Holographic Weyl entropy bounds

Andrew Chamblin* and Joshua Erlich†
Department of Physics, Queen Mary, University of London, Mile End Road, London E1 4NS, United Kingdom
Department of Physics, Box 1560, University of Washington, Seattle, Washington 98195, USA
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We consider the entropy bounds recently conjectured by Fischler, Susskind, and Bousso, and proven in certain cases by Flanagan, Marolf, and Wald (FMW). One of the FMW derivations supposes a covariant form of the Bekenstein entropy bound, the consequences of which we explore. The derivation also suggests that the entropy contained in a vacuum spacetime, e.g., Schwarzschild spacetime, may be related to the shear on congruences of null rays. We find evidence for this intuition, but in a surprising way. We compare the covariant entropy bound to certain earlier discussions of black hole entropy, and comment on the separate roles of quantum mechanics and gravity in the entropy bound.

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I. INTRODUCTION

Various authors have put forward the idea that a ''holographic principle'' should be incorporated into any attempt to construct a quantum theory of gravity. This principle, which was first developed in papers by 't Hooft [1] and Susskind [2], is on the surface a radical statement about how many degrees of freedom there are in nature. In essence, the principle asserts that a physical system can be completely described by information which is stored at the boundary of the system, without exceeding one bit of information per unit Planck area. Much study has been devoted to the related topics of black holes, entropy bounds, and holography, and we will not be able to do justice to the bulk of prior work in the field. For reviews see, for example, [3–6] and references therein.

For some time, there was no precise covariant statement of the holographic principle; however, this situation was rectified in a series of elegant papers by Fischler and Susskind [7] and Bousso [8–10]. In particular, by choosing appropriate lightlike surfaces (called ''light sheets'') where the entropy of a given system can reside, Bousso was able to develop a mathematically precise covariant entropy conjecture. Bousso defined a light sheet Γ associated with a spacelike two-surface B as a null congruence orthogonal to B. The light sheet is terminated at caustics, spacetime boundaries and singularities. A Bousso light sheet has the further property that the expansion θ (see [11], for example) is everywhere nonpositive. The covariant entropy bound is then the statement that the entropy contained in a Bousso light sheet is bounded by the area of its initial boundary B, or simply (suppressing $\hbar$ and Newton's constant G),

$$S(\Gamma)\leqslant A(B)/4. \tag{1}$$

Soon after the work of Fischler, Susskind, and Bousso (FSB), a proof of various classical versions of Bousso's bound was provided by Flanagan, Marolf, and Wald (FMW) [12]. In order to make a mathematically precise statement, which could consequently be proven, they took the step of introducing an entropy flux vector, denoted s^a. The total entropy through a given light sheet Γ is then defined to be the integral of s^a over the surface of the light sheet. They showed that FSB-type bounds could be proven, provided the entropy flux vector satisfied one of the following two sets of criteria:

Either,

$$s_\Gamma\cdot k\leqslant(1-\lambda)(\pi k_a T^{ab}k_b+\sigma_{ab}\sigma^{ab}/8), \tag{2}$$

or

$$(s_a k^a)^2\leqslant T_{ab}k^a k^b/(16\pi)+\sigma_{ab}\sigma^{ab}/(128\pi^2),$$

$$|k^a k^b\nabla_a s_b|\leqslant \pi T_{ab}k^a k^b/4+\sigma_{ab}\sigma^{ab}/32, \tag{3}$$

where k^a denotes the tangent vector to a given null geodesic generating the light sheet in question, T_{ab} denotes the stress-energy tensor, and σ_{ab} denotes the shear tensor of the null congruence [13]. The affine parameter λ is normalized to range from 0 to 1 over the light sheet. The first condition (2) is defined for each light sheet Γ, while the second set of conditions (3) are defined pointwise. The condition (2) is reminiscent of the Bekenstein bound,

$$S\leqslant 2\pi ER, \tag{4}$$

where E is the energy contained in a system of size R. (Reference [14] also discusses the Bekenstein bound and its relation to the Bousso bound. In particular, a version of the Bekenstein bound is derived from the Bousso bound, reversing the logic discussed here.) We will refer to the relation (2) as the covariant Bekenstein bound. Indeed, we will note in the next section that for spherically symmetric systems the condition (2) is quite similar to Eq. (4). A stronger form of the Bousso bound follows from Eq. (2), but not Eq. (3). If the light sheet Γ is terminated on some spacelike 2-surface B', then given Eq. (2) the entropy in the light sheet was shown in [12] to satisfy

*Email address: chamblin@mit.edu
†Email address: erlich@phys.washington.edu

$$S \leqslant \frac{A(B)-A(B')}{4}. \tag{5}$$

In fact, we will find that this bound can never be saturated unless $A(B')=0$.

