

# Potential for ill-posedness in several second-order formulations of the Einstein equations

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Second-order formulations of the 3+1 Einstein equations obtained by eliminating the extrinsic curvature in terms of the time derivative of the metric are examined with the aim of establishing whether they are well posed, in cases of somewhat wide interest, such as ADM, BSSN and generalized Einstein-Christoffel. The criterion for well-posedness of second-order systems employed is due to Kreiss and Ortiz. By this criterion, none of the three cases are strongly hyperbolic, but some of them are weakly hyperbolic, which means that they may yet be well posed but only under very restrictive conditions for the terms of order lower than second in the equations (which are not studied here). As a result, intuitive transferences of the property of well-posedness from first-order reductions of the Einstein equations to their originating second-order versions are unwarranted if not false.

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## I. INTRODUCTION

A common practice to study the well-posedness of a time-dependent second-order system of partial differential equations is to reduce the system to first order. This is done because first-order systems of partial differential equations (PDE) have been amply examined and are currently very well understood. By reducing to first-order, what is meant is to find a first-order system of PDE's whose solution space contains a subset that is equivalent to the solutions of the original second-order system by a trivial identification. If the first-order reduction is well posed, then the solutions of the first-order system are bounded in terms of the initial data of the first-order system, and theorems of existence, uniqueness and stability under small perturbations then follow.

It is tempting to presume that the well-posedness of a first-order reduction is a sign of well-posedness in the originating second-order system, especially because the originating second-order system is much smaller (in terms of the number of variables) and may be more amenable to a numerical implementation. However, in general relativity, the existence of constraints leads to first-order reductions that are equivalent to the original second order problem with regards to *constrained solutions* only. The evolution equations of two first-order reductions of the Einstein equations that differ by linear combinations with the Hamiltonian or momentum constraints are not equivalent. Thus, whether a well-posed first-order reduction of the Einstein equations guarantees the well-posedness of the corresponding second-order problem is not at all clear.

The quintessential model of a well-posed time-dependent second-order equation in three dimensions is the wave equation in flat space. By the well-posedness of the wave equation, more complicated systems of PDE's can be shown to be well posed if they consist of series of wave equations, as is the case of the 3+1 Einstein equations in the harmonic gauge or time-harmonic gauge [1]. But beyond the wave

equation, little is known about general well-posedness criteria for systems of second-order time dependent PDE's.

A criterion for the well-posedness of time-dependent second-order systems of PDE's has been developed recently by Kreiss and Ortiz [2]. Even though the criterion applies to cases that are much more general than what we are interested in, for our purposes it can be stated as follows. Consider a system of  $n$  second-order partial differential equations with constant coefficients. The system is of the form

$$\ddot{\mathbf{u}} = \sum_{jk} \mathbf{A}_{jk} \mathbf{u}_{,jk} \quad (1)$$

where  $\mathbf{u}$  is a vector containing the  $n$  fundamental variables and, for each value of  $(j,k)$ ,  $\mathbf{A}_{jk}$  is an  $n \times n$  constant matrix. An overdot denotes a partial derivative with respect to the time coordinate  $(\partial/\partial t)$ , so that  $\dot{f} \equiv \partial f / \partial t$ , and  $_{,j} \equiv \partial / \partial x^j$ . For any arbitrary unit covector  $\xi_i \equiv \omega_i / |\omega|$ , we define the  $n$ -dimensional matrix  $\mathbf{P}_0(\xi) \equiv \sum_{jk} \mathbf{A}_{jk} \xi_j \xi_k$ . The system is strongly hyperbolic if and only if the eigenvalues of  $\mathbf{P}_0(\xi)$  are strictly positive and  $\mathbf{P}_0(\xi)$  has a complete set of eigenvectors which is uniformly (in  $\xi$ ) linearly independent.

This criterion is entirely similar to strong hyperbolicity of first-order systems [3] except for the condition that the eigenvalues must be *strictly positive*, which excludes the value zero as well as all negative values. The reason for positivity of the eigenvalues is, essentially, that each eigenvalue must allow for two waves travelling with the same speed but in opposite directions (so, in a sense, the eigenvalues are the squares of real characteristic speeds of any sign). The reason for excluding the vanishing eigenvalue is that there is only one associated "wave" with zero speed instead of two. This criterion emulates fully the case of the three-dimensional wave equation. Any second-order system of PDE's that satisfies this criterion is well posed in the standard sense, that is: its solutions are bounded by the initial data irrespective of the spectral frequency of the data [2]. On the other hand, if the eigenvalues are real and non-negative and/or there is not a complete set of eigenvectors, the system is referred to as *weakly hyperbolic*. According to Ref. [2], weakly hyperbolic

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systems can develop “catastrophic exponential growth” when adding lower order terms or considering variable coefficients. It is inferred that if the linearization of a system of nonlinear second-order equations around a constant background is weakly hyperbolic, then the associated non-linear system itself is prone to “catastrophic exponential growth,” as is its linearization around any background that is not constant. Examples of what Kreiss and Ortiz refer to as “catastrophic growth” appear in Ref. [2].

