx^1\wedge dx^2\wedge V\varepsilon_{\mathrm{K3}},$$

and the various harmonic functions are given by

$$K=1-\frac{r_0}{r},$$
$$Z_2=1+\frac{\hat{r}_2}{r}\quad Z_6=1+\frac{\hat{r}_6}{r}. \tag{12}$$

Here

$$\hat{r}_6=-\frac{r_0}{2}+\sqrt{r_6^2+\left(\frac{r_0}{2}\right)^2}, \tag{13}$$

and $\alpha_6=\hat{r}_6/r_6$. There are two choices for $\hat{r}_2$ consistent with the equations of motion:

$$\hat{r}_2=-\frac{r_0}{2}\pm\sqrt{r_2^2+\left(\frac{r_0}{2}\right)^2}, \tag{14}$$

and $\alpha_2=\hat{r}_2/r_2$. Here, r_2 and r_6 are still given by Eq. (3). We have changed our conventions for Z_2 to facilitate comparison of formulas involving Z_2 and Z_6, so the repulson singularity, if there is one, is at $r=-\hat{r}_2$.

There are two branches of solutions. For the upper sign in Eq. (14), $\hat{r}_2>0$, so there is no repulson singularity, and the solution has a regular horizon at $r=r_0$. For the lower choice of sign, however, the repulson singularity always lies outside the would-be horizon, $|\hat{r}_2|>r_0$, and the geometry will be corrected by an enhançon shell. We therefore refer to the former as the "horizon branch" and the latter as the "shell branch." It is interesting that the appearance of a repulson singularity in the nonextremal solutions is not connected to whether $M>N$, but rather to a discrete choice. For $M>N$, the extremal solution is the same as the solution at $r_0=0$ on the horizon branch, and we regard the horizon branch as the only physical choice. For $M<N$, on the other hand, the extremal solution is the solution at $r_0=0$ on the shell branch, so we need to consider both branches of solutions. We will henceforth focus on the case where $M<N$.

The shell branch exterior solution is cut off by an enhançon shell at

$$r_{\mathrm{e}}=\frac{V_*\hat{r}_6-V\hat{r}_2}{V-V_*}. \tag{15}$$

As in the extremal case, this shell will contain $N-N'$ D6-branes, while the interior solution with M D2-branes and $N'<M$ D6-branes is

$$g_s^{1/2}ds^2=H_2^{-5/8}H_6^{-1/8}\left(-\frac{K(r_{\mathrm{e}})}{L(r_{\mathrm{e}})}Ldt^2+dx_1^2+dx_2^2\right)+H_2^{3/8}H_6^{7/8}(L^{-1}dr^2+r^2d\Omega)+V^{1/2}H_2^{3/8}H_6^{-1/8}ds_{K3}^2, \tag{16}$$

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[3] For $M>2N$, there is no enhançon, and we can form the above geometry by bringing the branes in individually from infinity. For $N<M<2N$, D6-branes on their own smear out at $r=r_{\mathrm{e}}$. We can still form the geometry (1) if we first form D2-D6 bound states, which can be brought to the origin.

with accompanying fields

$$e^{2\Phi}=g_s^2 H_2^{1/2} H_6^{-3/2},$$

$$C_{(3)}=\left(\frac{K(r_e)}{L(r_e)}\right)^{1/2}(g_s\alpha_2' H_2)^{-1}dt\wedge dx^1\wedge dx^2, \tag{17}$$

$$C_{(7)}=\left(\frac{K(r_e)}{L(r_e)}\right)^{1/2}(g_s\alpha_6' H_6)^{-1}dt\wedge dx^1\wedge dx^2\wedge V\varepsilon_{K3},$$

where

$$L=1-\frac{r_0'}{r},$$

$$H_2=1+\frac{\hat{r}_2-\hat{r}_2'}{r_e}+\frac{\hat{r}_2'}{r},$$

$$\hat{r}_2'=-\frac{r_0'}{2}+\sqrt{r_2'^2+\left(\frac{r_0'}{2}\right)^2},$$

$$r_2'=r_6\frac{V_*}{V}\frac{M-N'}{N}, \tag{18}$$

$$H_6=1+\frac{\hat{r}_6-\hat{r}_6'}{r_e}+\frac{\hat{r}_6'}{r},$$

$$\hat{r}_6'=-\frac{r_0'}{2}+\sqrt{r_6'^2+\left(\frac{r_0'}{2}\right)^2},$$

$$r_6'=r_6\frac{N'}{N},$$

and $\alpha_6'=\hat{r}_6'/r_6'$, $\alpha_2'=\hat{r}_2'/r_2'$. Note that we have introduced an independent nonextremality scale r_0' for the interior solution. Implicitly $r_0'<r_e$ in order that the interior black hole actually fits inside the shell. We have taken the horizon branch for the interior solution, as $N'<M$.

The shell branch solutions have an additional parameter, r_0', which is not determined by the asymptotic charges of the solution. It was argued in [17] that this was simply a weakness of the supergravity excision procedure, and that a better understanding of the physics of the shell should fix this parameter. We will not attempt to resolve this issue in this paper, but will simply consider the stability of the shell branch solutions for arbitrary $r_0'<r_0,r_e$.

III. THERMODYNAMICS

We would like to briefly compare the behaviors of the two branches. The ADM energy density for these solutions is

$$E=\frac{(2r_0+\hat{r}_2+\hat{r}_6)}{4G}, \tag{19}$$

where G is Newton's constant. For the horizon branch, this gives

$$E_{hb}=\frac{1}{4G}\left(r_0+\sqrt{\frac{r_0^2}{4}+r_2^2}+\sqrt{\frac{r_0^2}{4}+r_6^2}\right), \tag{20}$$

while for the shell branch,

$$E_{sb}=\frac{1}{4G}\left(r_0-\sqrt{\frac{r_0^2}{4}+r_2^2}+\sqrt{\frac{r_0^2}{4}+r_6^2}\right). \tag{21}$$

The difference between the $r_0=0$ solutions is $\Delta E=|r_2|/2G$. For $M<N$, we need to add this much energy to the extremal solution before we can get solutions on the horizon branch.

The entropy and temperature on the horizon branch are easily obtained from the metric (10), giving

$$S_{hb}=\frac{A}{4G}=\frac{\pi r_0}{G}(r_0+\hat{r}_2)^{3/8}(r_0+\hat{r}_6)^{7/8}, \tag{22}$$

$$T_{hb}=\frac{1}{4\pi(r_0+\hat{r}_2)^{1/2}(r_0+\hat{r}_6)^{1/2}}. \tag{23}$$

For the shell branch, we must use the interior solution (16), which gives

$$S_{sb}=\frac{A}{4G}=\frac{\pi r_0'^2}{G}H_2(r_0')^{3/8}H_6(r_0')^{7/8}, \tag{24}$$

$$T_{sb}=\frac{1}{4\pi r_0'}\left(\frac{K(r_e)}{L(r_e)H_2(r_0')H_6(r_0')}\right)^{1/2}. \tag{25}$$

On the horizon branch, we see that the temperature is a monotonic function of r_0, and hence the specific heat is always negative.[4] For the shell branch, we cannot evaluate the specific heat, as we do not know $r_0'(r_0)$.

