Particles and energy fluxes from a conformal field theory perspective

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We analyze the creation of particles in two dimensions under the action of conformal transformations. We focus our attention on Mobius transformations and compare the usual approach, based on the Bogoliubov coefficients, with an alternative but equivalent viewpoint based on correlation functions. In the latter approach the absence of particle production under full Mobius transformations is manifest. Moreover, we give examples, using the moving-mirror analogy, to illustrate the close relation between the production of quanta and energy.

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I. INTRODUCTION

One of the basic ingredients of quantum field theory in curved spacetime [1] is the Bogoliubov transformations. These reflect the absence, in general, of a privileged vacuum state, in parallel to the absence of global inertial frames. This framework is general and can be applied to a large number of physical situations, including flat spacetime backgrounds (like the Unruh-Fulling effect [1]). On the other hand, of particular physical interest are those field theories possessing the spacetime conformal symmetry $SO(d, 2)$, where d is the dimension of the Lorentzian spacetime. This symmetry is especially powerful in two dimensions, where the group $SO(2, 2)$ can be enlarged to an infinite-dimensional group [2]. However, this $SO(2, 2)$ subgroup, which includes dilatations, Poincaré and special conformal transformations, still plays an important role because it leaves the vacuum invariant [2]. From the point of view of Bogoliubov transformations this should imply that the β coefficients associated to them vanish. This is obvious for Poincaré and dilatations: they do not produce any mixing of positive and negative frequencies. However, this result is far from being obvious for special conformal transformations.

In addition, a restriction to one of the two branches of special conformal transformation produces, in the context of moving mirrors, a nonvanishing result [1,3]. Since Mobius transformations never produce local energy fluxes this has been interpreted as a manifestation of the fact that the production of quanta does not require presence of energy [1,3]. This claim has been criticized in [4] using an explicit particle detector.

The purpose of this paper is to clarify all the above issues. To this end we shall analyze the phenomena of quantal production in a different way, closer to the philosophy of conformal field theory (CFT). In the new perspective, the absence of particle production for the full set of Mobius transformations (including the special conformal transformation) is obvious from the very beginning, in sharp contrast to the usual approach based on the Bogoliubov transformations. We shall analyze the corresponding moving-mirror analogy for special conformal transformations (with one and two hyperbolic branches) to illustrate, in an easy way, how the production of energy and quanta are indeed closely related.

II. PARTICLE PRODUCTION AND BOGOLIUBOV COEFFICIENTS

Let us first briefly review the definition of the Bogoliubov coefficients for the two-dimensional massless scalar field f satisfying the wave equation

$$\nabla^2 f = 0. \tag{1}$$

In conformal gauge $ds^2 = -e^{2\rho} dx^+ dx^-$ we can decompose the field into positive and negative frequencies using a mode expansion:

$$f = \sum_i [\vec{a}_i u_i(x^-) + \vec{a}_i^\dagger u_i^*(x^-) + \overleftarrow{a}_i v_i(x^+) + \overleftarrow{a}_i^\dagger v_i^*(x^+)]. \tag{2}$$

These modes must form an orthonormal basis under the scalar product

$$(f_1, f_2) = -i \int_\Sigma d\Sigma^\mu (f_1 \partial_\mu f_2^* - \partial_\mu f_1 f_2^*), \tag{3}$$

where Σ is an appropriate Cauchy hypersurface. One can construct the Fock space from the commutation relations

$$[\vec{a}_i, \vec{a}_j^\dagger] = \delta_{ij}, \tag{4}$$

$$[\overleftarrow{a}_i, \overleftarrow{a}_j^\dagger] = \delta_{ij}. \tag{5}$$

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The vacuum state $|0_x\rangle$ is defined by

$$\vec{a}_i|0_x\rangle = 0, \overleftarrow{a}_i|0_x\rangle = 0, \tag{6}$$

and the excited states can be obtained by the application of creation operators $\vec{a}_i^\dagger, \overleftarrow{a}_i^\dagger$ out of the vacuum. We can perform an arbitrary conformal transformation

$$x^\pm \rightarrow y^\pm = y^\pm(x^\pm), \tag{7}$$

and consider the expansion

$$f = \sum_j [\vec{b}_j \tilde{u}_j(y^-) + \vec{b}_j^\dagger \tilde{u}_j^*(y^-) + \overleftarrow{b}_j \tilde{v}_j(y^+) + \overleftarrow{b}_j^\dagger \tilde{v}_j^*(y^+)]. \tag{8}$$

As both sets of modes are complete, the new modes $\tilde{u}_j(y^-)$, $\tilde{v}_j(y^+)$ can be expanded in terms of the old ones:

$$\begin{aligned} \tilde{u}_j(y^-) &= \sum_i [\alpha_{ji} u_i(x^-) + \beta_{ji} u_i^*(x^-)], \\ \tilde{v}_j(y^+) &= \sum_i [\gamma_{ji} v_i(x^+) + \eta_{ji} v_i^*(x^+)], \end{aligned} \tag{9}$$

where α_{ji}, β_{ji}, γ_{ji}, and η_{ji} are called Bogoliubov coefficients. These coefficients can be evaluated by the following scalar products:

$$\alpha_{ji} = (\tilde{u}_j, u_i), \quad \beta_{ji} = -(\tilde{u}_j, u_i^*), \quad \gamma_{ji} = (\tilde{v}_j, v_i), \quad \eta_{ji} = -(\tilde{v}_j, v_i^*). \tag{10}$$