As far as we are aware of, the criterion has been developed only for linear systems with constant coefficients, but there is reason to presume that it can be generalized to variable coefficients and quasilinear systems in a manner similar to the case of first-order systems. Therefore, in the following, we will use this criterion to analyze the well-posedness of second-order formulations of the Einstein equations insofar as they are linearized around flat space. The case of the standard ADM equations [4] is studied in Sec. II. The case of the widely used BSSN equations [5] is dealt with in Sec. III. Finally, the case of the generalized Einstein-Christoffel equations (EC) [6], which includes the case of the Einstein-Christoffel equations themselves [7], is developed in Sec. IV. We find that none of the second-order versions of the equations is strongly hyperbolic. The relevance of this result is summarized in Sec. V.

## II. ADM EQUATIONS IN SECOND ORDER FORM

Throughout the paper we assume the following form for the metric of spacetime  $g_{ab}$  in coordinates  $x^a=(x^i,t)$  in terms of the three-metric  $\gamma_{ij}$  of the slices at fixed value of  $t$ :

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij} dx^i dx^j, \quad (2)$$

where  $\alpha$  is the lapse function. The Einstein equations  $G_{ab}=0$  for the four-dimensional metric are equivalently expressed in the Arnowitt-Deser-Misner (ADM) form [4]:

$$\dot{\gamma}_{ij} = -2\alpha K_{ij}, \quad (3a)$$

$$\dot{K}_{ij} = \alpha(R_{ij} - 2K_{il}K^l_j + KK_{ij}) - D_i D_j \alpha, \quad (3b)$$

with the constraints

$$C \equiv \frac{1}{2}(R - K_{ij}K^{ij} + K^2) = 0, \quad (4a)$$

$$C_i \equiv D_j K^j_i - D_i K = 0, \quad (4b)$$

to be imposed on the initial data. Indices are raised with the inverse metric  $\gamma^{ij}$ ,  $D_i$  is the covariant three-derivative consistent with  $\gamma_{ij}$ ,  $R_{ij}$  is the Ricci curvature tensor of  $\gamma_{ij}$ ,  $R$  its Ricci scalar,  $K_{ij}$  is the extrinsic curvature of the slice at fixed value of  $t$  and  $K \equiv \gamma^{ij}K_{ij}$ . This system of evolution equations is a (partial) first-order reduction of the original second-order Einstein equations, which we can recover by substituting in Eq. (3b) the extrinsic curvature in terms of the time derivative of the metric as given by Eq. (3a):

$$\begin{aligned} \ddot{\gamma}_{ij} = & -2\alpha^2 R_{ij} + \gamma^{lm} \dot{\gamma}_{il} \dot{\gamma}_{mj} - \frac{1}{2} \gamma^{lm} \dot{\gamma}_{lm} \dot{\gamma}_{ij} + \frac{\dot{\alpha}}{\alpha} \dot{\gamma}_{ij} \\ & + 2\alpha D_i D_j \alpha. \end{aligned} \quad (5)$$

Linearizing around flat space, so that  $\gamma_{ij} = \delta_{ij} + h_{ij}$  and  $\alpha = 1 + \epsilon$ , these equations read

$$\ddot{h}_{ij} = \delta^{kl} (h_{kl,ij} - h_{il,kj} - h_{jl,ki} + h_{ij,kl}) + 2\epsilon_{,ij}. \quad (6)$$

Obviously the problem depends on the specification of the lapse function. We are here interested in two special cases: the case of constant unit lapse and the case of lapse equal to the square root of the determinant of the three-metric.

In the first place, then, suppose  $\epsilon=0$ . The equations then read

$$\ddot{h}_{ij} = \delta^{kl} (h_{kl,ij} - h_{il,kj} - h_{jl,ki} + h_{ij,kl}) \quad (7)$$

and conform to Eq. (1). Solving the eigenvalue problem of the principal symbol  $P_0(\xi)$  is equivalent to making the ansatz  $h_{ij} = V_{ij} \exp i(\xi_k x^k - st)$  for an arbitrary covector  $\xi_i$ , which yields the following:

$$s^2 V_{ij} = \delta^{kl} (V_{kl} \xi_i \xi_j - V_{il} \xi_k \xi_j - V_{jl} \xi_k \xi_i + V_{ij} \xi_k \xi_l) \quad (8)$$

or, equivalently

$$s^2 V_{ij} = V_{ij} + \xi_i \xi_j V - \xi_j V_{ik} \xi^k - \xi_i V_{jk} \xi^k, \quad (9)$$

where  $V$  denotes the trace of  $V_{ij}$  and indices are raised with  $\delta^{ij}$ . The eigenvalues are thus labeled by  $s^2$ . One can easily see that this problem admits solutions with vanishing eigenvalue, as follows. Assume  $s=0$ , which yields

$$0 = V_{ij} + \xi_i \xi_j V - \xi_j V_{ik} \xi^k - \xi_i V_{jk} \xi^k. \quad (10)$$

Contracting with  $\delta^{ij}$  we find  $V - \xi^i V_{ij} \xi^j = 0$ . Using this information back into Eq. (10) we have

$$0 = V_{ij} + \xi_i \xi_j \xi^l V_{lm} \xi^m - \xi_j V_{ik} \xi^k - \xi_i V_{jk} \xi^k. \quad (11)$$