The ambiguity in the interior solution on the shell branch prevents us from comparing the entropies of the two solutions for most values of the parameters. However, we can make a comparison at large energies, when $r_0\gg r_2,r_6$. Then

$$E_{hb}\approx\frac{r_0}{2G},\quad S_{hb}\approx\frac{\pi r_0^2}{G}\approx 4\pi G E_{hb}^2, \tag{26}$$

as for an uncharged black hole, while for the shell branch,

$$E_{sb}\approx\frac{r_0}{4G},\quad S_{sb}\approx\frac{\pi r_0'^2}{G}\left(\frac{V_*}{V}\right)^{3/8}\approx 16\pi G E_{sb}^2\frac{r_0'^2}{r_0^2}\left(\frac{V_*}{V}\right)^{3/8}. \tag{27}$$

[4] As a consequence, the conjecture of [23] presumably implies that if the x_1,x_2 directions are noncompact, this solution has a Gregory-Laflamme type instability [24]. This is not the instability we are interested in considering, as it seems unlikely to mediate a transition between our two families of translationally-invariant solutions.

Since $r_0' < r_0$ and V_*/V is a small parameter, we conclude that the entropy is larger on the horizon branch at large mass. Thus, at least for large fixed mass, we would expect the horizon branch to dominate.

It would also be interesting to compare the entropies at fixed low temperature (so again $r_0 \gg r_2, r_6$). Unfortunately, this is not so straightforward. On the horizon branch,

$$T_{hb} \approx \frac{1}{4\pi r_0}, \quad S_{hb} \approx \frac{1}{16\pi G T^2}, \tag{28}$$

but on the shell branch,

$$T_{sb} \approx \frac{1}{4\pi r_0'}\left(1-\frac{r_0'}{r_\mathrm{e}}\right)^{-1/2}, \tag{29}$$

so

$$S_{sb} \approx \frac{\pi r_0'^2}{G}\left(1-\frac{r_0}{r_\mathrm{e}}\right)^{3/8} \approx \frac{1}{16\pi G T^2}\left(1-\frac{r_0}{r_\mathrm{e}}\right)^{3/8}\left(1-\frac{r_0'}{r_\mathrm{e}}\right)^{-1}. \tag{30}$$

Thus, whether S_{sb} is smaller or larger than S_{hb} in this regime depends on how close r_0' can be to r_0. Surprisingly, if it is sufficiently close, S_{sb} can be the larger.

Thus, we see that thermodynamic considerations suggest that at least for large masses, the horizon branch should be the preferred one. Detailed investigation of the thermodynamics is hampered by the fact that we do not know how r_0' varies with r_0. Black hole thermodynamics depends on studying the static vacuum solutions as functions of the parameters, so the presence of an unphysical one-parameter ambiguity in our family of solutions is a serious impediment.

IV. PERTURBATION ANSATZ

We now turn to our main objective, the consideration of the stability of these solutions. We wish to consider the simplest set of linearized perturbations of the enhançon solutions which could provoke a transition between the two branches. We will therefore assume that the perturbations preserve many of the symmetries of the background (10). Specifically, we assume the spherical symmetry in the θ,ϕ directions, translational invariance in x_1 and x_2, and the discrete symmetries under $x_1 \to -x_1$, $x_2 \to -x_2$, $\phi \to -\phi$ are preserved. By a suitable choice of coordinates, the most general perturbation consistent with these symmetries can be written as the metric

$$g_s^{1/2} ds^2 = e^{-\psi_1/2}[\bar{Z}_2^{-1/2}\bar{Z}_6^{-1/2}(-\bar{K}e^{\delta\psi_2}dt^2 + e^{-(1/2)\delta\psi_2+\delta\psi_3}dx_1^2 + e^{-(1/2)\delta\psi_2-\delta\psi_3}dx_2^2) + \bar{Z}_2^{1/2}\bar{Z}_6^{1/2}(\bar{K}^{-1}dr^2 + r^2 d\Omega_2^2) + V^{1/2}\bar{Z}_2^{1/2}\bar{Z}_6^{-1/2}ds_{\mathrm{K3}}^2], \tag{31}$$

dilaton

$$\bar{\phi} = \phi + \delta\phi, \tag{32}$$

and RR fields

$$\bar{C}_{(3)} = C_{(3)} + \delta C_{(3)}, \quad \bar{C}_{(7)} = C_{(7)} + \delta C_{(7)}. \tag{33}$$

Here

$$\psi_1 = \phi + \delta\psi_1, \quad \bar{Z}_2 = Z_2(1+\delta Z_2), \quad \bar{Z}_6 = Z_6(1+\delta Z_6), \quad \bar{K} = K(1+\delta K), \tag{34}$$

the harmonic functions Z_2, Z_6, K are as in Eq. (12), the unperturbed dilaton ϕ is as in Eq. (11), and the RR potentials are as in Eq. (11). The first-order perturbations are all functions of (r,t) only, while the background quantities are functions only of r. We look for perturbations of the form $f(r)e^{i\omega t}$.

Our ansatz is slightly more general than the ansatz adopted in the study of perturbations of the extremal enhançon geometry in [18]. We have introduced three new perturbation functions, $\delta\psi_2$, $\delta\psi_3$, and δK. As we will see shortly, we can choose to set $\delta K = 0$ by a gauge transformation. The first-order function $\delta\psi_3$ is the only thing that breaks the rotational symmetry between x_1 and x_2. As a consequence, it decouples from the other perturbations. We could set it to zero without affecting the other modes; instead, we retain it, and study it independently of the others. This provides us with a single simple (but nontrivial) perturbation equation, which we study in Sec. V. One might think that $\delta\psi_2$ would also decouple, as it breaks the boost symmetry between t and x_1, x_2 which Eq. (10) respects. However, the assumption that the perturbations depend on t and not on x_1, x_2 also breaks this symmetry, so we will find that $\delta\psi_2$ couples to the time derivatives of the other perturbations, and it is not possible to set it to zero. That is, it is necessary to consider the more general ansatz containing $\delta\psi_2$ to satisfy all the field equations, even in the extreme case.