The relation between creation and annihilation operators in the two basis is

$$\vec{b}_j = \sum_i (\alpha_{ji}^* \vec{a}_i - \beta_{ji}^* \vec{a}_i^\dagger), \qquad \overleftarrow{b}_j = \sum_i (\gamma_{ji}^* \overleftarrow{a}_i - \eta_{ji}^* \overleftarrow{a}_i^\dagger) \tag{11}$$

along with the corresponding ones for $\vec{b}_j^\dagger$ and $\overleftarrow{b}_j^\dagger$. Therefore the expectation value of the (right-mover sector) particle number operator $\vec{N}_j \equiv \vec{b}_j^\dagger \vec{b}_j$ is given by the expression

$$\langle 0_x|\vec{N}_j|0_x\rangle = \sum_i |\beta_{ji}|^2. \tag{12}$$

If we consider Mobius transformations, the quantities $\langle 0_x|N_i|0_x\rangle$ should vanish since, in a conformal field theory, the vacuum is invariant under these transformations. This means that the corresponding β coefficients should also vanish. An explicit analysis shows that this is far from being obvious (see below), despite the fact that the invariance of the vacuum under Mobius transformations is almost an axiom in CFT [2]. In the next section we shall analyze the particle production from a different perspective. We shall give a different expression for $\langle 0_x|N_i|0_x\rangle$, in terms of which the invariance of the vacuum under full Mobius transformation, one of the basic cornerstones of CFT, will be manifest.

III. PARTICLE PRODUCTION AND CORRELATIONS

Let us now show how to obtain an expression for $\langle 0_x|\vec{N}_j|0_x\rangle$ without introducing explicitly the Bogoliubov coefficients. Our starting point is the two-point correlation function for the derivatives of the field f

$$\langle 0_x|\partial_\pm f(x^\pm)\partial_\pm f(x'^\pm)|0_x\rangle = -\frac{1}{4\pi}\frac{1}{(x^\pm - x'^\pm)^2}. \tag{13}$$

Under conformal transformations $x^\pm \rightarrow y^\pm = y^\pm(x^\pm)$, the above correlation functions transform according to the rule for primary fields [2]:

$$\langle 0_x|\partial_\pm f(y^\pm)\partial_\pm f(y'^\pm)|0_x\rangle = -\frac{1}{4\pi}\left(\frac{dx^\pm(y^\pm)}{dy^\pm}\right)\left(\frac{dx^\pm(y'^\pm)}{dy^\pm}\right) \times \frac{1}{[x^\pm(y^\pm) - x'^\pm(y^\pm)]^2}. \tag{14}$$

These relations are fundamental to construct the normal-ordered stress tensor $:T_{\pm\pm}:$, but also for the particle number operator. In the coordinates $\{x^\pm\}$ the normal-ordered stress tensor operator can be defined via point splitting

$$:T_{\pm\pm}(x^\pm): = \lim_{x^\pm \rightarrow x'^\pm} :\partial_\pm f(x^\pm)\partial_\pm f(x'^\pm):, \tag{15}$$

where

$$:\partial_\pm f(x^\pm)\partial_\pm f(x'^\pm): = \partial_\pm f(x^\pm)\partial_\pm f(x'^\pm) + \frac{1}{4\pi} \times \frac{1}{(x^\pm - x'^\pm)^2}. \tag{16}$$

Similar relations hold in the coordinates $\{y^\pm\}$. It is easy to relate $:T_{\pm\pm}(y^\pm):$ with $:T_{\pm\pm}(x^\pm):$ since

$$:\partial_\pm f(y^\pm)\partial_\pm f(y'^\pm): = \frac{dx^\pm(y^\pm)}{dy^\pm}\frac{dx^\pm(y'^\pm)}{dy^\pm} \times \partial_\pm f(x^\pm)\partial_\pm f(x'^\pm) + \frac{1}{4\pi}\frac{1}{(y^\pm - y'^\pm)^2}, \tag{17}$$

and the result is

$$:T_{\pm\pm}(y^\pm): = \left(\frac{dx^\pm}{dy^\pm}\right)^2 :T_{\pm\pm}(x^\pm): - \frac{1}{24\pi}\{x^\pm, y^\pm\}, \tag{18}$$

where

$$\{x^\pm, y^\pm\} = \frac{d^3x^\pm}{dy^{\pm 3}} \Big/ \frac{dx^\pm}{dy^\pm} - \frac{3}{2}\left(\frac{d^2x^\pm}{dy^{\pm 2}} \Big/ \frac{dx^\pm}{dy^\pm}\right)^2 \tag{19}$$

is the Schwarzian derivative. Note that the normal-ordered operator $:T_{\pm\pm}(x^\pm):$ does not transform as a tensor. Normal ordering breaks the classical covariant transformation law under conformal transformations

$$T_{\pm\pm}(y^{\pm}) = \left(\frac{dx^{\pm}}{dy^{\pm}}\right)^2 T_{\pm\pm}(x^{\pm}). \qquad (20)$$

Indeed, normal ordering requires a selection of modes, and therefore of coordinates. For instance, $:T_{\pm\pm}(x^{\pm}):$ can be defined from the plane-wave modes $u_w = (4\pi w)^{-1/2}e^{-iwx^{\pm}}$ and $:T_{\pm\pm}(y^{\pm}):$ can be defined from the modes $\tilde{u}_w = (4\pi w)^{-1/2}e^{-iwy^{\pm}}$.