Now contracting with  $\xi^j$  yields

$$0 = \xi_i (V - \xi^l V_{lm} \xi^m), \quad (12)$$

which are three identities. Therefore, three of the six components of  $V_{ij}$  are free, the remaining three being given by Eq. (11). One can pick the three free components to be the three projections  $V_{ij} \xi^j$ , in which case, by Eq. (11), the other three are vanishing. This observation leads directly to the fact that there are three linearly independent eigenvectors with vanishing eigenvalue, the eigenvectors being the six-dimensional unit vectors along the directions of the three projections  $V_{ij} \xi^j$ . This is so for every arbitrary direction  $\xi_i$ . As an illustration, representing the six-dimensional eigenvectors in the form  $u \equiv (V_{xx}, V_{xy}, V_{xz}, V_{yy}, V_{yz}, V_{zz})$ , for  $\xi_i = \delta_i^x = (1, 0, 0)$  we have that  $V_{ik} \xi^k = V_{ix}$ ,  $V_{xy}$ ,  $V_{xz}$  are free, whereas, by Eq. (11), the remaining components  $V_{yy}$ ,  $V_{yz}$  and  $V_{zz}$  vanish. An arbitrary eigenvector in the degenerate space of eigenvalue 0 has thus the form

$$\begin{aligned}
{}^0\mathbf{u} &= (V_{xx}, V_{xy}, V_{xz}, 0, 0, 0) \\
&= V_{xx}(1, 0, 0, 0, 0, 0) + V_{xy}(0, 1, 0, 0, 0, 0) \\
&\quad + V_{xz}(0, 0, 1, 0, 0, 0),
\end{aligned}$$

where  $V_{xx}$ ,  $V_{xy}$  and  $V_{xz}$  are arbitrary real numbers. There are thus three zero-speed eigenvectors associated with the  $x$  direction which can be chosen as

$$\begin{aligned}
{}^0\mathbf{u}_1 &= (1, 0, 0, 0, 0, 0), \\
{}^0\mathbf{u}_2 &= (0, 1, 0, 0, 0, 0), \\
{}^0\mathbf{u}_3 &= (0, 0, 1, 0, 0, 0),
\end{aligned}$$

which are manifestly linearly independent. In accordance with the criterion of Sec. I, this is enough to demonstrate that the evolution equations (7) are not strongly hyperbolic. Nonetheless, for completeness, one can calculate the remainder of the eigenvalues and eigenvectors in a similar manner (or one could, of course, proceed by any standard methods of linear algebra to calculate all the eigenvectors and eigenvalues at the same time). The reader can verify that the only other eigenvalue is  $s^2 = 1$ , for which Eq. (9) reads

$$0 = \xi_i \xi_j V - \xi_j V_{ik} \xi^k - \xi_i V_{jk} \xi^k. \quad (13)$$

These six equations are all satisfied if and only if

$$V_{ik} \xi^k = \xi_i \frac{V}{2}. \quad (14)$$

This in turn implies that the three components of  $V_{ij}$  other than the projections  $V_{ik} \xi^k$  are arbitrary. Denoting them by  $V_{ij}^\perp$  they are given by

$$V_{ij}^\perp \equiv V_{ij} - \xi_i V_{jk} \xi^k - \xi_j V_{ik} \xi^k + \xi_i \xi_j V_{kl} \xi^k \xi^l \quad (15)$$

and are such that  $V_{ij}^\perp \xi^j = 0$  by construction (which means that there are only three independent components in  $V_{ij}^\perp$ ). This leads directly to the conclusion that there are three linearly independent eigenvectors with light speed, labeled by the three components of  $V_{ij}$  other than  $V_{ik} \xi^k$ , and they are, clearly, linearly independent from the zero-speed eigenvectors (labeled by  $V_{ik} \xi^k$ ). For instance, in the case that  $\xi_i = (1, 0, 0)$  as above, by Eq. (15) the three arbitrary components of  $V_{ij}$  are  $V_{yy}$ ,  $V_{yz}$  and  $V_{zz}$ , which, if combined with Eq. (14), leads to the fact that an arbitrary eigenvector in the three-dimensional degenerate space of eigenvalue 1 is of the form

$$\begin{aligned}
{}^1\mathbf{u} &= (V_{yy} + V_{zz}, 0, 0, V_{yy}, V_{yz}, V_{zz}) \\
&= V_{yy}(1, 0, 0, 1, 0, 0) + V_{yz}(0, 0, 0, 0, 1, 0) \\
&\quad + V_{zz}(1, 0, 0, 0, 0, 1),
\end{aligned}$$

where  $V_{yy}$ ,  $V_{yz}$  and  $V_{zz}$  are completely arbitrary real numbers. This shows that three unit eigenvectors can be chosen as

$${}^1\mathbf{u}_1 = \frac{1}{\sqrt{2}}(1, 0, 0, 1, 0, 0),$$

$${}^1\mathbf{u}_2 = (0, 0, 0, 0, 1, 0),$$

$${}^1\mathbf{u}_3 = \frac{1}{\sqrt{2}}(1, 0, 0, 0, 0, 1),$$

which are manifestly linearly independent of each other and of  ${}^0\mathbf{u}_i$ .