We will now consider the full set of linearized equations for the perturbations. The gauge field equations give

$$\partial_r \delta C_{(3)} = -\frac{1}{2}(4\,\delta Z_2 + \delta\phi - \delta\psi_1)\partial_r C_{(3)} \tag{35}$$

and

$$\partial_r \delta C_{(7)} = -\frac{1}{2}(4\,\delta Z_6 - 3\,\delta\phi + 3\,\delta\psi_1)\partial_r C_{(7)}. \tag{36}$$

The linear part of the stress tensor only involves $\partial_r \delta C_{(3)}$ and $\partial_r \delta C_{(7)}$, so we can substitute Eqs. (35),(36) directly into the stress tensor.

There are seven distinct equations coming from the linearized Einstein's equations: six different diagonal components, and an off-diagonal $[tr]$ component. With the dilaton equation, this gives us eight equations; but with the gauge field perturbations fixed by Eqs. (35),(36), there are only seven undetermined functions in our ansatz. The problem seems overdetermined, so it is important to ask whether there will be any nontrivial solutions of the full set of equations. We have written down the most general perturbation consistent with the symmetries we have assumed, so we expect there is sufficient redundancy in the equations to admit nontrivial solutions.

In fact, we can see directly that there are nontrivial solutions to these equations, using a trick from [25]. We observe that the ansatz (31) does not completely fix the gauge, as there are infinitesimal diffeomorphisms which preserve its form. Namely,

$$t \to t' = t + e^{i\omega t}\delta t(r), \quad r \to r' = r + e^{i\omega t}\delta r(r), \tag{37}$$

with

$$\partial_r \delta t = i\omega \frac{Z_2 Z_6}{K^2}\delta r. \tag{38}$$

If we apply this diffeomorphism to the nonextremal enhançon geometry (10), we obtain a metric of the form (31) with

$$\begin{aligned}
\delta\psi_1^d &= \left(\phi' - \frac{4}{3r}\right)\delta r - \frac{2}{3}\partial_r \delta r - \frac{2}{3} i\omega\delta t,\\
\delta\psi_2^d &= -\frac{4}{3r}\delta r + \frac{4}{3}\partial_r\delta r + \frac{4}{3} i\omega \delta t,\\
\delta Z_6^d &= \left(\frac{Z_6'}{Z_6} + \frac{2}{r}\right)\delta r,\\
\delta Z_2^d &= \left(\frac{Z_2'}{Z_2} + \frac{2}{3r}\right)\delta r - \frac{2}{3}\partial_r\delta r - \frac{2}{3} i\omega\delta t,\\
\delta K^d &= \left(\frac{K'}{K} + \frac{2}{r}\right)\delta r - 2\partial_r \delta r,\\
\delta\phi^d &= \phi' \,\delta r.
\end{aligned} \tag{39}$$

Since this particular perturbation comes from a coordinate transformation, it must solve the equations of motion. Thus, there are nontrivial solutions of these equations. Of course, we are not interested in solutions which are pure gauge, but this serves to demonstrate that there is some redundancy in the equations.

This diffeomorphism contains an arbitrary function; since we are not interested in pure gauge perturbations, we should fix this additional gauge symmetry. We can do so by setting one of the perturbations to zero. It is convenient to choose $\delta K = 0$. There remain diffeomorphisms which will preserve $\delta K = 0$: these have

$$\delta r = a r K^{1/2} \tag{40}$$

and

$$\delta t = i\omega a\left[(r_0 + \hat{r}_2)(r_0 + \hat{r}_6)\ln\left(\frac{r}{r_0} - 1\right) + (\hat{r}_2 + \hat{r}_6 + r_0) r + \frac{1}{2} r^2\right] + i\omega b, \tag{41}$$

where a and b are arbitrary constants. The perturbations (39) with this δt, δr then give a two-parameter family of solutions of the linearized equations with $\delta K = 0$. We will exploit this remaining coordinate freedom to simplify the equations later.

Having set $\delta K = 0$, the contributions to the Ricci tensor linear in the perturbations are

$$\begin{aligned}
\delta R_{tt} = {}& \frac{1}{4}(2\delta\ddot{\psi}_2 + 9\delta\ddot{\psi}_1 - 5\delta\ddot{Z}_2 + 3\delta\ddot{Z}_6) + \frac{K^2}{32 Z_2 Z_6}\Bigg[16\left(\delta\psi_2'' + \frac{2}{r}\delta\psi_2'\right) - 8\left(\delta\psi_1'' + \frac{2}{r}\delta\psi_1'\right) - 8\left(\delta Z_2'' + \frac{2}{r}\delta Z_2'\right)\\
& - 8\left(\delta Z_6'' + \frac{2}{r}\delta Z_6'\right) + 16\delta\psi_2'\frac{K'}{K} + 4\delta\psi_1'\left(-10\frac{K'}{K} + 5\frac{Z_2'}{Z_2} + \frac{Z_6'}{Z_6}\right) - \delta Z_2'\left(5\frac{Z_2'}{Z_2} + \frac{Z_6'}{Z_6}\right) + \delta Z_6'\left(-32\frac{K'}{K} + 15\frac{Z_2'}{Z_2} + 3\frac{Z_6'}{Z_6}\right)\\
& + (\delta\psi_2 - \delta Z_2 - \delta Z_6)\left(10\frac{Z_2'^2}{Z_2^2} - 10\frac{K'}{K}\frac{Z_2'}{Z_2} + 2\frac{Z_6'^2}{Z_6^2} - 2\frac{K'}{K}\frac{Z_6'}{Z_6}\right)\Bigg],
\end{aligned} \tag{42}$$

$$\begin{aligned}
\delta_+ = \delta R_{tt} + \frac{1}{2}K(\delta R_{11} + \delta R_{22}) = {}& \frac{1}{4}(\delta\ddot{\psi}_2 + 8\delta\ddot{\psi}_1 - 6\delta\ddot{Z}_2 + 2\delta\ddot{Z}_6) + \frac{K^2}{32 Z_2 Z_6}\Bigg[24\left(\delta\psi_2'' + \frac{2}{r}\delta\psi_2'\right)\\
& + \frac{K'}{K}(24\delta\psi_2' - 32\delta\psi_1' - 24\delta Z_6' + 8\delta Z_2')\,\delta\psi_2\left(15\frac{Z_2'^2}{Z_2^2} - 15\frac{K'}{K}\frac{Z_2'}{Z_2} + 3\frac{Z_6'^2}{Z_6^2} - 3\frac{K'}{K}\frac{Z_6'}{Z_6}\right)\Bigg],
\end{aligned} \tag{43}$$

$$\delta_- = K(\delta R_{11} - \delta R_{22}) = \delta\ddot{\psi}_3 + \frac{K^2}{8 Z_2 Z_6}\left[-8\left(\delta\psi_3'' + \frac{2}{r}\delta\psi_3'\right) - 8\delta\psi_3'\frac{K'}{K} + \delta\psi_3\left(-5\frac{Z_2'^2}{Z_2^2} + 5\frac{K'}{K}\frac{Z_2'}{Z_2} - \frac{Z_6'^2}{Z_6^2} + \frac{K'}{K}\frac{Z_6'}{Z_6}\right)\right], \tag{44}$$