Flanagan, Marolf, and Wald chose to consider somewhat stronger conditions on the entropy flux vector than Eq. (2) or Eq. (3) by suppressing the terms proportional to σ_{ab}^2, although the neglected terms can be included without violating the Bousso bound. It is suggested in [12] that the neglected terms might be related to gravitational entropy, an idea which we will explore further in this paper. The possible relation between the shearing of null congruences and gravitational entropy is reminiscent of Penrose's suggestion that the shear should be interpreted as a measure of gravitational *energy* [15,16]. The Weyl tensor acts as a source for the shear tensor, and hence the shearing of null rays can naturally be interpreted as due to a gravitational contribution to the energy. Amusingly, some of the ideas in Penrose's paper are tantalizingly similar to those of Fischler, Susskind, and Bousso if one replaces the concept of entropy with energy. Indeed, quoting from [15]:

"...it is suggested that the resultant focusing power of spacetime curvature along a null ray is a good measure of the total energy flux (matter plus gravitation) across the ray ... ."

In other words, Penrose was interested in the idea of measuring the total energy (as opposed to entropy) which can flow through a given light sheet. Generically there are two different types of focusing which may occur along a Bousso light sheet: anastigmatic focusing due to the stress-energy tensor T_{ab}, and purely astigmatic focusing due to the shear tensor σ_{ab}. Here, Penrose borrowed the terminology from elementary optics. Anastigmatic focusing means that the conjugate points will appear before the central point at which the rays would ordinarily have focused (in the absence of any stress-energy). On the other hand, if we fire a graviton through a light sheet then we will see astigmatic focusing: The congruence of curves will acquire shear. Physically, this means that an initially spherical ingoing array of photons will be deformed into an ellipsoid towards the center of the regions bounded by B. The magnitude of the distortion and the angle of the ellipse measure the amplitude and polarization properties of the incident graviton [15].

According to Penrose, astigmatic focusing is to be interpreted as due to gravitational energy, and indeed a relation similar to Eq. (4) would then imply a relation between the shear and gravitational entropy. The intuition that the Weyl tensor should somehow count gravitational degrees of freedom is also reflected in [17].

The physical interpretation of the entropy flux vector is somewhat obscure, and in general it is certainly not clear that a quasilocal entropy current should exist. However, if an entropy flux vector can be suitably defined, both sets of conditions (2) and (3) are reasonable under a large class of situations [12]. It should also be noted that these conditions could be violated in situations in which the number of species of matter is large enough. Hence, by assuming either of the above sets of conditions the species problem is swept under the rug. Be that as it may, in the hope that it will indeed follow from a fundamental theory including gravity, at least in a large class of situations, we would like to take the Bekenstein-like relation (2) seriously and further explore its consequences.

In particular, we will investigate how the shear tensor of a given light sheet might be a measure of the number of gravitational degrees of freedom. On the surface, this seems counterintuitive because many null geodesic congruences will have vanishing shear, even though the spacetime may have large curvature. Indeed, spherically symmetric light sheets in Schwarzschild spacetime are shear-free. As a result, the potential relationship between shear and entropy is *a priori* dubious, and leads us to consider less symmetric Bousso light sheets. We introduce the concept of a maximal entropy light sheet, and are led to a type of ultraviolet-infrared duality between matter and gravitational entropy.

II. CONSEQUENCES OF THE COVARIANT BEKENSTEIN BOUND

In this section we will consider the properties of the covariant Bekenstein bound (2) and its relation to the Bousso bound. Flanagan, Marolf, and Wald [12] derived the Bousso bound (1) from the covariant Bekenstein bound (2). It is interesting that saturation of the inequality (2) does not imply saturation of the Bousso bound (1). We begin by studying the conditions for saturation of the Bousso bound. Black holes are expected to saturate "useful" entropy bounds, and as black holes provide the motivation for most of these ideas, we would like to learn what we can about them from reasonable assumptions like the covariant Bekenstein bound. To this end, we first review the derivation of the Bousso bound from Eq. (2), following [12].