In summary, the principal symbol  $P_0(\xi)$  of Eq. (7) admits a complete set of eigenvectors and has real but not strictly positive eigenvalues. The linearized second-order ADM equations with constant unit lapse are, thus, weakly hyperbolic. Consequently, the original nonlinear second-order ADM equations (5) are prone to catastrophic exponential growth, as is their linearization around any background but flat.

Suppose now that the lapse function is equal to the square root of the determinant of the three-metric. In the linearization, this means that  $2\epsilon_{,ij} = \delta^{kl} h_{kl,ij}$ . With this choice of lapse, Eq. (6) reads

$$\ddot{h}_{ij} = \delta^{kl}(2h_{kl,ij} - h_{il,kj} - h_{jl,ki} + h_{ij,kl}). \quad (16)$$

The associated eigenvalue problem is

$$s^2 V_{ij} = V_{ij} + 2\xi_i \xi_j V - \xi_j V_{ik} \xi^k - \xi_i V_{jk} \xi^k. \quad (17)$$

One can see that this eigenvalue problem admits two linearly independent solutions for  $s=0$ . To see this quickly (without necessarily using standard algebraic methods to solve the problem completely), set  $s=0$  and contract with  $\delta^{ij}$ , which yields  $V = (2/3)\xi^i V_{ij} \xi^j$ . Substituting this back and contracting this time with  $\xi^j$  one has

$$0 = \frac{1}{3} \xi_i \xi^l V_{lm} \xi^m. \quad (18)$$

Thus four out of the six equations are solved by setting  $\xi^i V_{ij} \xi^j = 0$  and consequently also  $V=0$ . This implies that two components of  $V_{ij}$  are free, which can be taken as the two  $V_{ij} \xi^j$  other than  $\xi^i V_{ij} \xi^j$ . The remaining two equations in the set fix the remaining two components of  $V_{ij}$  in terms of these:

$$0 = V_{ij} - \xi_j V_{ik} \xi^k - \xi_i V_{jk} \xi^k. \quad (19)$$

As in any eigenvalue problem, the fact that two components of  $V_{ij}$  are left arbitrary by Eq. (17) with  $s=0$  leads directly to the conclusion that there are two linearly independent eigenvectors (which can be calculated in any way the reader finds appealing) associated with zero speed and any direction  $\xi_i$ . For instance, if  $\xi_i = (1, 0, 0)$ , then  $V_{xy}$  and  $V_{xz}$  are free, but  $V_{xx}=0$  by Eq. (18), and  $V_{yy}$ ,  $V_{yz}$  and  $V_{zz}$  also vanish by virtue of Eq. (19). The generic eigenvector in the degenerate space of  $s=0$  is thus

$${}^0\mathbf{u} = (0, V_{xy}, V_{xz}, 0, 0, 0)$$

$$= V_{xy}(0, 1, 0, 0, 0, 0) + V_{xz}(0, 0, 1, 0, 0, 0),$$

where  $V_{xy}$  and  $V_{xz}$  are arbitrary real numbers, so there are only two linearly independent eigenvectors and they can be chosen as

$${}^0\mathbf{u}_1 = (0, 1, 0, 0, 0, 0),$$

$${}^0\mathbf{u}_2 = (0, 0, 1, 0, 0, 0).$$

This is enough to conclude that the linearized second-order ADM equations with “harmonic” lapse, namely Eq. (16), are not strongly hyperbolic.

For completeness, one can calculate the remaining eigenvalues and eigenvectors. The reader can verify that the only eigenvalue other than zero is  $s^2 = 1$ , which has three eigenvectors associated with it, namely the components of  $V_{ij}$  other than  $V_{ik}\xi^k$ . In keeping up with our illustration but skipping over the procedure, which is entirely similar to the one used three times in the preceding, for  $\xi_i = (1, 0, 0)$  the three eigenvectors associated with light speed are (i.e., can be chosen as)

$${}^1\mathbf{u}_1 = \frac{1}{\sqrt{3}}(1, 1, 1, 0, 0, 0),$$

$${}^1\mathbf{u}_2 = (0, 0, 0, 0, 1, 0),$$

$${}^1\mathbf{u}_3 = \frac{1}{\sqrt{2}}(0, 0, 0, 1, 0, -1),$$

which are manifestly linearly independent of each other and of  ${}^0\mathbf{u}_1, {}^0\mathbf{u}_2$ . Because the eigenvalues are real and non-negative and there is not a complete set of eigenvectors, Eqs. (16) are weakly hyperbolic. One can infer that the second-order nonlinear ADM equations (5) are prone to “catastrophic exponential growth” even in the case of a lapse function proportional to the determinant of the three-metric, as is their linearization around any nonflat background.

This result does not contradict Ref. [2]. In Ref. [2], the authors do consider the linearized second-order ADM equations and conclude that in the case of a lapse function that is proportional to the determinant of the three metric the problem is strongly hyperbolic. However, they impose, additionally, the linearized constraints on the evolution equations, which, as a consequence, do not have the form (16). Thus the problem that Kreiss and Ortiz found to be well posed is equivalent to a constrained evolution problem for the ADM equations in second order with “harmonic” lapse, where the solutions are constrained at every time slice. In contrast, we have demonstrated here that the corresponding unconstrained evolution problem is not well posed.