$$\delta R_{tr}=\frac{1}{8}\left[4\,\delta\dot{\psi}_2'+16\delta\dot{\psi}_1'-8\,\delta\dot{Z}_2'+8\,\delta\dot{Z}_6'-2\,\delta\dot{\psi}_2\frac{K'}{K}+\delta\dot{\psi}_1\left(-8\frac{K'}{K}+5\frac{Z_2'}{Z_2}+\frac{Z_6'}{Z_6}\right)+\delta\dot{Z}_2\left(4\frac{K'}{K}-5\frac{Z_2'}{Z_2}-\frac{Z_6'}{Z_6}\right)\right.$$
$$\left.+\,\delta\dot{Z}_6\left(-4\frac{K'}{K}+3\frac{Z_2'}{Z_2}-\frac{Z_6'}{Z_6}\right)\right], \tag{45}$$

$$\delta R_{rr}=\frac{Z_2Z_6}{4K^2}(-\delta\ddot{\psi}_1+\delta\ddot{Z}_2+\delta\ddot{Z}_6)+\frac{1}{32}\left[72\,\delta\psi_1''-24\delta Z_2''+40\delta Z_6''-16\delta\psi_2'\frac{K'}{K}+\delta\psi_1'\left(\frac{16}{r}+40\frac{K'}{K}-12\frac{Z_2'}{Z_2}-28\frac{Z_6'}{Z_6}\right)\right.$$
$$\left.+\,\delta Z_2'\left(-\frac{16}{r}-27\frac{Z_2'}{Z_2}+\frac{Z_6'}{Z_6}\right)+\delta Z_6'\left(-\frac{16}{r}+32\frac{K'}{K}-15\frac{Z_2'}{Z_2}-35\frac{Z_6'}{Z_6}\right)\right], \tag{46}$$

$$\delta R_{\theta\theta}=\frac{r^2Z_2Z_6}{4K}(-\delta\ddot{\psi}_1+\delta\ddot{Z}_2+\delta\ddot{Z}_6)+\frac{r^2K}{32}\left[8\,\delta\psi_1''-8\,\delta Z_2''-8\,\delta Z_6''+\delta\psi_1'\left(\frac{80}{r}+8\frac{K'}{K}+12\frac{Z_2'}{Z_2}+28\frac{Z_6'}{Z_6}\right)\right.$$
$$\left.+\,\delta Z_2'\left(-\frac{32}{r}-8\frac{K'}{K}-3\frac{Z_2'}{Z_2}-7\frac{Z_6'}{Z_6}\right)+\delta Z_6'\left(\frac{32}{r}-8\frac{K'}{K}+9\frac{Z_2'}{Z_2}+21\frac{Z_6'}{Z_6}\right)\right], \tag{47}$$

and

$$\delta R_{mn}[K3]=\frac{\sqrt{V}Z_2}{4K}(-\delta\ddot{\psi}_1+\delta\ddot{Z}_2-\delta\ddot{Z}_6)+\frac{\sqrt{V}K}{32Z_6}\left[8\left(\delta\psi_1''+\frac{2}{r}\,\delta\psi_1'\right)-8\left(\delta Z_2''+\frac{2}{r}\,\delta Z_2'\right)+8\left(\delta Z_6''+\frac{2}{r}\,\delta Z_6'\right)\right.$$
$$+\,\delta\psi_1'\left(8\frac{K'}{K}+12\frac{Z_2'}{Z_2}-4\frac{Z_6'}{Z_6}\right)+\delta Z_2'\left(-8\frac{K'}{K}-3\frac{Z_2'}{Z_2}+\frac{Z_6'}{Z_6}\right)+\delta Z_6'\left(8\frac{K'}{K}+9\frac{Z_2'}{Z_2}-3\frac{Z_6'}{Z_6}\right)$$
$$\left.+\,\delta Z_6\left(-6\frac{Z_2'^2}{Z_2^2}+6\frac{K'}{K}\frac{Z_2'}{Z_2}+2\frac{Z_6'^2}{Z_6^2}-2\frac{K'}{K}\frac{Z_6'}{Z_6}\right)\right], \tag{48}$$

where we have introduced certain combinations which simplify the resulting equations, $\dot{}$ signifies ∂_t, and $'$ signifies ∂_r.

The linearized Einstein's equations give seven equations. First, there are the simple equations from δ_- and δ_+, which are, respectively,

$$\frac{Z_2Z_6}{K^2}\delta\ddot{\psi}_3-\delta\psi_3''-\delta\psi_3'\left(\frac{2}{r}+\frac{K'}{K}\right)=0 \tag{49}$$

and

$$\frac{Z_2Z_6}{K^2}(\,\delta\ddot{\psi}_2+8\,\delta\ddot{\psi}_1-6\,\delta\ddot{Z}_2+2\,\delta\ddot{Z}_6)+3\,\delta\psi_2''+3\,\delta\psi_2'\left(\frac{2}{r}+\frac{K'}{K}\right)+(-4\,\delta\psi_1'-3\,\delta Z_6'+\delta Z_2')\frac{K'}{K}=0. \tag{50}$$

There are three more independent second-order equations,

$$-2\frac{Z_2Z_6}{K^2}\delta\ddot{Z}_6+2\,\delta Z_6''+\delta Z_6'\left(-\frac{2}{r}+2\frac{K'}{K}-3\frac{Z_6'}{Z_6}\right)+\delta\psi_1'\left(-\frac{8}{r}-4\frac{Z_6'}{Z_6}\right)+\delta Z_2'\left(\frac{2}{r}+\frac{K'}{K}+\frac{Z_6'}{Z_6}\right)+\delta Z_6\frac{3}{4}\left(\frac{Z_6'^2}{Z_6^2}-\frac{K'}{K}\frac{Z_6'}{Z_6}\right)$$
$$+(3\,\delta\phi+\delta\psi_1-\delta Z_2)\left(\frac{Z_6'^2}{Z_6^2}-\frac{K'}{K}\frac{Z_6'}{Z_6}\right)=0, \tag{51}$$

$$-2\frac{Z_2Z_6}{K^2}(4\,\delta\ddot{\psi}_1+2\,\delta\ddot{\psi}_2+\delta\ddot{Z}_6)+6\left(\delta Z_2''+\frac{2}{r}\,\delta Z_2'\right)+\delta\psi_1'\left(4\frac{K'}{K}-12\frac{Z_2'}{Z_2}\right)+\delta Z_2'\left(5\frac{K'}{K}+3\frac{Z_2'}{Z_2}\right)+\delta Z_6'\left(3\frac{K'}{K}-9\frac{Z_2'}{Z_2}\right)$$
$$-3(\,\delta\phi-5\,\delta\psi_1+5\,\delta Z_2-3\,\delta Z_6)\left(\frac{Z_2'^2}{Z_2^2}-\frac{K'}{K}\frac{Z_2'}{Z_2}\right)=0, \tag{52}$$