The two-point correlation function $\langle 0_x|:\partial_{\pm}f(y^{\pm})\partial_{\pm}f(y'^{\pm}):|0_x\rangle$ also serves to construct the particle number operator. We start from the explicit form of it in terms of creation and annihilation operators (for simplicity we shall consider only the right-mover sector)

$$\begin{aligned}\langle 0_x|:\partial_{y_1^-}f(y_1^-)\partial_{y_2^-}f(y_2^-):|0_x\rangle = \sum_{ji}\{&\langle 0_x|\vec{b}_j^{\dagger}\vec{b}_i|0_x\rangle \\ &\times(\partial_{y_1^-}\tilde{u}_i\partial_{y_2^-}\tilde{u}_j^* \\ &+\partial_{y_1^-}\tilde{u}_j^*\partial_{y_2^-}\tilde{u}_i) \\ &+(\langle 0_x|\vec{b}_j\vec{b}_i|0_x\rangle \\ &\times\partial_{y_1^-}\tilde{u}_j\partial_{y_2^-}\tilde{u}_i + \text{c.c.})\}. \qquad (21)\end{aligned}$$

Now, instead of taking the limit $y_1^- \to y_2^-$, as in the construction of the stress tensor, we shall perform the following transform:

$$\begin{aligned}&\int_{-\infty}^{+\infty}dy_1^-dy_2^-\tilde{u}_k(y_1^-)\tilde{u}_{k'}^*(y_2^-)\langle 0_x|:\partial_{y_1^-}f(y_1^-)\partial_{y_2^-}f(y_2^-):|0_x\rangle \\ &\quad = \frac{1}{4}\langle 0_x| \to b_k^{\dagger} \to b_{k'}|0_x\rangle. \qquad (22)\end{aligned}$$

We can evaluate this expression in terms of the particle number operator. To this end we shall use (21) together with the relations

$$\begin{aligned}(\tilde{u}_i, \tilde{u}_j) &= -2i\int_{-\infty}^{\infty}dy^-\tilde{u}_i\partial_{y^-}\tilde{u}_j^* = \delta_{ij}, \\ (\tilde{u}_i^*, \tilde{u}_j^*) &= -2i\int_{-\infty}^{\infty}dy^-\tilde{u}_i^*\partial_{y^-}\tilde{u}_j = -\delta_{ij}, \qquad (23) \\ (\tilde{u}_i, \tilde{u}_j^*) &= -2i\int_{-\infty}^{\infty}dy^-\tilde{u}_i\partial_{y^-}\tilde{u}_j = 0.\end{aligned}$$

The result is as follows: $\to b_{k'}$

$$\begin{aligned}&\int_{-\infty}^{+\infty}dy_1^-dy_2^-\tilde{u}_k(y_1^-)\tilde{u}_{k'}^*(y_2^-)\langle 0_x|:\partial_{y_1^-}f(y_1^-)\partial_{y_2^-}f(y_2^-):|0_x\rangle \\ &\quad = \frac{1}{4}\langle 0_x| \to \vec{b}_k \to b_{k'}|0_x\rangle. \qquad (24)\end{aligned}$$

We then immediately get an expression for the expectation value of the particle number operator $\vec{N}_k = \vec{b}_k^{\dagger}\vec{b}_k$ associated to the right-moving mode k :

$$\begin{aligned}\langle 0_x|\vec{N}_k|0_x\rangle = 4\int_{-\infty}^{+\infty}&dy_1^-dy_2^-\tilde{u}_k(y_1^-)\tilde{u}_k^*(y_2^-) \\ &\times\langle 0_x|:\partial_{y_1^-}f(y_1^-)\partial_{y_2^-}f(y_2^-):|0_x\rangle. \qquad (25)\end{aligned}$$

Taking into account (17) we obtain

$$\begin{aligned}\langle 0_x|\vec{N}_k|0_x\rangle = -\frac{1}{\pi}\int_{-\infty}^{+\infty}&dy_1^-dy_2^-\tilde{u}_k(y_1^-)\tilde{u}_k^*(y_2^-) \\ &\times\left[\left(\frac{dx^-(y_1^-)}{dy^-}\right)\left(\frac{dx^-(y_2^-)}{dy^-}\right)\right. \\ &\left.\times\frac{1}{[x^-(y_1^-) - x^-(y_2^-)]^2} - \frac{1}{(y_1^- - y_2^-)^2}\right]. \qquad (26)\end{aligned}$$