### III. BSSN EQUATIONS IN SECOND ORDER FORM

In the case of vanishing shift vector, the formulation of the Einstein equations referred to as BSSN [5] consists of the

following evolution equations:

$$\dot{\tilde{\gamma}}_{ij} = -2\alpha\tilde{\alpha}\tilde{A}_{ij}, \quad (20a)$$

$$\dot{\phi} = -\frac{\alpha}{6}K, \quad (20b)$$

$$\dot{K} = -\gamma^{ij}D_i D_j \alpha + \alpha\left(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2\right), \quad (20c)$$

$$\dot{\tilde{\Gamma}}^i = 2\alpha\left(\tilde{\Gamma}^i_{jk}\tilde{A}^{kj} + \frac{2}{3}\tilde{\gamma}^{ij}K_{,j} + 6\tilde{A}^{ij}\phi_{,j}\right) - 2\tilde{A}^{ij}\alpha_{,j}, \quad (20d)$$

$$\begin{aligned} \dot{\tilde{A}}_{ij} = & \alpha e^{-4\phi}\left(-\frac{1}{2}\tilde{\gamma}^{lm}\tilde{\gamma}_{ij,lm} + \tilde{\gamma}_{k(i}\tilde{\Gamma}^k_{j)}\right. \\ & + \tilde{\Gamma}^k\tilde{\Gamma}_{(ij)k} + 2\tilde{\Gamma}^{kl}_{(i}\tilde{\Gamma}_{j)kl} + \tilde{\Gamma}^{kl}_i\tilde{\Gamma}_{klj} - 2\tilde{D}_i\tilde{D}_j\phi \\ & + 4\tilde{D}_i\phi\tilde{D}_j\phi - \frac{1}{3}\tilde{\gamma}_{ij}(\tilde{\Gamma}^k_{,k} + \tilde{\Gamma}^{kli}(2\tilde{\Gamma}_{ikl} + \tilde{\Gamma}_{kli})) \\ & + \frac{2}{3}\tilde{\gamma}_{ij}(\tilde{D}^l\tilde{D}_l\phi - 2\tilde{D}^l\phi\tilde{D}_l\phi) - \frac{(D_i D_j \alpha)_{TF}}{\alpha}\Big) \\ & + K\tilde{A}_{ij} - 2\tilde{A}_{il}\tilde{A}^l_j \end{aligned} \quad (20e)$$

for the 15 variables

$$\phi \equiv \frac{1}{12}\ln(\det \gamma_{ij}), \quad (21a)$$

$$\tilde{\gamma}_{ij} \equiv e^{-4\phi}\gamma_{ij}, \quad (21b)$$

$$K \equiv \gamma^{ij}K_{ij}, \quad (21c)$$

$$\tilde{A}_{ij} \equiv e^{-4\phi}\left(K_{ij} - \frac{1}{3}\gamma_{ij}K\right), \quad (21d)$$

$$\tilde{\Gamma}^i \equiv -\tilde{\gamma}^{ij}_{,j}. \quad (21e)$$

As in the ADM case, the initial data for these evolution equations must be chosen to satisfy the constraints (4). Other than as applied to the initial data, we will disregard the constraints.

We obtain a six-dimensional second order system for the metric variables by substituting  $A_{ij}$  in terms of  $\tilde{\gamma}_{ij}$  back into Eq. (20e), and  $K$  in terms of  $\dot{\phi}$  into Eq. (20c). Additionally, we substitute  $\tilde{\Gamma}^i$  back in terms of  $\tilde{\gamma}^{ij}_{,j}$  into Eq. (20e).

The linearization around flat space  $\gamma_{ij} = \delta_{ij} + h_{ij}$  implies that  $\tilde{\gamma}_{ij} = \delta_{ij} + \tilde{h}_{ij}$  and  $\phi = h/12$ , where  $\tilde{h}_{ij} = h_{ij} - (1/2)\delta_{ij}h$  and  $h = \delta^{ij}h_{ij}$ . After linearization around flat space, the result is the following second-order system:

$$\ddot{\phi} = \frac{1}{6}\delta^{ij}\epsilon_{,ij}, \quad (22a)$$



$$\begin{aligned} \ddot{h}_{ij} = & \delta^{kl} \left( \tilde{h}_{ij,kl} - \tilde{h}_{il,kj} - \tilde{h}_{jl,ki} + \frac{2}{3} \delta_{ij} \delta^{rs} \tilde{h}_{ls,kr} \right) + 4 \left( \phi_{,ij} \right. \\ & \left. - \frac{1}{3} \delta_{ij} \delta^{kl} \phi_{,kl} \right) + 2 \left( \epsilon_{,ij} - \frac{1}{3} \delta_{ij} \delta^{kl} \epsilon_{,kl} \right). \end{aligned} \quad (22b)$$