and

$$-8\frac{Z_2Z_6}{K^2}(7\,\delta\ddot{\psi}_1+2\,\delta\ddot{\psi}_2-3\,\delta\ddot{Z}_2+\delta\ddot{Z}_6)+24\delta\psi_1''+\delta\psi_1'\left(\frac{144}{r}+40\frac{K'}{K}-12\frac{Z_2'}{Z_2}+36\frac{Z_6'}{Z_6}\right)+\delta Z_2'\left(-\frac{24}{r}-4\frac{K'}{K}+3\frac{Z_2'}{Z_2}-9\frac{Z_6'}{Z_6}\right)$$
$$+\delta Z_6'\left(\frac{72}{r}+12\frac{K'}{K}-9\frac{Z_2'}{Z_2}+27\frac{Z_6'}{Z_6}\right)+3(-\delta\phi+5\,\delta\psi_1-5\,\delta Z_2+3\,\delta Z_6)\left(\frac{Z_2'^2}{Z_2^2}-\frac{K'}{K}\frac{Z_2'}{Z_2}\right)+9(-3\,\delta\phi-\delta\psi_1+\delta Z_2+\delta Z_6)$$
$$\times\left(\frac{Z_6'^2}{Z_6^2}-\frac{K'}{K}\frac{Z_6'}{Z_6}\right)=0. \tag{53}$$

The remaining Einstein's equations give us two equations which are first order in ∂_r. Integrating the tr equation in t gives

$$4\,\delta\psi_2'+16\,\delta\psi_1'-8\,\delta Z_2'+8\,\delta Z_6'-2\,\delta\psi_2\frac{K'}{K}+\delta\psi_1\left(-8\frac{K'}{K}+5\frac{Z_2'}{Z_2}+\frac{Z_6'}{Z_6}\right)+\delta Z_2\left(4\frac{K'}{K}-5\frac{Z_2'}{Z_2}-\frac{Z_6'}{Z_6}\right)+\delta Z_6\left(-4\frac{K'}{K}+3\frac{Z_2'}{Z_2}-\frac{Z_6'}{Z_6}\right)$$
$$+\delta\phi\left(-\frac{Z_2'}{Z_2}+3\frac{Z_6'}{Z_6}\right)=f(t), \tag{54}$$

and a suitable combination gives the equation

$$16\frac{Z_2Z_6}{K^2}(4\,\delta\ddot{\psi}_1+\delta\ddot{\psi}_2-2\,\delta\ddot{Z}_2+2\,\delta\ddot{Z}_6)-8\,\delta\psi_2'\frac{K'}{K}+\delta\psi_1'\left(-\frac{128}{r}-32\frac{K'}{K}-12\frac{Z_2'}{Z_2}-28\frac{Z_6'}{Z_6}\right)+\delta Z_2'\left(\frac{32}{r}+16\frac{K'}{K}-12\frac{Z_2'}{Z_2}+4\frac{Z_6'}{Z_6}\right)$$
$$+\delta Z_6'\left(-\frac{96}{r}-16\frac{K'}{K}-12\frac{Z_2'}{Z_2}-28\frac{Z_6'}{Z_6}\right)+\delta\phi'\left(-4\frac{Z_2'}{Z_2}+12\frac{Z_6'}{Z_6}\right)+(\delta\psi_1-\delta Z_2)\left(20\frac{Z_2'^2}{Z_2^2}-20\frac{K'}{K}\frac{Z_2'}{Z_2}+4\frac{Z_6'^2}{Z_6^2}-4\frac{K'}{K}\frac{Z_6'}{Z_6}\right)$$
$$+\delta Z_6\left(-3\frac{Z_2'^2}{Z_2^2}+3\frac{K'}{K}\frac{Z_2'}{Z_2}-4\frac{Z_6'^2}{Z_6^2}+4\frac{K'}{K}\frac{Z_6'}{Z_6}\right)+\delta\phi\left(-4\frac{Z_2'^2}{Z_2^2}+4\frac{K'}{K}\frac{Z_2'}{Z_2}+24\frac{Z_6'^2}{Z_6^2}-24\frac{K'}{K}\frac{Z_6'}{Z_6}\right)=0. \tag{55}$$

Finally, there is the dilaton equation

$$-8\frac{Z_2Z_6}{K^2}\delta\ddot{\phi}+\frac{8}{r^2}\partial_r(Kr^2\partial_r\delta\phi)=(4\,\delta\psi_1'-\delta Z_2'+3\,\delta Z_6')\left(\frac{Z_2'}{Z_2}-3\frac{Z_6'}{Z_6}\right)+3(3\,\delta\phi+\delta\psi_1-\delta Z_6-\delta Z_2)\left(\frac{Z_6'^2}{Z_6^2}-\frac{K'}{K}\frac{Z_6'}{Z_6}\right)$$
$$+(\delta\phi-5\,\delta\psi_1+5\,\delta Z_2-3\,\delta Z_6)\left(\frac{Z_2'^2}{Z_2^2}-\frac{K'}{K}\frac{Z_2'}{Z_2}\right). \tag{56}$$

These equations are coupled in a complicated fashion, but we see that as mentioned earlier, there is one simple equation (49). In fact, this is the free wave equation. We will discuss the analysis of this decoupled mode in detail in Sec V.

To simplify the other equations, we exploit the remaining two-parameter family of diffeomorphisms (40), (41). These can be used to construct a change of variables which will simplify the equations: we replace a and b by functions $a(r)$ and $b(r)$, and set

$$\delta\psi_1=\delta\psi_1^d(a(r),b(r)),$$

$$\delta\psi_2=\delta\psi_2^d(a(r),b(r))+\Psi_2,$$

$$\delta Z_6=\delta Z_6^d(a(r),b(r))+\mathcal{Z}_6, \tag{57}$$

$$\delta Z_2=\delta Z_2^d(a(r),b(r)),$$

$$\delta\phi=\delta\phi^d(a(r),b(r))+\Phi,$$

with $\delta K=0$. The first term on the right-hand sides is the diffeomorphism-induced perturbation (39) for the diffeomorphism (40),(41), but with a and b now functions. Since the diffeomorphism satisfies the equations of motion for arbitrary constants a and b, the linearized equations will only involve derivatives of $a(r)$ and $b(r)$. The two first-order equations (54),(55) can then be solved for $\partial_r a(r)$ and $\partial_r b(r)$. Inserting these values into the other four second-order equations (50)–(53) and the dilaton equation (56) gives two equations which are trivially satisfied, and a coupled set of three second-order equations for Ψ_2, $\mathcal{Z}_6$, and Φ.