This expression has a nice physical interpretation. The production of quanta, as measured by an observer with coordinates $y^{\pm}$, is clearly associated to the deviation of the correlations $\langle 0_x|\partial_{y_1^-}f(y_1^-)\partial_{y_2^-}f(y_2^-)|0_x\rangle$ from their corresponding value in the vacuum $|0_y\rangle$. Moreover, the correlations contributing to the production of quanta in the mode k are those supported at the set of points y_1^- and y_2^- where the mode is located. This is clearer when one introduces finite-normalization wave packet modes, instead of the usual plane-wave modes:

$$\tilde{u}_w = \frac{e^{-iwy^-}}{\sqrt{4\pi w}}, \qquad (27)$$

for which $(\tilde{u}_w, \tilde{u}_{w'}) = \delta(w - w')$.[1] The wave packet modes can be defined as follows [5]:

$$\tilde{u}_{jn} = \frac{1}{\sqrt{\epsilon}}\int_{j\epsilon}^{(j+1)\epsilon}dwe^{2\pi inw/\epsilon}\tilde{u}_w, \qquad (28)$$

with integers $j \geq 0$, n. These wave packets are peaked about $y^- = 2\pi n/\epsilon$ with width $2\pi/\epsilon$. Taking ϵ small ensures that the modes are narrowly centered around $w \simeq w_j = j\epsilon$. Therefore the main contribution to $\langle 0_x|\vec{N}_{jn}|0_x\rangle$ comes from correlations, of range similar to the support of the wave packet, around the point $y^- = 2\pi n/\epsilon$.

It is interesting to remark that the difference of two-point functions in (26) at $y_1^- = y_2^-$ is not singular. In fact, for $y_1^- = y_2^- + \epsilon$ and $|\epsilon| \ll 1$, it is proportional to

$$\begin{aligned}-4\pi\langle 0_x|:T_{--}(y^-):|0_x\rangle - 2\pi\frac{d}{dy^-}& \\ \times\langle 0_x|:T_{--}(y^-):|0_x\rangle\epsilon + O(\epsilon^2),& \qquad (29)\end{aligned}$$

which clearly shows a smooth behavior at the coincidence limit.

Finally we mention that the expression (26) must be equivalent, by construction, to that given in terms of Bogoliubov coefficients (12). The aim of the next sections is to show that (26) offers an interesting perspective to understand better the phenomena of particle production.

[1] Note that the integral $\int_0^{\infty}dww\langle 0_x|\vec{N}_w|0_x\rangle$ gives the integrated flux $\int dy^-\langle 0_x|:T_{--}(y^-):|0_x\rangle$.

IV. THERMAL RADIATION

As an illustrative example we shall show how the expression (26) reproduces the thermal properties associated to the conformal transformation

$$x^{\pm} = \pm\kappa^{-1}e^{\pm\kappa y^{\pm}}. \tag{30}$$

We can think of this transformation as relating the Minkowskian $x^{\pm}$ and Rindler $y^{\pm}$ null coordinates, where κ is the acceleration parameter (the same relation holds for the Schwarzschild black hole between the Kruskal and Eddington-Finkelstein null coordinates with $\kappa = 1/4M$). As an intermediate step we shall first make use of plane waves and work out an expression for $\langle 0_x|\vec{b}^{\dagger}_w\vec{b}_{w'}|0_x\rangle$,

$$\langle 0_x|\vec{b}^{\dagger}_w\vec{b}_{w'}|0_x\rangle = -\frac{1}{4\pi^2\sqrt{ww'}}\int_{-\infty}^{+\infty}dy^- dy'^-\bigg[\frac{dx^-}{dy^-}(y^-)\times\frac{dx^-}{dy^-}(y'^-)\frac{1}{(x^- - x'^-)^2} - \frac{1}{(y^- - y'^-)^2}\bigg]e^{-iwy^- + iw'y'^-}. \tag{31}$$

Substitution of the relations (30) yields

$$\langle 0_x|\vec{b}^{\dagger}_w\vec{b}_{w'}|0_x\rangle = -\frac{1}{2\pi w}\delta(w - w')\times\int_{-\infty}^{+\infty}dz\bigg[\frac{\kappa^2 e^{-\kappa z}}{(1 - e^{-\kappa z})^2} - \frac{1}{z^2}\bigg]e^{-iwz}, \tag{32}$$

where $z = y^- - y'^-$. Evaluation of the integral gives

$$\langle 0_x|\vec{b}^{\dagger}_w\vec{b}_{w'}|0_x\rangle = \delta(w - w')\frac{1}{e^{2\pi w/\kappa} - 1}. \tag{33}$$

The delta function leads to a divergent result for the emitted number of particles $\langle 0_x|\vec{N}_w|0_x\rangle = \langle 0_x|\vec{b}^{\dagger}_w\vec{b}_w|0_x\rangle$. As usual, this divergence can be cured introducing a basis of finite-normalization wave packet modes. If we evaluate, instead, $\langle 0_x|\vec{N}_{jn}|0_x\rangle$, using again (26), it turns out that

$$\langle 0_x|\vec{N}_{jn}|0_x\rangle = \frac{1}{\epsilon}\int_{j\epsilon}^{(j+1)\epsilon}dw e^{2\pi iwn/\epsilon}\times\int_{j\epsilon}^{(j+1)\epsilon}dw' e^{-2\pi iw'n/\epsilon}\langle 0_x|\vec{b}^{\dagger}_w\vec{b}_{w'}|0_x\rangle = \frac{1}{\epsilon}\int_{j\epsilon}^{(j+1)\epsilon}dw\frac{1}{e^{8\pi Mw} - 1} = \frac{1}{e^{8\pi Mw_j} - 1}, \tag{34}$$

where in the last step we have assumed that the wave packets are sharply peaked around the frequencies w_j. This corresponds to the Planckian spectrum of radiation at the temperature $T = \frac{\kappa}{2\pi}$. Similar results hold for the left-mover sector. Evaluation of the expectation value of the stress tensor using (18), taking into account that $\langle 0_x|:T_{\pm\pm}(x^{\pm}):|0_x\rangle = 0$, gives

$$\langle 0_x|:T_{\pm\pm}(y^{\pm}):|0_x\rangle = \frac{\kappa^2}{48\pi} = \frac{\pi T^2}{12}. \tag{35}$$