A. Saturation of the Bousso bound

The area factor $\mathcal{A}(\lambda)$ is given by

$$\mathcal{A}(\lambda)=\exp\int_0^\lambda d\bar{\lambda}\,\theta(\bar{\lambda}), \tag{6}$$

where θ is the expansion along a null ray in the congruence (see, for example, [11]). $\mathcal{A}(\lambda)$ measures the ratio of the area of the spacelike slice of the light sheet at affine parameter λ to the area of the boundary 2-surface, $A(B)$. It is helpful to define, as in [12],

$$G(\lambda)=\sqrt{\mathcal{A}(\lambda)}. \tag{7}$$

The entropy $S(\Gamma)$ is given by [12]

$$S(\Gamma)=A(B)\int_0^1 d\lambda\, s_a k^a \mathcal{A}(\lambda). \tag{8}$$

From this and Eq. (2), the generalized Bousso bound (5) is equivalent to the statement that along each null ray k in the congruence,

$$I_\gamma \equiv \frac{1}{8}\int_0^1 d\lambda(1-\lambda)(\sigma^2+8\pi kTk)\mathcal{A}(\lambda) < \frac{1}{4}[1-\mathcal{A}(1)]. \tag{9}$$

Using the Raychaudhuri equation,

$$-\frac{d\theta}{d\lambda} = \frac{1}{2}\theta^2 + 8\pi k_a T^{ab} k_b + \sigma_{ab}^2, \tag{10}$$

and integrating by parts, we can rewrite I_γ as

$$I_\gamma = -\frac{1}{4} d\lambda(1-\lambda)G''(\lambda)G(\lambda) \tag{11}$$

$$= -\frac{1}{4}\int_0^1 d\lambda(1-\lambda)G''(\lambda) + \frac{1}{4}\int_0^1 d\lambda(1-\lambda)G''(\lambda)[1-G(\lambda)] \tag{12}$$

$$= \frac{1}{4}[G(0)-G(1)+G'(0)] + \frac{1}{4}\int_0^1 d\lambda(1-\lambda)G''(\lambda)[1-G(\lambda)]. \tag{13}$$

But $G(0)=1$ and $G(1)=\sqrt{\mathcal{A}(1)}$, so we can now write I_γ as

$$I_\gamma = \frac{1}{4}[1-\mathcal{A}(1)] + \frac{1}{4}G'(0) - \frac{1}{4}(\sqrt{\mathcal{A}(1)}-\mathcal{A}(1)) - \frac{1}{4}\int_0^1 d\lambda(1-\lambda)G''(\lambda)[G(\lambda)-1]. \tag{14}$$

The term in Eq. (14) proportional to $G'(0)$ is nonpositive because $G' = 1/2\theta G$ is manifestly nonpositive along ingoing null rays. The next to last term in Eq. (14) is negative because $\mathcal{A}(1)<1$. The last term is negative if we assume the null energy condition,

$$k_a T^{ab} k_b \geq 0, \tag{15}$$

because then $G'' = -1/2(\sigma^2+8\pi kTk)G$ is manifestly negative. Equation (9) follows.

This demonstrates that the Bousso bound follows from the covariant Bekenstein bound with the additional assumption of the null energy condition, and also demonstrates under what conditions the Bousso bound can be saturated. Namely, to saturate the Bousso bound in this situaion it is necessary that:

(1) $\mathcal{A}(1)=0$.
(2) $\theta|_{\lambda=0}=0$.
(3) $\sigma^2+8\pi kTk$ vanishes if $\lambda \neq 1$ or $\mathcal{A}(\lambda)\neq 1$.