It is convenient to write the coupled equations so that each one only involves second derivatives of one of the functions. Then the equation which involves Φ'' is (where $'$ again denotes ∂_r, and we assume that all the perturbations behave as $e^{i\omega t}$)

$$D\left(\Phi''+\frac{2r-r_0}{r^2K}\Phi'+\frac{Z_2Z_6}{K^2}\omega^2\Phi\right)+P_2^1(\Psi_2'+2\mathcal{Z}_6')+Q_1^1\Phi+Q_2^1\Psi_2+Q_3^1\mathcal{Z}_6=0, \tag{58}$$

with the polynomial coefficients

$$D=r^2K(8r^2+5r\hat{r}_2+5r\hat{r}_6+2\hat{r}_2\hat{r}_6)\times(4r^2+3r\hat{r}_2+3r\hat{r}_6+2\hat{r}_2\hat{r}_6), \tag{59}$$

$$P_2^1=-2r^2K(-2r^2\hat{r}_2+6r^2\hat{r}_6+8r\hat{r}_2\hat{r}_6+3\hat{r}_2^2\hat{r}_6+\hat{r}_2\hat{r}_6^2), \tag{60}$$

$$Q_1^1=-r^2(4r_0\hat{r}_2+36r_0\hat{r}_6+3\hat{r}_2^2+6\hat{r}_2\hat{r}_6+27\hat{r}_6^2)-r(40r_0\hat{r}_2\hat{r}_6+2\hat{r}_2^2\hat{r}_6+30\hat{r}_2\hat{r}_6^2)-12r_0\hat{r}_2^2\hat{r}_6-8\hat{r}_2^2\hat{r}_6^2, \tag{61}$$

$$Q_2^1=r_0(-2r^2\hat{r}_2+6r^2\hat{r}_6+8r\hat{r}_2\hat{r}_6+3\hat{r}_2^2\hat{r}_6+\hat{r}_2\hat{r}_6^2), \tag{62}$$

$$Q_3^1=r^2(8r_0\hat{r}_2+24r_0\hat{r}_6+9\hat{r}_2^2+10\hat{r}_2\hat{r}_6+9\hat{r}_6^2)+r(24r_0\hat{r}_2\hat{r}_6+6\hat{r}_2^2\hat{r}_6+10\hat{r}_2\hat{r}_6^2)+6r_0\hat{r}_2^2\hat{r}_6-2r_0\hat{r}_2\hat{r}_6^2. \tag{63}$$

The equation involving Ψ_2'' is

$$D\left(\Psi_2''+\frac{Z_2Z_6}{K^2}\omega^2\Psi_2\right)+P_2^2\Psi_2'+P_3^2\mathcal{Z}_6'+Q_1^2\Phi+Q_2^2\Psi_2+Q_3^2\mathcal{Z}_6=0, \tag{64}$$

where D is as before, and the other polynomial coefficients are

$$P_2^2=64r^5+r^4(-32r_0+120\hat{r}_2+88\hat{r}_6)+r^3(-76r_0\hat{r}_2-44r_0\hat{r}_6+30\hat{r}_2^2+172\hat{r}_2\hat{r}_6+30\hat{r}_6^2)+r^2(-15r_0\hat{r}_2^2-118r_0\hat{r}_2\hat{r}_6-15r_0\hat{r}_6^2+44\hat{r}_2^2\hat{r}_6+52\hat{r}_2\hat{r}_6^2)+r(-28r_0\hat{r}_2^2\hat{r}_6-36r_0\hat{r}_2\hat{r}_6^2+8\hat{r}_2^2\hat{r}_6^2)-4r_0\hat{r}_2^2\hat{r}_6^2, \tag{65}$$

$$P_3^2=-8r^2\hat{r}_2K(8r^2+16r\hat{r}_6+3\hat{r}_2\hat{r}_6+5\hat{r}_6^2), \tag{66}$$

$$Q_1^2=4\hat{r}_2(r^2(-8r_0-6\hat{r}_2-6\hat{r}_6)+r(4r_0\hat{r}_6-7\hat{r}_2\hat{r}_6+3\hat{r}_6^2)+6r_0\hat{r}_2\hat{r}_6+2\hat{r}_2\hat{r}_6^2), \tag{67}$$

$$Q_2^2=-2r_0\hat{r}_2(8r^2+16r\hat{r}_6+3\hat{r}_2\hat{r}_6+5\hat{r}_6^2), \tag{68}$$

$$Q_3^2=4\hat{r}_2(r^2(16r_0+18\hat{r}_2+2\hat{r}_6)+r(12r_0\hat{r}_6+21\hat{r}_2\hat{r}_6-\hat{r}_6^2)-3r_0\hat{r}_2\hat{r}_6+5r_0\hat{r}_6^2+6\hat{r}_2\hat{r}_6^2). \tag{69}$$

The equation involving $\mathcal{Z}_6''$ is

$$D\left(\mathcal{Z}_6''+\frac{Z_2Z_6}{K^2}\omega^2\mathcal{Z}_6\right)+P_2^3\Psi_2'+P_3^3\mathcal{Z}_6'+Q_1^3\Phi+Q_2^3\Psi_2+Q_3^3\mathcal{Z}_6=0, \tag{70}$$

where D is as before, and the other polynomial coefficients are

$$P_2^3=-2r^2K(6r^2\hat{r}_2-2r^2\hat{r}_6+8r\hat{r}_2\hat{r}_6+\hat{r}_2^2\hat{r}_6+3\hat{r}_2\hat{r}_6^2), \tag{71}$$

$$P_3^3=64r^5+r^4(-32r_0+64\hat{r}_2+96\hat{r}_6)+r^3(-20r_0\hat{r}_2-52r_0\hat{r}_6+30\hat{r}_2^2+76\hat{r}_2\hat{r}_6+30\hat{r}_6^2)+r^2(-15r_0\hat{r}_2^2-22r_0\hat{r}_2\hat{r}_6-15r_0\hat{r}_6^2+28\hat{r}_2^2\hat{r}_6+20\hat{r}_2\hat{r}_6^2)+r(-12r_0\hat{r}_2^2\hat{r}_6-4r_0\hat{r}_2\hat{r}_6^2+8\hat{r}_2^2\hat{r}_6^2)-4r_0\hat{r}_2^2\hat{r}_6^2, \tag{72}$$

$$Q_1^3=r^2(12r_0\hat{r}_2+12r_0\hat{r}_6+9\hat{r}_2^2+10\hat{r}_2\hat{r}_6+9\hat{r}_6^2)+r(8r_0\hat{r}_2\hat{r}_6+10\hat{r}_2^2\hat{r}_6+6\hat{r}_2\hat{r}_6^2)-4r_0\hat{r}_2^2\hat{r}_6, \tag{73}$$

$$Q_2^3=r_0(6r^2\hat{r}_2-2r^2\hat{r}_6+8r\hat{r}_2\hat{r}_6+\hat{r}_2^2\hat{r}_6+3\hat{r}_2\hat{r}_6^2), \tag{74}$$

$$Q_3^3=-r^2(24r_0\hat{r}_2+8r_0\hat{r}_6+27\hat{r}_2^2+6\hat{r}_2\hat{r}_6+3\hat{r}_6^2)-r(24r_0\hat{r}_2\hat{r}_6+30\hat{r}_2^2\hat{r}_6+2\hat{r}_2\hat{r}_6^2)+2r_0\hat{r}_2^2\hat{r}_6-6r_0\hat{r}_2\hat{r}_6^2-8\hat{r}_2^2\hat{r}_6^2. \tag{75}$$

