Temperature inversion symmetry in the Casimir effect with an antiperiodic boundary condition

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We present explicitly another example of a temperature inversion symmetry in the Casimir effect for a nonsymmetric boundary condition. We also give an interpretation for our result.

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This Brief Report was motivated by a recent paper published by Santos *et al.* [1], in which they discuss the temperature inversion symmetry in the Casimir effect [2] for mixed boundary conditions (for a detailed discussion on the Casimir effect see [3,4] and references therein). In an earlier paper, Ravndal and Tollefsen [5] showed that for the usual setup of two parallel plates a simple inversion symmetry arises in the Casimir effect at finite temperature. Temperature inversion symmetry also appeared in the Brown-Maclay work [6] where they related directly the zero-temperature Casimir energy to the energy density of blackbody radiation at temperature *T*. A few other papers on this kind of symmetry have also been published [7–11]. Until the publication of Ref. [1], this kind of inversion symmetry had appeared only in calculations of Casimir energy involving symmetric boundary conditions. In 1999, Santos *et al.* [1] showed, for the case of a massless scalar field submitted to mixed boundary conditions (Dirichlet-Neumann), that the Helmholtz free energy per unit area could be written as a sum of two terms, each of them obeying separately a temperature inversion symmetry. Our purpose here is to present another kind of nonsymmetric boundary condition for which there exists such a symmetry. We show explicitly that for the massless scalar field under an antiperiodic boundary condition the Helmholtz free energy per unit area can also be cast as a sum of two terms, where each one satisfies a temperature inversion symmetry.

The Casimir effect for a massless scalar field under an antiperiodic boundary condition (compactification of R^1 to S^1) in a 3+1 spacetime is given by

$$\phi(\tau,x,y,z)=-\phi(\tau,x,y,z+a). \tag{1}$$

We will use the imaginary time formalism, which means that

$$\phi(\tau,x,y,z)=\phi(\tau+\beta,x,y,z) \tag{2}$$

where $\beta=T^{-1}$, the reciprocal of the temperature. The conditions (1) and (2) lead us to the following eigenvalues for the Euclidean operator $\partial_E^2=\partial^2/\partial\tau^2+\nabla^2$:

$$\left\{\kappa^2+(2n+1)^2\left(\frac{\pi}{a}\right)^2+\left(\frac{2\pi\ell}{\beta}\right)^2 \right. \\ \left. \text{where } \kappa^2=k_1^2+k_2^2,\ n \text{ and } \ell=0,\pm1,\pm2,\cdots\right\}. \tag{3}$$

The partition function for a massless scalar field at finite temperature is given by

$$\mathcal{Z}(\beta)=N\int_{periodic}[\mathcal{D}\phi]\exp\int_0^\beta\int d^3x\mathcal{L}, \tag{4}$$

where N is a normalization constant and $\mathcal{L}$ is the Lagrangian density for the theory under study. The Helmholtz free energy $F(\beta)$ can then be written in terms of the corresponding generalized ζ function as

$$F(\beta)=-\frac{1}{2\beta}\frac{d}{ds}\zeta(s,-\partial_E^2)|_{s=0}. \tag{5}$$

After some algebra, we can write the zeta function as

$$\zeta(s,-\partial_E^2)=\frac{L^2}{4\pi}\frac{\Gamma(s-1)}{\Gamma(s)}\left\{2\left(\frac{\pi}{a}\right)^{2-2s}(1-2^{2-2s})\zeta_R(2s-2) \right. \\ +4\pi^{2-2s}E_2\left(s-1,\frac{1}{a^2},\frac{4}{\beta^2}\right)-4\pi^{2-2s}E_2 \\ \left.\times\left(s-1,\frac{4}{a^2},\frac{4}{\beta^2}\right)\right\}, \tag{6}$$

where $\zeta_R(z)$ is the Riemann zeta function and $E_2(z,a_1,a_2)$ is an Epstein function. Using the same methods of Ref. [1], we can write the Helmholtz free energy per unit area as

$$\frac{F(\beta)}{L^2}=\frac{7}{720}\frac{\pi}{a^3}-\frac{1}{\pi\beta^3}f(\xi), \tag{7}$$

where we defined $\xi=a/\pi\beta$ and

$$f(\xi)=\frac{1}{2\pi^4\xi^3}\left\{\sum_{\ell,n=-\infty}^{\infty}\frac{(-1)^n\pi^4\xi^4}{[\ell^2+\pi^2\xi^2n^2]^2}-\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{n^4}\right\}. \tag{8}$$

Note that the first term on the right-hand side (RHS) of Eq. (7) represents the Casimir energy at zero temperature for the antiperiodic boundary condition. We should now eliminate

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from Eq. (8) the terms $\ell = n = 0$ in the first summation and $n=0$ in the second one (see Ref. [1] for details). Once it has been done, we have (after convenient manipulations)

$$\frac{F(\beta)}{L^2} = \frac{F_1(\beta)}{L^2} - \frac{F_2(\beta)}{L^2}, \tag{9}$$

where the functions $F_1(\beta)/L^2$ and $F_2(\beta)/L^2$ are defined by

$$\frac{F_1(\beta)}{L^2} = -\frac{1}{16\pi^2 a^3} \sum_{\ell,n=-\infty}^{\infty} \frac{(2\pi\xi)^4}{[\ell^2+(2\pi\xi n)^2]^2} \tag{10}$$

and

$$\frac{F_2(\beta)}{L^2} = -\frac{1}{2\pi^2 a^3} \sum_{\ell,n=-\infty}^{\infty} \frac{(\pi\xi)^4}{[\ell^2+(\pi\xi n)^2]^2}, \tag{11}$$

which satisfy the following temperature inversion symmetry relations:

$$F_1(\xi) = (2\pi\xi)^4 F_1\left(\frac{1}{4\pi^2\xi}\right) \quad \text{and} \quad F_2(\xi) = (\pi\xi)^4 F_1\left(\frac{1}{\pi^2\xi}\right). \tag{12}$$

The temperature inversion symmetry just presented for the case with an antiperiodic boundary condition may be interpreted following the same lines as that appearing in Ref. [1]. In this reference, Santos *et al.* showed that the Helmholtz free energy per unit area for a massless scalar field under mixed boundary condition may be written as a sum of two terms corresponding, each one, to a pair of uncharged parallel perfectly conducting plates kept at a distance $2d$ and d apart, respectively. In the present case, we have an analogous situation, namely, the Helmholtz free energy per unit area for a massless scalar field under an antiperiodic boundary condition (with spatial "period" a) may also be written as a sum of two terms corresponding, each one, to a periodic boundary condition, but with spatial periods $2a$ and a, respectively [see Eqs. (10) and (11)].

Temperature inversion symmetry directly relates the Casimir effect at zero temperature to its high temperature limit, where the Stefan-Boltzmann term dominates and hence may be viewed as one of the simplest examples of duality. Our result provides one more explicit example of such a phenomenon. In addition, this kind of symmetry can be useful to derive approximate expressions for the Helmholtz free energy. For instance, we can derive the low temperature limit, that is, we may calculate the first thermal corrections to the zero temperature Casimir energy, from the high temperature limit, which is in general much easier to obtain.

The consideration of massive fields is not an easy task. It is not obvious whether this duality symmetry will remain valid for massive fields, but this will be left for a future investigation.

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