This is nothing other than the stress tensor corresponding to a two-dimensional thermal bath of radiation at the temperature T.[2]

Note that our derivation of the Planckian spectrum bypasses the explicit evaluation of the Bogoliubov coefficients. Instead, it is based on the explicit form of the two-point correlation function, and the evaluation of the corresponding integral leads directly to the thermal result.

V. MOBIUS TRANSFORMATIONS

We shall now analyze the case associated to the Mobius transformations

$$x^{\pm} \to y^{\pm} = \frac{a^{\pm}x^{\pm} + b^{\pm}}{c^{\pm}x^{\pm} + d^{\pm}}, \tag{36}$$

where $a^{\pm}d^{\pm} - b^{\pm}c^{\pm} = 1$. These form the so-called global conformal group $[[\mathrm{SL}(2,R)\otimes\mathrm{SL}(2,R)]/Z_2 \approx \mathrm{SO}(2,2)]$. The physical meaning of these transformations can be found in [2]. In addition, a nice physical interpretation of the special conformal transformations was given in terms of a uniformly accelerating mirror [1,3].

The Mobius transformations have the property of giving a vanishing Schwarzian derivative. Therefore, under the action of the Mobius transformations the flux of radiation in the vacuum $|0_x\rangle$ for the observer $\{y^{\pm}\}$ vanishes:

$$\langle 0_x|:T_{\pm\pm}(y^{\pm}):|0_x\rangle = 0. \tag{37}$$

Moreover, since the two-point function (13) is invariant under (36) it is clear from (26) that the expectation value of the particle number operator also vanishes,

$$\langle 0_x|\vec{N}_k|0_x\rangle = 0 = \langle 0_x|\overleftarrow{N}_k|0_x\rangle, \tag{38}$$

irrespective of the particular mode basis. This is indeed what we expect in the context of CFT, since the vacuum is invariant under Mobius transformations (see [6] for a different approach). However, this conclusion is not obvious from the point of view of Bogoliubov coefficients. Let us consider those Mobius transformations which are not dilatations nor Poincaré, such as

[2] We must remark, nevertheless, that the covariant quantum stress tensor [1] $\langle 0_x|T_{\pm\pm}(y^{\pm})|0_x\rangle \equiv \langle 0_x|:T_{\pm\pm}(y^{\pm}):|0_x\rangle - (12\pi)^{-1}(\partial_{\pm}\rho\partial_{\pm}\rho - \partial^2_{\pm}\rho)$, where the metric is $ds^2 = -e^{2\rho}dy^+dy^- = -e^{\kappa(y^+ - y^-)}dy^+dy^-$, vanishes. Despite the existence of particle production in Rindler space ($\langle 0_x|\vec{N}_{jn}|0_x\rangle \neq 0 \neq \langle 0_x|:T_{\pm\pm}(y^{\pm}):|0_x\rangle$), the vanishing of the expectation values of the covariant stress tensor operator $\langle 0_x|T_{\pm\pm}(y^{\pm})|0_x\rangle = 0$ implies the absence of backreaction effects on the background flat metric.

$$x^- = -\frac{1}{a^2 y^-}, \tag{39}$$

where a is an arbitrary constant. We mention that this transformation originally appeared in the moving-mirror model of Davies and Fulling [3] (the parameter a^2 is related to the acceleration of the mirror) and more recently in the analysis of extremal black holes [7,8], where it gives the (leading order) relation between the Eddington-Finkelstein and Kruskal coordinates [which is instead given by (30) in the case of Schwarzschild and nonextremal Reissner-Nordström], and in the late-time behavior of evaporating near-extremal Reissner-Nordström black holes [9]. The Bogoliubov coefficients associated to the standard plane-wave basis are

$$\begin{aligned} \alpha_{ww'} &= \frac{1}{2\pi}\sqrt{\frac{w}{w'}}\int_{-\infty}^{+\infty} dy^- e^{-iwy^- - iw'/a^2y^-}, \\ \beta_{ww'} &= -\frac{1}{2\pi}\sqrt{\frac{w}{w'}}\int_{-\infty}^{+\infty} dy^- e^{-iwy^- + iw'/a^2y^-}. \end{aligned} \tag{40}$$