Leaving aside the decoupled mode $\delta\psi_3$, which will be discussed in the next section (and which we will find leads to no instabilities), we have now reduced the perturbation problem to these three second-order equations. The background whose stability we are mainly interested in addressing is the shell branch solution, so we will also need to formulate appropriate matching conditions at the shell. The determination of the matching conditions and the numerical investigation of the existence of suitable solutions of the equations for negative ω^2 will be the subject of a forthcoming companion publication [22].

V. STABILITY OF THE FREE WAVE EQUATION

We will now discuss the stability of the solution against perturbation by just turning on $\delta\psi_3$. We make the ansatz $\delta\psi_3(t,r)=\Psi_3(r)e^{i\omega t}$. Then Eq. (49) implies

$$\frac{K^2}{Z_2Z_6}\left[\partial_r^2\Psi_3+\left(\frac{K'}{K}+\frac{2}{r}\right)\partial_r\Psi_3\right]+\omega^2\Psi_3=0. \tag{76}$$

If we consider the horizon branch, we need simply look for solutions of this equation regular on the horizon and at infinity. For the shell branch, Eq. (76) applies for $r>r_e$, and

$$\frac{L^2}{H_2H_6}\left[\partial_r^2\Psi_3+\left(\frac{L'}{L}+\frac{2}{r}\right)\partial_r\Psi_3\right]+\frac{L(r_e)}{K(r_e)}\omega^2\Psi_3=0 \quad (77)$$

applies for $r<r_e$. Since the shell does not couple to $\delta\psi_3$, the appropriate boundary conditions at the shell are that Ψ_3 and $\partial_r\Psi_3$ are continuous.

We translate this into a standard one-dimensional bound state problem, by introducing the tortoise coordinate

$$r_*=\begin{cases}\sqrt{\frac{L(r_e)}{K(r_e)}}\int_{r_e}^{r}\frac{\sqrt{H_2H_6}}{L}d\bar{r}, & r<r_e,\\ \int_{r_e}^{r}\frac{\sqrt{Z_2Z_6}}{K}d\bar{r}, & r>r_e.\end{cases} \quad (78)$$

This coordinate runs from $-\infty$ at $r=r_0'$ to $+\infty$ at $r=\infty$. We make a change of variable[5]

$$\Psi_3=\begin{cases}\frac{1}{(Z_2Z_6)^{1/4}r}\psi, & r<r_e,\\ \frac{1}{(H_2H_6)^{1/4}r}\psi, & r>r_e.\end{cases} \quad (79)$$

Then Eq. (77) becomes

$$\partial_{r_*}^2\psi+\omega^2\psi+W\psi=0, \quad (80)$$

where for $r>r_e$,

$$W(r)=W_>\equiv\frac{K^2}{Z_2Z_6}\left[\frac{1}{4}\frac{Z_2''}{Z_2}+\frac{1}{4}\frac{Z_6''}{Z_6}-\frac{5}{16}\left(\frac{Z_2'}{Z_2}\right)^2-\frac{5}{16}\left(\frac{Z_6'}{Z_6}\right)^2-\frac{1}{8}\frac{Z_2'}{Z_2}\frac{Z_6'}{Z_6}+\frac{1}{4}\left(\frac{Z_2'}{Z_2}+\frac{Z_6'}{Z_6}\right)\frac{K'}{K}+\frac{1}{r}\frac{K'}{K}\right], \quad (81)$$

while for $r<r_e$,

$$W(r)=W_<\equiv\frac{K(r_e)}{L(r_e)}\frac{L^2}{H_2H_6}\left[\frac{1}{4}\frac{H_2''}{H_2}+\frac{1}{4}\frac{H_6''}{H_6}-\frac{5}{16}\left(\frac{H_2'}{H_2}\right)^2-\frac{5}{16}\left(\frac{H_6'}{H_6}\right)^2-\frac{1}{8}\frac{H_2'}{H_2}\frac{H_6'}{H_6}+\frac{1}{4}\left(\frac{H_2'}{H_2}+\frac{H_6'}{H_6}\right)\frac{L'}{L}+\frac{1}{r}\frac{L'}{L}\right]. \quad (82)$$

Plugging in the functions from Eq. (12), we have

$$W_>=\frac{K}{16Z_2^3Z_6^3}\{[8(\hat{r}_2+\hat{r}_6)+16r_0]r^{-3}+[3\hat{r}_2^2+3\hat{r}_6^2+30\hat{r}_2\hat{r}_6+20r_0(\hat{r}_2+\hat{r}_6)]r^{-4}+[12\hat{r}_2\hat{r}_6(\hat{r}_2+\hat{r}_6)+9r_0(\hat{r}_2+\hat{r}_6)^2]r^{-5}+[4\hat{r}_2^2\hat{r}_6^2+8r_0\hat{r}_2\hat{r}_6(\hat{r}_2\hat{r}_6)]r^{-6}+4r_0\hat{r}_2^2\hat{r}_6^2r^{-7}\}. \quad (83)$$

The general form of $W_<$ is complicated, but in the case $M=0$, where we have simply an uncharged black hole inside the shell,

$$W_<=\frac{K(r_e)}{Z_2(r_e)Z_6(r_e)}\frac{L}{L(r_e)}\frac{r_0'}{r^3}. \quad (84)$$

On the horizon branch, where $\hat{r}_2>0$, $W>0$ everywhere, and there can be no instability associated with this mode. This is as we would expect; the horizon branch looks like a normal charged black hole solution, and the free wave equation has no non-constant solutions regular both on the horizon and at infinity. However, on the shell branch, there may be a region with $W_><0$. (Since we take the horizon branch for the solution inside the shell, $W_<$ is always positive.) The leading term is always positive, as

$$\hat{r}_2+\hat{r}_6+r_0=\frac{1}{2}\sqrt{4r_6^2+r_0^2}\pm\frac{1}{2}\sqrt{4r_2^2+r_0^2}>0, \quad (85)$$

since $|r_2|<r_6$. On the other hand, $W_>$ is always negative near $r=-\hat{r}_2$. As $r\to-\hat{r}_2$,

$$W_>\approx-\frac{5}{16}\frac{\hat{r}_2^2K^2}{r^4Z_2^3Z_6}<0. \quad (86)$$

If we considered just the pure repulson solution, this divergence would lead us to suspect the solution is unstable to a perturbation by $\delta\psi_3$. Although one would need to consider the issue of boundary conditions at the singularity, $W_>$ diverges sufficiently quickly that there could be bound states supported away from $r=-\hat{r}_2$. The question, then, is whether the enhançon excises this instability, along with the various other undesirable features of the geometry.