The first requirement implies that the Bousso bound can be saturated only in its weak form, Eq. (1), and not its stronger form, Eq. (5) [except when the cutoff 2-surface B' vanishes so that Eqs. (5) and (1) are equivalent]. Hence, we consider light sheets which are terminated only at caustics. The second requirement is that the expansion vanish at the light sheet boundary B. Notice that the second requirement is a condition on the choice of the boundary two-surface, and is violated infinitesimally if a two-surface satisfying the condition is deformed infinitesimally. This will be important when we consider the contribution of shear to the entropy bounds. Finally, the third requirement necessitates that there be no "source" of entropy except at points of vanishing expansion. This is reminiscent of the membrane paradigm [18], and also of an operational definition of black hole entropy by Pretorius, Vollick, and Israel (PVI) [19]. PVI define the entropy of a black hole as the entropy that must be given to a thin shell of matter brought from infinity to its Schwarzschild horizon in order to maintain mechanical and thermal equilibrium (with the local acceleration temperature on the shell) during the process. It may be possible to formulate such a definition covariantly making use of these ideas, although we will not be more precise about such a relation here. We also point out Ref. [20], where it was also argued that a thin spherical shell held in mechanical and thermodynamic equilibrium at its horizon would have entropy $S_{BH}=A/4$.

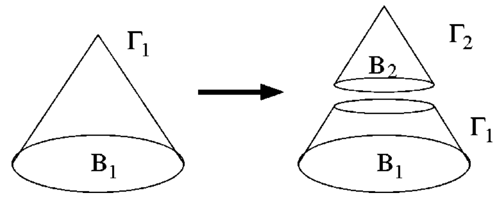

FIG. 1. The entropy is additive over the light sheet, so the light sheet can be broken up into sections.

B. Dependence of the Bousso bound on choice of light sheet

It is necessary *a priori* to distinguish between the Bousso bound (1) and the area law for black holes,

$$S_{BH} = A_h/4, \tag{16}$$

where A_h is the area of the black hole horizon. In a black hole spacetime we would like it to be the case that an entropy bound somehow be related to the horizon area A_h, as opposed to the light sheet boundary area $A(B)$. Note that the entropy, $S=\int_\Gamma s\cdot k d\lambda d^2x$, can be broken up into sections as in Fig. 1, so that $S=\Sigma_i S_i$, where S_i is the entropy in the ith section, $S_i = A(B_i)\int_{\Gamma_i} s\cdot k\mathcal{A}(\lambda)d\lambda d^2x$. Intuitively, if matter and black hole horizons are confined to a particular Γ_i then we would expect the entropy in a light sheet which contains Γ_i to depend only on Γ_i. It would be still better if the maximal entropy satisfying the covariant Bekenstein bound, and hence I_γ in Eq. (9), depended only on Γ_i. If we include the shear in our analysis this is not strictly true, as we will discuss, but in the absence of shear this result follows immediately from Eq. (2). To see how this works explicitly in a specific case, consider a static, spherically symmetric, thin shell of matter (Fig. 2). We will assume the geometry is Minkowski space inside the shell and Schwarzschild outside. Both geometries have metrics of the form,

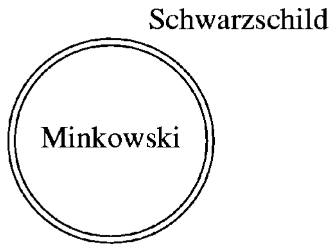

FIG. 2. A thin shell of matter separates regions of Schwarzschild and flat spacetimes.

$$ds^2=f(r)dt^2-h(r)dr^2-r^2d\Omega^2. \tag{17}$$

We assume the shell of matter is at $r=R$, and we require that the induced metric be equivalent on both sides of the shell. For the Schwarzschild region we have

$$f_{Sch}(r)=1-\frac{2M}{r}, \quad h_{Sch}(r)=1/f_{Sch}(r), \tag{18}$$

and for the Minkowski region we have

$$f_{Mink}(r)=1-\frac{2M}{R}\equiv f_0, \quad h_{Mink}(r)=1. \tag{19}$$

The existence of a Killing vector ∂_t implies a constant of the motion e, such that

$$\frac{dt}{d\lambda}=eg^{tt}=e/f. \tag{20}$$

Vanishing of ds^2 along the null path then implies that

$$\frac{dr}{d\lambda}=\frac{e}{\sqrt{f(r)h(r)}}. \tag{21}$$

Suppose the spherically symmetric null congruence reaches the shell at $r=R$ when the affine parameter takes the value $\lambda=\lambda_0$. Then r goes from R to 0 when λ goes from λ_0 to 1. Hence