Consider first the case of constant lapse equal to 1. The equations reduce to

$$\dot{\phi} = 0, \quad (23a)$$

$$\begin{aligned} \ddot{h}_{ij} = & \delta^{kl} \left( \tilde{h}_{ij,kl} - \tilde{h}_{il,kj} - \tilde{h}_{jl,ki} + \frac{2}{3} \delta_{ij} \delta^{rs} \tilde{h}_{ls,kr} \right) \\ & + 4 \left( \phi_{,ij} - \frac{1}{3} \delta_{ij} \delta^{kl} \phi_{,kl} \right), \end{aligned} \quad (23b)$$

and the corresponding eigenvalue problem, with  $\tilde{h}_{ij} = \tilde{V}_{ij} \exp i(\xi_k x^k - st)$  and  $\phi = \psi \exp i(\xi_k x^k - st)$ , is

$$s^2 \psi = 0, \quad (24a)$$

$$\begin{aligned} s^2 \tilde{V}_{ij} = & \tilde{V}_{ij} - \xi_i \tilde{V}_{jl} \xi^l - \xi_j \tilde{V}_{il} \xi^l + \frac{2}{3} \delta_{ij} \xi^k \tilde{V}_{kl} \xi^l \\ & + 4 \psi \xi_i \xi_j - \frac{4}{3} \delta_{ij} \psi. \end{aligned} \quad (24b)$$

This system has a vanishing eigenvalue  $s^2=0$  with three linearly independent associated eigenvectors, labeled by the three components  $\tilde{V}_{ij} \xi^j$ . In order to see this, set  $s^2=0$  in Eq. (24b) and contract with  $\xi^i$ . This yields  $\xi_j [\psi - (1/8) \xi^k \tilde{V}_{kl} \xi^l] = 0$ . So three equations are satisfied by the choice  $\psi = (1/8) \xi^k \tilde{V}_{kl} \xi^l$ . Additionally, since  $s^2=0$ , then also Eq. (24a) is identically satisfied. Four out of the six equations are thus satisfied with this one choice, which means that three of the fields are free. This is enough to conclude that the second-order BSSN equations with unit lapse, namely Eq. (23), are not strongly hyperbolic. We can calculate the remaining eigenvalues and eigenvectors. The reader can verify that there are two other eigenvalues. One is  $s^2=1$ , with two associated eigenvectors, which are the components of  $\tilde{V}_{ij}$  other than  $\tilde{V}_{il} \xi^l$ , with  $\tilde{V}_{il} \xi^l = 0$  and  $\psi=0$ . The other eigenvalue is negative  $s^2=-1/3$ , and has one eigenvector associated with it, which is given by  $\psi=0, \tilde{V}_{il} \xi^l = 0$  except for  $\xi^i \tilde{V}_{il} \xi^l$  which is free, and  $\tilde{V}_{ij} = (3/2) [\xi_i \xi_j - (1/3) \delta_{ij} \xi^k \tilde{V}_{kl} \xi^l]$ . Thus there is a complete set of eigenvectors and all eigenvalues are real, but are not non-negative, and the equations are not even weakly hyperbolic. This means that the full nonlinear second-order BSSN equations with unit lapse have the potential for ‘‘catastrophic growth,’’ as does their linearization around any background other than flat.

Things change very little if one considers now the case of  $\alpha = \sqrt{\det \gamma_{ij}}$ , or  $\epsilon_{,ij} = 6 \phi_{,ij}$ . Equations (22) reduce to

$$\dot{\phi} = \delta^{ij} \phi_{,ij}, \quad (25a)$$

$$\begin{aligned} \ddot{h}_{ij} = & \delta^{kl} \left( \tilde{h}_{ij,kl} - \tilde{h}_{il,kj} - \tilde{h}_{jl,ki} + \frac{2}{3} \delta_{ij} \delta^{rs} \tilde{h}_{ls,kr} \right) \\ & + 16 \left( \phi_{,ij} - \frac{1}{3} \delta_{ij} \delta^{kl} \phi_{,kl} \right). \end{aligned} \quad (25b)$$

This system has two eigenvectors with  $s^2=0$ , which are given by  $\xi^i \tilde{V}_{il} \xi^l = 0$ ,  $\psi=0$  and  $\tilde{V}_{ij} = \xi_i \tilde{V}_{jl} \xi^l + \xi_j \tilde{V}_{il} \xi^l$  with arbitrary values for the two components  $\tilde{V}_{il} \xi^l$  other than  $\xi^i \tilde{V}_{il} \xi^l$ . There are two other eigenvalues different from zero. One is  $s^2=1$ , with three associated eigenvectors which have free values of  $\psi$  and of the components of  $\tilde{V}_{ij}$  other than  $\tilde{V}_{ij} \xi^j = 8 \xi_i \psi$ . The other one is negative  $s^2=-1$ , and has one eigenvector associated with it which has  $\psi=0$  and  $\tilde{V}_{ij} = (3/2) [\xi_i \xi_j - (1/3) \delta_{ij} \xi^k \tilde{V}_{kl} \xi^l]$  with  $\xi^k \tilde{V}_{kl} \xi^l$  arbitrary. So Eq. (25) has a complete set of eigenvectors and its eigenvalues are real but not non-negative, and thus it is not even weakly hyperbolic. The consequences to the full nonlinear BSSN equations in second-order form and with ‘‘harmonic’’ lapse are the same as in the case of unit lapse.