In Fig. 1, we plot the potential for some representative values of the parameters. We see that there is a substantial region outside the shell where the potential is negative, and might suspect that this signals an instability.

However, there is a general argument which says that there can never be an instability in this case [26]. First, we note that as Eq. (76) is simply the free wave equation in this background, it always has $\delta\psi_3=$ constant as a solution. In terms of the bound state problem (80), this translates into the statement that there is a zero energy ($\omega=0$) eigenmode ψ_0 of Eq. (80) which is of the same sign and is bounded every-

[5]Note that $\partial_r\Psi_3$ being continuous is not equivalent to $\partial_r\psi$ being continuous.

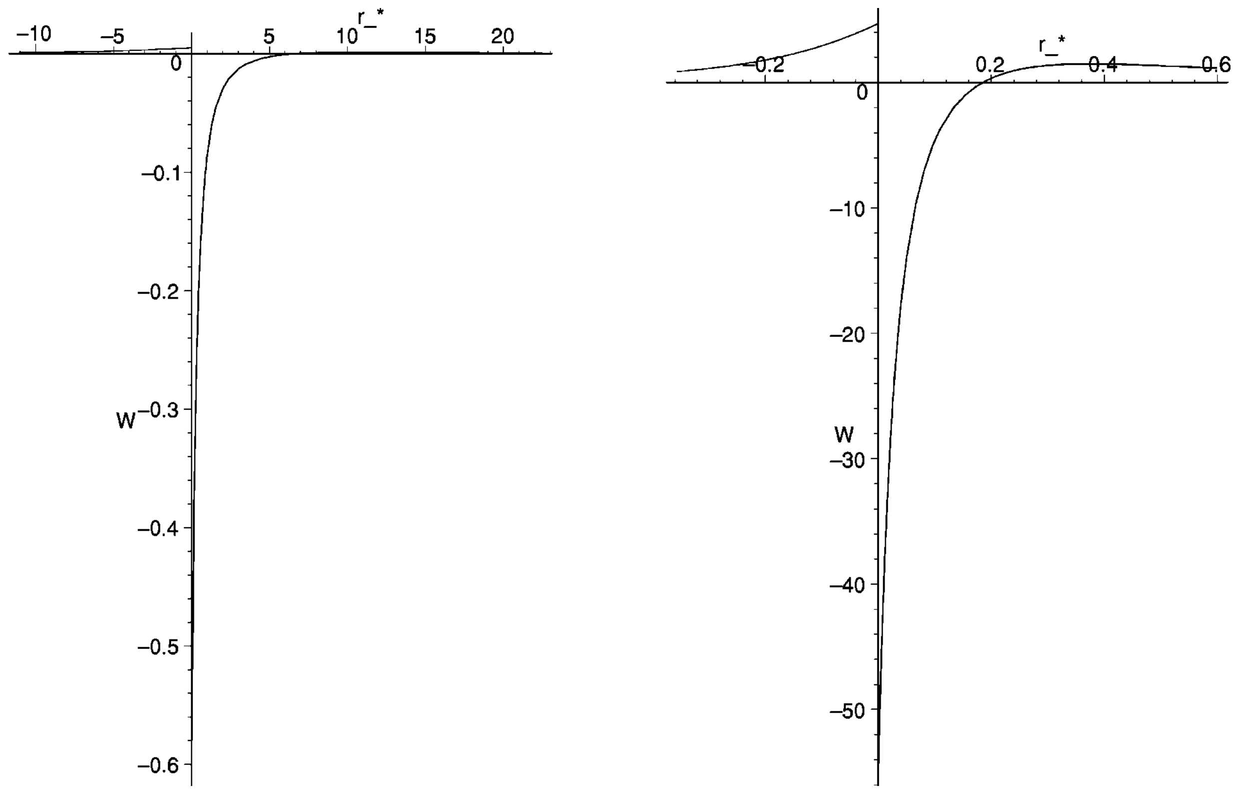

FIG. 1. $r_6^2 W$ plotted against r_*/r_6 for (left) $r_0=10r_6$, $V=1000V_*$, $M=0$, and (right) $r_0=r_6/10$, $V=1000V_*$, $M=0$.

where; we can take it to be always positive. This zero mode ψ_0 does not vanish at the boundaries, so it is not a physical perturbation but it is still an acceptable mathematical solution of this equation.

Now assume there is a discrete spectrum of bound states ψ_ω with negative energy. These are our hypothetical unstable modes with $\omega^2<0$. We can see from the form of Eq. (80) that they go to zero as $r_*\rightarrow\pm\infty$. This means that they are bounded solutions and physical perturbations of our problem. The standard "node rule" for the number of nodes of the eigenfunctions of the discrete bound states says that in order of increasing energy, the nth eigenmode has $n-1$ nodes (without including the boundary ones). Thus, the lowest negative mode $\psi_{\omega_{max}}$ must have no nodes in the sense of the above rule: we can take it to be everywhere positive.

Both ψ_0 and $\psi_{\omega_{max}}$ are solutions of the wave equation (80). By multiplying the equation for each mode by the other and taking the difference, and integrating over r_*, we can obtain the equation

$$(\psi_{\omega_{max}}\partial_{r_*}\psi_0-\psi_0\partial_{r_*}\psi_{\omega_{max}})\big|_{r_*=\pm\infty} = -\omega_{max}^2\int_{-\infty}^{\infty}\psi_{\omega_{max}}\psi_0 dr_* . \quad (87)$$

The left-hand side is the difference of the Wronskians calculated at the boundaries. Since the eigenmode ψ_0 approaches a positive constant at the boundaries $r_*=\pm\infty$, while the eigenmode $\psi_{\omega_{max}}$ goes to zero, the Wronskian vanishes at each boundary. Hence, the left-hand side is zero. On the other hand, since both $\psi_{\omega_{max}}$ and ψ_0 are supposed to be everywhere positive, the right-hand side cannot be zero unless $\omega_{max}=0$. Thus, assuming the existence of eigenmodes ψ_ω with $\omega^2<0$ produces a contradiction. Hence there can be no such modes, implying that the geometry is stable to perturbation by $\delta\psi_3$.

ACKNOWLEDGMENTS

We are grateful for discussions with James Gregory, Clifford Johnson and Matt Strassler. This work was supported in part by the EPSRC, and by funds from the University of Durham.

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