$$e=\frac{R\sqrt{f_0}}{(1-\lambda_0)}. \tag{22}$$

The stress tensor has the form

$$8\pi T_a^b=S_a^b\frac{\delta(r-R)}{\sqrt{g_{rr}}}. \tag{23}$$

The Israel junction condition [21] determines S_a^b. If h_{ij} is the induced metric on the shell, and K_{ij} is the extrinsic curvature at the shell, we find that the combination $K_{ij}-Kh_{ij}$ (where $K=K_{ij}h^{ij}$) takes the values

$$K_{ij}-Kh_{ij} \simeq \mathrm{diag}\left(\frac{2f}{r\sqrt{h}},0,\frac{-r(2f+rf')}{2f\sqrt{h}},\frac{-r\sin^2(\theta)(2f+rf')}{2f\sqrt{h}}\right), \tag{24}$$

in the spherical basis (t,r,θ,ϕ). The Israel junction condition relates the change in the extrinsic curvature across the shell to the localized stress tensor on the shell:

$$\Delta(K_{ij}-Kh_{ij})=S_{ij}, \tag{25}$$

where $\Delta(\cdots)$ refers to the difference in the quantity $(\cdots)$ evaluated just inside and just outside the shell. Using Eq. (24) we find

$$S_t^t=\frac{2}{R}(h_{Mink}^{-1/2}(R)-h_{Sch}^{-1/2}(R)). \tag{26}$$

Note also that the shear vanishes on a spherically symmetric light sheet in a spherically symmetric spacetime. We now have all of the information required to calculate I_γ defined in Eq. (9), assuming saturation of Eq. (2). The result is

$$I_\gamma/\mathcal{A}(\lambda_0)=\frac{1}{8}\int_0^1 d\lambda\, 8\pi k^a T_{ab}k^b=\frac{1}{4}(1-\sqrt{f_0}) \tag{27}$$

$$=\frac{1}{4}\left(1-\sqrt{1-\frac{2M}{R}}\right). \tag{28}$$

As expected the Bousso bound is saturated, i.e. $I_\gamma=1/4$, when the covariant entropy bound is saturated and the matter shell approaches its Schwarzschild radius. The Bousso bound cannot be saturated for the shell away from the Schwarzschild radius. Note also that this result is independent of the size of the boundary of the Bousso light sheet (as long as it is larger than the matter shell). This is consistent with our expectation that only the region "containing" the source of entropy contributes to the entropy in the light sheet.

In this setting, we can also be more precise as to the relation between the original Bekenstein bound and its covariant version. For the thin static shell the entropy is written in terms of the entropy flux vector as

$$\frac{S}{A}=\int d\lambda\, s^t k_t \tag{29}$$

$$=\int dr\frac{s^t k_t}{k^r}, \tag{30}$$

where in Eq. (30) we used $k^r=dr/d\lambda$. Using Eqs. (20) and (21) we can write

$$s^t=\frac{S}{A}\frac{1}{\sqrt{f}}\frac{\delta(r-R)}{\sqrt{g_{rr}}}. \tag{31}$$

In the limiting case that the shell forms a black hole $R=2M$. Comparing S_t^t with the black hole energy $E_{BH}=M$ in that case, we define the energy of the shell E, for any $R \geqslant 2M$, via

$$S_t^t=\frac{16\pi E}{A}. \tag{32}$$

With this definition of E, and using Eqs. (20) and (22), the covariant Bekenstein bound (2) integrated over the light sheet can be written exactly as the original Bekenstein bound (4). The original Bekenstein bound is saturated precisely when the covariant Bekenstein bound is saturated.

Although it is nice to have rederived some familiar results, we are immediately led to a puzzle. What if there was no matter shell? In Schwarzschild spacetime there is no stress tensor, and so no source of entropy if we assume the covariant Bekenstein bound. Then where might the entropy be, and how is this situation related to the case of a matter shell sitting at its horizon? We will suggest a solution to this puzzle without denouncing the covariant Bekenstein bound in the next section.

III. SHEAR AND GRAVITATIONAL ENTROPY

At the end of the last section we argued that if we assume a covariant Bekenstein bound of the form (2) then the entropy contained in a spherically symmetric light sheet in Schwarzschild spacetime vanishes. Thus, the horizon, and all other possible light sheets with the same symmetry, will remain shear-free. One can also imagine arranging a spherically symmetric impulsive gravitational wave front [22] which passes through the light sheet. However, this wavefront would also not impart any shear to a spherically symmetric null geodesic congruence, and hence could not contribute any entropy to a spherically symmetric light sheet by the argument above.