These integrals do not converge, as it usually happens for the plane-wave basis. Therefore one should introduce wave packets. The Bogoliubov coefficients can then be computed from the expressions

$$\begin{aligned} \beta_{jn,w'} &= -(\tilde{u}_{jn}, u^*_{w'}) = 2i\int_{-\infty}^{+\infty} dy^- \tilde{u}_{jn}\partial_{y^-} u_{w'}, \\ \alpha_{jn,w'} &= -(\tilde{u}_{jn}, u_{w'}) = -2i\int_{-\infty}^{+\infty} dy^- \tilde{u}_{jn}\partial_{y^-} u^*_{w'}, \end{aligned} \tag{41}$$

where

$$\begin{aligned} u_{w'} &= \frac{1}{\sqrt{4\pi w'}} e^{-iw'x^-(y^-)}, \\ \tilde{u}_{jn} &= \frac{1}{\sqrt{\epsilon}}\int_{j\epsilon}^{(j+1)\epsilon} dw e^{2\pi i n w/\epsilon}\tilde{u}_w. \end{aligned} \tag{42}$$

Since $\tilde{u}_w = (1/\sqrt{4\pi w})e^{-iwy^-}$ we have

$$\tilde{u}_{jn} = \frac{1}{\sqrt{4\pi\epsilon w_j}} e^{iw_jL}\frac{\sin L\epsilon/2}{L/2}, \tag{43}$$

where

$$L = \frac{2\pi n}{\epsilon} - y^-. \tag{44}$$

We then get

$$\begin{aligned} \beta_{jn,w'} &= \frac{1}{\pi\sqrt{\epsilon}}\sqrt{\frac{w'}{w_j}}\int_{-\infty}^{+\infty} dy^- \frac{\sin L\epsilon/2}{a^2(y^-)^2L} e^{iw_jL}e^{iw'/a^2y^-}, \\ \alpha_{jn,w'} &= -\frac{1}{\pi\sqrt{\epsilon}}\sqrt{\frac{w'}{w_j}}\int_{-\infty}^{+\infty} dy^- \frac{\sin L\epsilon/2}{a^2(y^-)^2L} e^{iw_jL}e^{-iw'/a^2y^-}. \end{aligned} \tag{45}$$

According to our previous discussion the first integral above should vanish, to agree with the result obtained using the Mobius invariance of the two-point correlation function,

$$\langle 0_x|\vec{N}_{jn}|0_x\rangle = \int_0^{+\infty} dw'|\beta_{jn,w'}|^2 = 0. \tag{46}$$

However, to show that the first integral vanishes is not easy, due to the singularity at $y^- = 0$.

Summarizing, the absence of particle production is immediate according to (26), but requires a lengthy elaboration using the Bogoliubov coefficients. We regard this as an indication of the advantage of using the expression (26) to analyze the production of quanta. At this respect we want to remark that the analysis leading to the expression (26) is based on the use of correlation functions of the "primary" field $\partial_\pm f$, rather than f itself. This avoids the infrared divergence of the scalar field in two dimensions. In fact,

$$\langle 0_x|f(x)f(x')|0_x\rangle = -\frac{\hbar}{4\pi}[2\gamma + \ln\lambda^2(x-x')^2], \tag{47}$$

where γ is the Euler constant and λ is an infrared cutoff for frequencies. This infrared difficulty is cured when one considers instead correlations of the field $\partial_\pm f$ [see (13)]. In contrast, the Bogoliubov coefficients are defined using mode solutions of the field f itself. Therefore, it should not be a complete surprise that the result (38), which is straightforward using (26), is not so obvious in terms of the Bogoliubov coefficients.

VI. INTERPRETATION IN TERMS OF MOVING MIRRORS

All the above discussion can be reinterpreted in terms of the so-called moving-mirror analogy. The idea is the following. Instead of having a two-dimensional flat spacetime with two different sets of modes $(u_i(x^-), v_i(x^+))$ and $(\tilde{u}_i(y^-), \tilde{v}_i(y^+))$, where the coordinates $y^\pm$ and $x^\pm$ are related by a conformal transformation,

$$y^- = y^-(x^-), y^+ = y^+(x^+), \tag{48}$$

one can introduce a boundary in the spacetime to produce the same physical consequences. The effect of the boundary is to disturb the modes in such a way that modes that at past null infinity behave as $(u_i(x^-), v_i(x^+))$, once evolved to future null infinity will take a form similar to $(\tilde{u}_i(y^-), \tilde{v}_i(y^+))$. This is the main property of a mirror model [1,10,11]: it can nicely mimic the physics in a nontrivial background (i.e., Hawking radiation in a black hole geometry), or the effect of having two different physically relevant sets of modes in a fixed background (as in the Fulling-Unruh construction).

The basic ingredient to define a moving-mirror model is the introduction of a (time-dependent) reflecting boundary in the space such that the field is assumed to satisfy the boundary condition $f = 0$ along its world line. It is convenient to parametrize the trajectory of the mirror in terms of null coordinates

$$x^+ = p(x^-). \quad (49)$$

Therefore the boundary condition is just

$$f(x^-, x^+ = p(x^-)) = 0. \quad (50)$$

A null ray at fixed x^+ which reflects off the mirror becomes a null ray of fixed x^-. The concrete relation between the coordinates of this null ray is given by the mirror's trajectory $x^+ = p(x^-)$. In terms of mode functions it is easy to construct plane-wave solutions of the equation $\nabla^2 f = 0$ vanishing on the world line of the wall:

$$u_w^{\rm in} = \frac{1}{\sqrt{4\pi w}}(e^{-iwx^+} - e^{-iwp(x^-)}). \quad (51)$$