#### IV. GENERALIZED EC EQUATIONS IN SECOND ORDER FORM

The generalized Einstein-Christoffel (EC) formulation is first introduced in Ref. [6], where the method of derivation from the ADM equations is described without the details of the resulting equations themselves. The principal part of the evolution equations of the system appear explicitly in Ref. [8] as follows:

$$\dot{\gamma}_{ij} = -2\alpha K_{ij}, \quad (26a)$$

$$\dot{K}_{ij} = -\alpha \gamma^{kl} \partial_l f_{kij} + \dots, \quad (26b)$$

$$\dot{f}_{kij} = -\alpha \partial_k K_{ij} + \dots \quad (26c)$$

where  $f_{kij}$  constitute a set of 18 first-order variables defined by the following relationship with the first derivatives of the three-metric [8]:

$$\gamma_{ij,k} \equiv 2f_{kij} + \eta \gamma_{k(i} (f_{j)s}^s - f_{j)s}^s) + \frac{\eta-4}{4} \gamma_{ij} (f_{ks}^s - f_{ks}^s). \quad (27)$$

Here  $\eta$  is a free parameter. The generalized EC family requires the lapse function to be densitized, that is: to be proportional to the square root of the determinant of the three-metric. The original (standard) EC formulation obtained by Anderson and York [7] corresponds to the choice of  $\eta=4$ .

We are interested in the second-order system of equations for  $\gamma_{ij}$  that is implied by this 30-dimensional first-order problem. We start by inverting Eq. (27) in order to have an expression for  $f_{kij}$  that we can use to substitute in terms of  $\gamma_{ij}$ . The inversion yields

$$2f_{kij} = \gamma_{ij,k} + \gamma_{ik}(\gamma^{ls}\gamma_{ls,j} - \gamma^{ls}\gamma_{lj,s}) + \gamma_{jk}(\gamma^{ls}\gamma_{ls,i} - \gamma^{ls}\gamma_{li,s}) + \frac{\eta-4}{2\eta}\gamma_{ij}(\gamma^{ls}\gamma_{ls,k} - \gamma^{ls}\gamma_{lk,s}). \quad (28)$$

If one uses Eq. (28) in its left-hand side, Eq. (26c) reduces to a linear combination of the components of the momentum constraint  $C_i$ . Therefore, this equation is redundant for the problem that we are interested in. However, eliminating  $f_{kij}$  from the right-hand side of Eq. (26b) by means of Eq. (28) yields

$$\dot{K}_{ij} = \alpha(R_{ij} - 2K_{il}K^l_j + KK_{ij}) - D_i D_j \alpha + \frac{\eta-4}{2\eta}\alpha\gamma_{ij}C. \quad (29)$$

One can see that, at  $\eta=4$ , the evolution equation for the extrinsic curvature in the standard EC formulation is an exact transcription of the ADM evolution equation without mixing of the constraints, whereas for any other value of the parameter  $\eta$ , the Hamiltonian constraint is involved. This means that the second-order version of the standard EC is indistinguishable from the second-order version of the ADM equations, dealt with in Sec. II. Additionally, one can also show that for  $\eta=12/7$ , the second-order version of the generalized EC formulation is indistinguishable from the second-order version of the BSSN formulation, which is dealt with in Sec. III. Yet, for  $\eta \neq 4, 12/7$ , the second-order version of the generalized EC is a genuinely different problem which requires separate study.

The second-order problem that we seek is found by substituting  $K_{ij}$  in terms of  $\dot{\gamma}_{ij}$  in the left-hand side of Eq. (29) and writing explicitly the right-hand side in terms of  $\gamma_{ij,kl}$ :

$$\alpha^{-2}\ddot{\gamma}_{ij} = \gamma^{kl}\gamma_{ij,kl} - \gamma^{kl}\gamma_{il,kj} - \gamma^{kl}\gamma_{kj,il} + 2\gamma^{kl}\gamma_{kl,ij} + \frac{\eta-4}{2\eta}\gamma_{ij}(\gamma^{kl}\gamma^{ms}\gamma_{ms,kl} - \gamma^{kl}\gamma^{ms}\gamma_{ml,ks}) + \dots \quad (30)$$

In the linearization around flat space ( $\gamma_{ij} = \delta_{ij} + h_{ij}$ ) we have

$$\ddot{h}_{ij} = \delta^{kl}h_{ij,kl} - \delta^{kl}h_{il,kj} - \delta^{kl}h_{kj,il} + 2\delta^{kl}h_{kl,ij} + \frac{\eta-4}{2\eta}\delta_{ij}(\delta^{kl}\delta^{ms}h_{ms,kl} - \delta^{kl}\delta^{ms}h_{ml,ks}), \quad (31)$$

which has the following eigenvalue problem in terms of  $h_{ij} = V_{ij}\exp i(\xi_k x^k - st)$ :

$$s^2 V_{ij} = V_{ij} - V_{ik}\xi^k \xi_j - V_{jk}\xi^k \xi_i + 2V\xi_i \xi_j + \frac{\eta-4}{2\eta}\delta_{ij}(V - \xi^k V_{kl}\xi^l). \quad (32)$$

This problem has the following eigenvalues. First we have  $s^2=1$  with three linearly independent eigenvectors which are the three components of  $V_{ij}$  other than the projections  $V_{ij}\xi^i (= \xi_i V)$ . Then we have  $s^2=2(\eta-2)/\eta$  with one eigen-

vector given by  $V_{ij}=2\eta V[2\xi_i \xi_j + \delta_{ij}(\eta-4)/2\eta]/(7\eta-12)$  with arbitrary  $V$ , which is linearly independent of the other three except at  $\eta=4$ . Finally we have  $s^2=0$  with two eigenvectors given by  $V_{ij}=\xi_i V_{jk}\xi^k + \xi_j V_{ik}\xi^k$  with  $V=\xi^k V_{kl}\xi^l=0$ .