This would be in contradiction to the intuition that such a light sheet which ''contains'' a black hole should measure its entropy. One might be concerned that the singularity should somehow contribute to the Bousso bound in the presence of a black hole, and indeed whether or not this is the case relies on the interpretation of the Bousso bound in such situations. One solution to the puzzle would then be to simply define the entropy content of the singularity to be $A_h/4$. We suggest that such a modification to the Bousso bound is not necessary. One thing which we can immediately check is that the vanishing of the entropy as calculated above is not an accident of spherical symmetry. After all, one might worry that the portion of the light sheet near the singularity might give a large contribution to the entropy for a nonspherical light sheet even in the limit of spherical symmetry. One can consider a deformation of the spherically symmetric Bousso light sheet which avoids the singularity at all but a finite number of points, as caustics will generically be reached before the singularity along nonspherical light sheets.

The easiest way to see that the entropy vanishes in the spherical limit is by using the derivation of the Bousso bound described in the previous section and in [12]. For the case of the spherically symmetric light sheet we find $G'(0)=-1$, which exactly cancels the first term in Eq. (14). [We assume that $\mathcal{A}(1)=0$ on our light sheet.] Because $G'(0)$ varies smoothly as the light sheet is deformed, it must approach its spherical value of -1 in the spherical limit. I_γ is manifestly positive, so the last term in Eq. (14) must therefore vanish in the spherical limit. Hence, avoiding the singularity by an infinitesimal deformation of the light sheet boundary does not alter the result that the entropy ''inside'' the black hole vanishes assuming the covariant Bekenstein bound. Notice that this analysis relied only on the behavior of the light sheet at its boundary, and we were able to eliminate all reference to the bulk of the light sheet.

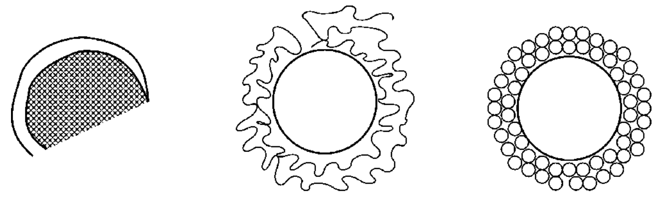

FIG. 3. (a) A portion of a folded light sheet boundary. The shaded region represents the portion of the corresponding light sheet which is overcounted in the sense described in the text. (b) A portion of the boundary of a ''crinkly'' light sheet surrounding a black hole horizon. (c) The crinkly light sheet is approximated by a ''space-filling'' set of smaller light sheets.

Given this result, rather than reject the covariant Bekenstein bound we would like to point out that if two light sheets ''contain'' all matter and black holes in a given spacetime, and if for both light sheets the covariant Bekenstein bound is saturated, then the two light sheets need not contain the same entropy. In this sense, the entropy of a spacetime is observer dependent. Indeed, a generic light sheet in the Schwarzschild geometry will be sheared, and may have a nonvanishing entropy according to Eq. (2). This line of thinking begs the question, is there a maximal entropy light sheet, perhaps with entropy given by the black hole horizon, in the case of a black hole spacetime? In this general sense the answer is clearly no, because even if we only consider Bousso light sheets with connected boundaries, the light sheet may fold back on itself [as in Fig. 3(a)] and overcount the entropy in a given spacelike region. Hence, a poor choice of Bousso light sheet can have as large an entropy as desired, while still satisfying the Bousso bound. We would like to consider light sheets which do not overcount the entropy. Similar concerns also appear in [23]. One choice of light sheet which satisfies this intuitive restriction is a crinkly light sheet which wraps back on itself many times, extending out to scri [Fig. 3(b)]. The light sheet can be approximated by a large number of smaller light sheets, which are space-filling in the appropriate sense as the light sheet becomes suitably crinkled [Fig. 3(c)]. Alternatively, at least for static spacetimes we can consider a Bousso light sheet formed from a disconnected set of boundary balls in a space-filling limit. Perhaps such a light sheet could have an entropy given by the black hole entropy. We will argue that, assuming saturation of the covariant Bekenstein bound, the entropy in such a light sheet indeed scales with the black hole area. Then we will discuss the interpretation of this result.