They represent a positive frequency wave e^{-iwx^+}, coming from I_R^-, that reflects on the curve $x^+ = p(x^-)$ and becomes an outgoing wave $e^{-iwp(x^-)}$, which in general is not a pure positive frequency wave at I_R^+, but rather a superposition of positive and negative frequency components. In addition we also have modes representing a pure outgoing positive frequency wave e^{-iwx^-} at I_R^+, which is produced by the reflection of a wave $e^{-ip^{-1}(x^+)}$ from I_R^-:

$$u_w^{\rm out} = \frac{1}{\sqrt{4\pi w}}(e^{-iwx^-} - e^{-iwp^{-1}(x^+)}). \quad (52)$$

The above two sets of modes are the natural mode basis for inertial observers at I_R^- and I_R^+ and allow one to define the corresponding IN and OUT vacuum states. This concerns the dynamics of the field at the right-hand side of the mirror. A similar basis can be constructed to describe the dynamics at the left of the mirror, but we shall restrict our discussion, as usual, to the right region. Moreover, we can construct a wave packet basis from the plane-wave modes and rederive the same results obtained in Sec. III. The expectation value of the particle number operator in the mode k is given by

$$\begin{aligned}\langle 0_{\rm in}|N_k^{\rm out}|0_{\rm in}\rangle &= 4\int_{I_R^+} dx_1^- dx_2^- u_k^{\rm out}(x_1^-) u_k^{\rm out *}(x_2^-) \\ &\quad\times \langle 0_{\rm in}|{:}\partial_{x_1^-} f(x_1^-)\partial_{x_2^-} f(x_2^-){:}|0_{\rm in}\rangle \\ &= -\frac{1}{\pi}\int_{I_R^+} dx_1^- dx_2^- u_k^{\rm out}(x_1^-) u_k^{\rm out *}(x_2^-) \\ &\quad\times \left\{\frac{p'(x_1^-)p'(x_2^-)}{[p(x_1^-) - p(x_2^-)]^2} - \frac{1}{(x_1^- - x_2^-)^2}\right\},\end{aligned} \quad (53)$$

and the flux of energy radiated to the right is given by the Schwarzian derivative

$$\langle 0_{\rm in}|{:}T_{--}(x^-){:}|0_{\rm in}\rangle = -\frac{1}{24\pi}\{p(x^-), x^-\}. \quad (54)$$

The results concerning thermal radiation obtained in Sec. IV can be rederived in this context by considering the mirror trajectory $x^+ = -\kappa^{-1} e^{-\kappa x^-}$.

We shall now illustrate our previous discussion on Mobius transformations with the use of the moving-mirror analogy.

A. Two hyperbolic mirrors

Our first example will be a mirror with two hyperbolic branches

$$p(x^-) = -\frac{1}{a^2 x^-}. \quad (55)$$

One branch with $x^- < 0$ and the other with $x^- > 0$ (see Fig. 1.)

The above function $p(x^-)$ can be regarded as associated to a special conformal transformation of coordinates. Note that all the modes supported on I_R^- are reflected to I_R^+. The IN modes supported on the interval $x^+ \in]-\infty, 0[$ reach I_R^+ on $x^- \in]0, \infty[$; the IN modes supported on the interval $x^+ \in]0, \infty[$ reach I_R^+ on $x^- \in]-\infty, 0[$. In this way, the correlations existing between positive and negative x^+ are transferred to correlations between positive and negative x^-. Moreover, since the two-point correlation function on I_R^+ is the same as that of the vacuum

$$\begin{aligned}\langle 0_{\rm in}|\partial_{x^-}\phi(x_1^-)\partial_{x^-}\phi(x_2^-)|0_{\rm in}\rangle &= -\frac{1}{4\pi}\frac{p'(x_1^-)p'(x_2^-)}{[p(x_1^-) - p(x_2^-)]^2} \\ &= -\frac{1}{4\pi}\frac{1}{(x_1^- - x_2^-)^2} \\ &\text{for all } x_1^-, x_2^-,\end{aligned} \quad (56)$$

there is neither particle production $\langle 0_{\rm in}|N_k^{\rm out}|0_{\rm in}\rangle = 0$ nor energy flux $\langle 0_{\rm in}|{:}T_{--}(x^-){:}|0_{\rm in}\rangle = 0$.

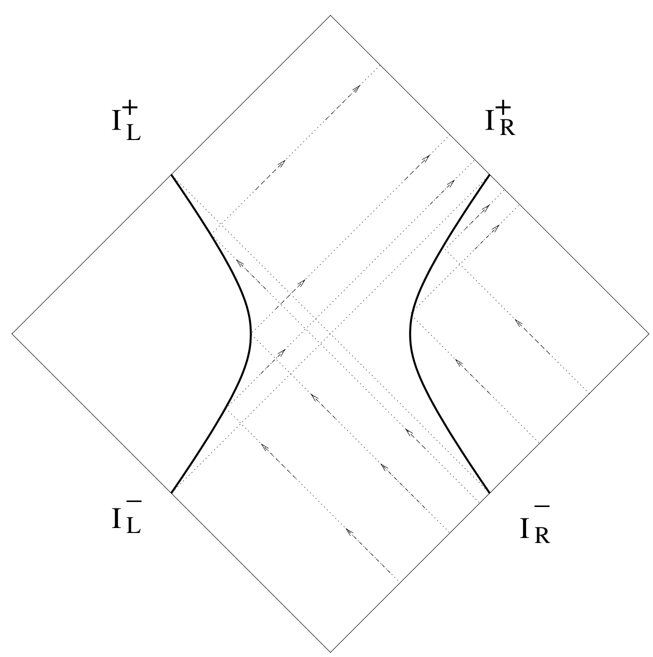

FIG. 1. A mirror with two hyperbolic branches.