So there is a complete set of eigenvectors for all values of  $\eta \neq 4, 12/7$ . However, there are two eigenvectors with  $s^2=0$  for all  $\eta \neq 2, 4, 12/7$ . Furthermore, if  $\eta=2$  then  $s^2=0$  has multiplicity 3, but if  $\eta < 2$  then there is a negative eigenvalue. Collectively, this means that for no values of  $\eta$  is the second-order problem implied by the generalized EC formulation strongly hyperbolic. The second-order problem is weakly hyperbolic for  $\eta \geq 2$ , which includes the ADM case, and this is consistent with Sec. II. Yet the second-order problem is not even weakly hyperbolic in the range  $\eta < 2$  considered in Ref. [8], which includes the BSSN case, and this is consistent with Sec. III.

## V. CONCLUDING REMARKS

Summarizing, we have found that the second-order equations for the three-metric that are implied by a large number of first-order reductions of the 3+1 Einstein equations whose evolution equations differ by the addition of different multiples of the Hamiltonian constraint are potentially ill posed irrespective of whether the lapse is constant or densitized. We say *potentially* only because the linearized argument that we use does in one way rule out the possibility that the lower-order terms that are present in the nonlinear equations may yet (fortuitously) result in a well-posed nonlinear problem. This should raise a flag for numerical efforts seeking formulations for general relativity with the smallest possible number of variables.

On the other hand, studying the second-order equations implied by formulations with such disparity of properties as ADM, BSSN and generalized EC gives us yet another perspective for what it is about such formulations that characterizes their properties.

In the case of the ADM equations and their associated second-order version, we have verified a fact that has been suspected for a long time, starting with Ref. [9], where the second-order version itself is found unfit for standard hyperbolicity studies. Subsequently, many authors have concluded that the ADM equations themselves are ill posed on the basis of the existence of full first-order reductions not involving constraint mixing in the evolution equations of the first-order variables. This intuitive line of reasoning, whether rigorously justified or not, naturally leads to the conclusion that the second-order version should also be ill posed. Our results of Sec. II provide a rigorous basis for this intuitive inference.

In the case of the BSSN equations the implications are less straightforward. In essence, the BSSN equations themselves constitute a partial first-order reduction of the ADM equations with constraint mixing in two senses. First, there is the use of a multiple of the Hamiltonian constraint in the evolution equation for the extrinsic curvature. Second, there is a use of the momentum constraint in the evolution equations for the three new first-order variables  $\tilde{\Gamma}^i$ . According to Ref. [10], the latter play a critical role in the well-posed

properties of the BSSN equations in a pseudospectral sense, whereas the former is irrelevant to well-posedness in such a sense. This accounts for the difference in the well-posedness of the first-order reduction of BSSN as compared to ADM (but of course, the comparison is not completely fair because BSSN is already a partial reduction). Yet, the constraint mixing in the evolution equations of the new first-order variables plays no role whatsoever in the well-posedness of the second-order problem for the three-metric components because it simply leads to a redundancy, as explained in the main body of this article. Therefore, it is disappointing but hardly surprising that the second-order version of the BSSN equations does not turn out to be well posed. In fact, this verifies the idea put forward in Ref. [10] that the constraint mixing in the evolution of the first-order variables is crucial to the well-posedness of a first-order reduction of the Einstein equations.

This idea is most strongly upheld by the results in the case of the generalized EC formulation and its second-order version. Here is a family of first-order reductions all of which are well posed for any multiple of the Hamiltonian constraint added to the evolution equation for the extrinsic curvature. But they all have momentum-constraint mixing in the evolution equations for the new first-order variables  $f_{kij}$ . Yet none

of the second-order versions of these formulations turns out to be well posed. What the second-order versions are missing is precisely the evolution equations for the first-order variables (which become redundant in the second-order version). For several reasons including the present ones, the constraints seem to be surfacing as key players in the initial value problem of the Einstein equations, contrary to their long-standing reputation as choosers of physical initial data but otherwise ignorable.

Additionally, reflecting on the fact that the standard second-order version of the Einstein equations with harmonic slicing where all the components of the metric evolve according to wave equations is indeed well posed, one is led to conclude that the shift vector must play a crucial role in the well-posedness of second-order versions of the 3+1 Einstein equations, in agreement with the intuition of the authors of Ref. [11] where a dynamical shift choice is used to analyze some nonstandard hyperbolicity properties of the BSSN equations. This remains an open problem.

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