Assuming that the covariant Bekenstein bound (2) is saturated for the light sheet, the entropy through the light sheet is determined by I_γ as defined in Eq. (9). The explicit calculation of I_γ for closely packed small light sheets depends on two regulators, and potentially the shape of the light sheet boundaries. The two regulators are the size of the small light sheets, and how close to the black hole horizon the light sheets should be allowed to probe. We will choose both regulators to be Planck size in the ordinary Schwarzschild coordinates. Note that this is different from taking the corresponding proper sizes to be Planck scale, as we will comment on later. For notation, we take the size of the small light sheets to be L, and we will integrate over such light sheets to the position $r=2M+\delta$ in Schwarzschild coordinates. In that case, as an order of magnitude $k_t \sim L$ because each null ray traverses a distance on the order of L when the affine parameter goes from 0 to 1. In Schwarzschild metric, the tensor $B_{ab} \equiv \nabla_a k_b$, which contains the shear, contains the term

$$B_{tr} \sim \frac{LM}{r^2} \frac{1}{(1-2M/r)}. \tag{33}$$

The leading term in the shear squared is

$$(\sigma_{ab})^2 \sim B_{tr} B_{tr} g^{tt} g^{rr} = B_{tr}^2. \tag{34}$$

Multiplying this by the number of balls in a shell of radius r gives a factor of r^2/L^2, and integrating over the size of a ball gives a factor L^2, so assuming the covariant Bekenstein bound is saturated for the light sheet we then have

$$S_{shell} \sim (\sigma_{ab})^2 \frac{r^2}{L^2} L^2 \sim \frac{L^2 M^2}{r^2 (1-2M/r)^2}. \tag{35}$$

Now we want to integrate S_{shell} out to infinity. If we regulate the light sheets as suggested above, then integrating over shells gives

$$S \sim \int_{2M+\delta}^{\infty} \frac{dr}{L} S_{shell} \tag{36}$$

$$= M^2 L/\delta. \tag{37}$$

Note that we chose to regulate the size of the light sheets in the Schwarzschild coordinates, with no additional metric factors in the integral. If we choose the two regulators δ and L to be the Planck length in these coordinates, then from Eq. (37) it follows that

$$S \sim M^2 \sim A_h. \tag{38}$$

We should note that this result is sensitive to the choice of regulators, so that for example choosing the cutoff δ to be a proper distance of a Planck length away from the horizon, as opposed to defining the cutoff in the Schwarzschild coordinates, would yield a different result. At this point we do not have any further justification for our choice of regulators. However, we also point out that the result is only sensitive to the ratio of regulators L/δ, and therefore is insensitive to an identical rescaling of both regulators.

Notice that the shear (or Weyl) contribution to S comes entirely from the region with Schwarzschild coordinate r (as opposed to the proper length) within a few Planck lengths of the black hole horizon. We could have chosen the size of the outermost shell arbitrarily, and as long as it is much larger than the Planck scale in these coordinates the resulting entropy in our approximation would be unchanged. This is consistent with the intuitive notion that the entropy should be contained within a thin shell around the horizon (the membrane paradigm). In this sense, the calculation of the entropy for the thin shell of matter in the previous section is analogous to the calculation in this section of the Weyl entropy.

We have argued that the entropy in a space-filling light sheet (in the sense described above) is proportional to the horizon area, but we have not calculated the coefficient. It would be nice to understand under which circumstances the black hole entropy $S_{BH}=A_h/4$ would be obtained. The dependence of such a result on the shape of the light sheets would also be interesting to explore, but is beyond the scope of this paper. A numerical exploration of these issues is in progress [24].

It is worth mentioning that there is another logical possibility concerning the entropy contained on this space-filling light sheet: it could be the case that this light sheet measures purely gravitational degrees of freedom, which sit just outside the horizon, and which have not been properly included in previous discussions of black hole entropy. This interpretation is suggested by the example of the thin spherically symmetric shell of matter sitting just at the Schwarzschild horizon (which we considered above). In addition to the contribution from the matter shell on a very thin spherically symmetric Bousso light sheet, we could include the contributions from the small, closely packed light sheets exterior to the shell. We would then have two contributions to the entropy, both of which scale precisely like the area of the horizon. If this is the correct way to think about the contribution to the entropy from the small, closely packed light sheets, then it suggests a "new" version of the generalized second law (GSL): In addition to the usual matter (S_{matter}) and horizon (S_{BH}) contributions to the total entropy, S_{total}, perhaps we should also include a purely gravitational term S_{grav}:

$$S_{total} = S_{matter} + S_{BH} + S_{grav}. \tag{39}$$

The statement of the GSL would then be that S_{total} can never decrease. Note that this would imply that the process of Hawking evaporation does not necessarily generate a huge amount of entropy. This is because a lot of entropy could already be contained in S_{grav}, and hence both S_{grav} and S_{BH} could be converted to pure S_{matter} (thermal radiation) at the endpoint of the evaporation process.

IV. DISCUSSION

We have studied Bousso's covariant entropy bound and its relation to the covariant Bekenstein bound. We found that a thin spherically symmetric shell of matter, which saturates the covariant Bekenstein bound and sits at its Schwarzschild horizon, gives rise to the expected black hole entropy on a large spherically symmetric light sheet. We also found that a more fine grained light sheet which explores the region outside the black hole gives a result proportional to the black hole area under certain assumptions regarding choice of regulators and saturation of the covariant Bekenstein bound, and we conjecture that an appropriate choice of light sheet would give the correct coefficient of 1/4 in the entropy-area relation.

If this is indeed correct, and if we are to interpret this result as due to gravitational entropy as suggested in the last section, then we are led to a remarkable conclusion. In the formation of a black hole by a thin shell of matter, the question of where the entropy of the black hole is contained is ambiguous. The question depends on a choice of light sheet and is not the same for all light sheets, even for space-filling light sheets (when the notion of space-filling is well defined). The black hole entropy can be interpreted either as gravitational entropy, which is bounded by the shear on fine grained light sheets; or it can be interpreted as due to matter entropy in the case of the thin shell discussed above. The latter interpretation is similar to the operational definition of entropy given by Pretorius, Vollick, and Israel [19], as discussed in the text. On the other hand, the gravitational interpretation suggests that in order to probe the entropy gravitationally short distance probes are required, as opposed to the long distance probes which measure the matter entropy. This indicates a sort of ultraviolet-infrared duality, although different in nature to the ultraviolet-infrared duality of [25].

Alternatively, as discussed at the end of the previous section, it may be the case that the shear entropy through the little light sheets is to be interpreted as an addition to the usual black hole entropy. In this case the generalized second law should be further generalized to reflect this gravitational contribution to the entropy. It would be interesting to make such a statement more precise, by finding an appropriate class of light sheets and studying the time evolution of the entropy through those light sheets, assuming the covariant Bekenstein bound as we did in this paper. Even if in a particular time slicing and a clever choice of light sheet a generalized second law could be deduced, the challenge will be making such a statement generally covariant.

To be fair, we have not precisely calculated the contribution of the shear to the Bousso bound, but only argued that this contribution may be proportional to the area of the black hole horizon (given certain assumptions as discussed above). It would be interesting to do a more explicit calculation, and also to study the effect of modifying the shape and size of the fine-grained light sheets. There are many ''derivations'' of the black hole entropy law and various formulations of entropy bounds in the literature. Most of them are not covariant. It is necessary to compare older approaches to black hole entropy to modern covariant approaches in the hope of better understanding gravity. Much remains to be done in this regard.

In addition, it is worth commenting that there is a nice separation of ''quantum'' and ''gravitational'' effects in the covariant Bekenstein bound. Putting $\hbar$ and G back into Eq. (2) gives

$$\hbar s_\Gamma \cdot k \leq (1-\lambda)(\pi k_a T^{ab} k_b + \sigma_{ab}\sigma^{ab}/8G). \tag{40}$$

Although the parameters $\hbar$ and G are dimensionful and can be rescaled to one, it is tempting to interpret the covariant entropy bound as due to a purely quantum mechanical constraint on the entropy of matter and a quantum gravitational constraint with regards to the gravitational Weyl entropy. As there is no G appearing in the part of the covariant Bekenstein bound related to the stress tensor, the entropy-area relation we found for the spherical matter shell relies on gravity only classically. This is similar in spirit to previous studies of spherical shells [19,20]. We also note that such a separation of $\hbar$ and G does not follow from the second set of constraints under which the Bousso bound has been proven, Eq. (3).

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