B. Hyperbolic mirror accelerating from rest

Let us consider a mirror that from rest accelerates to the left following an hyperbolic trajectory, i.e.,

$$p(x^-) = \begin{cases} x^- & \text{if } x^- \le 0 \\ \frac{x^-}{1+a^2x^-} & \text{if } x^- \ge 0. \end{cases} \quad (57)$$

We can write this trajectory in the following compact notation:

$$p(x^-) = x^-\theta(-x^-) + \frac{x^-}{1+a^2x^-}\theta(x^-). \quad (58)$$

In this case (see Fig. 2) the mirror starts at i^- and ends up on I_L^+. This gives rise to the appearance of a horizon: the IN modes in the range $x^+ \in [1/a^2, \infty[$ do not reach I_R^+. On the other hand, the modes supported in the range $x^+ \in [-\infty, 1/a^2[$, upon reflection off the mirror, will reach I_R^+. The corresponding correlation function along I_R^+ becomes

$$\langle 0_{\rm in}|\partial_{x^-}\phi(x_1)\partial_{x^-}\phi(x_2)|0_{\rm in}\rangle = \begin{cases} -\frac{1}{4\pi}\frac{1}{(x_1^- - x_2^-)^2} & \text{if } x_1^- \times x_2^- > 0 \\ -\frac{1}{4\pi}\frac{1}{(x_1^- - x_2^- + a^2x_1^-x_2^-)^2} & \text{if } x_1^- \times x_2^- < 0. \end{cases} \quad (59)$$

From this we learn that the identity transformation for $x^- < 0$ and the "single-branch" special conformal transformation for $x^- > 0$ make it impossible to distinguish the IN vacuum state from the OUT vacuum by means of measurements restricted to $x_1^-, x_2^- > 0$ or to $x_1^-, x_2^- < 0$. Only the mixed correlations $x_1^- > 0$, $x_2^- < 0$ and $x_1^- < 0$, $x_2^- > 0$ allow one to distinguish the IN vacuum from the OUT vacuum. Moreover, even though the flux is zero for $x^- < 0$ and $x^- > 0$ (since in these regions the normal-ordered two-point correlation function is identically zero), there is a divergence at $x^- = 0$. The evaluation of the Schwarzian derivative for the trajectory (58) gives

$$\langle 0_{\rm in}|T_{--}|0_{\rm in}\rangle = \frac{a^2}{12\pi}\delta(x^-). \quad (60)$$

The origin of this nonvanishing flux at $x^- = 0$ can be attributed to the deviation of the correlation function (for $x_1^- < 0$ and $x_2^- > 0$) from that of the vacuum. This is not surprising, since the mirror has undergone a sudden acceleration just at $x^- = 0$. The same reason underlies the (nonvanishing) production of quanta, which according to Eq. (53) turns out to be

$$\begin{aligned}\langle 0_{\rm in}|N_k|0_{\rm in}\rangle = -\frac{1}{\pi}\int_{-\infty}^{0} dx_1^- \int_0^\infty dx_2^- [u_k^{\rm out}(x_1^-)u_k^{\rm out*}(x_2^-) \\ + u_k^{\rm out}(x_2^-)u_k^{\rm out*}(x_1^-)] \\ \times\left[\frac{1}{(x_1^- - x_2^- + a^2x_1^-x_2^-)^2} - \frac{1}{(x_1^- - x_2^-)^2}\right].\end{aligned} \quad (61)$$

We observe that for modes k supported in the region $x^- > 0$ the expectation value $\langle 0_{\rm in}|N_k|0_{\rm in}\rangle$ vanishes. Therefore, a particle detector which is switched on at late times $x^- \gg 0$ (or early times $x^- \ll 0$) will never detect the emission of quanta, since there $\langle 0_{\rm in}|N_k|0_{\rm in}\rangle = 0$. The detection of quanta will take place only through the region $x^- = 0$ where the flux is nonvanishing. In other words, the measured quanta needs to correspond to a wave packet mode with support around the point $x^- = 0$.

C. Single-branch hyperbolic mirror

Let us consider again, as in the first example, a pure hyperbolic mirror, but this time only a single branch,

$$p(x^-) = -\frac{1}{a^2x^-}\theta(x^-). \quad (62)$$

For $x^- < 0$ there is no reflecting wall. This case is more involved since right and left are not disconnected. According to Fig. 3, the IN modes reaching I_R^+ are of two types: those coming from the $(-\infty, 0)$ segment of I_R^- and those coming from the $(-\infty, 0)$ segment of I_L^-.

Since there are no correlations between I_R^- and I_L^-, it is easy to see that the two-point correlation function on I_R^+ is given by

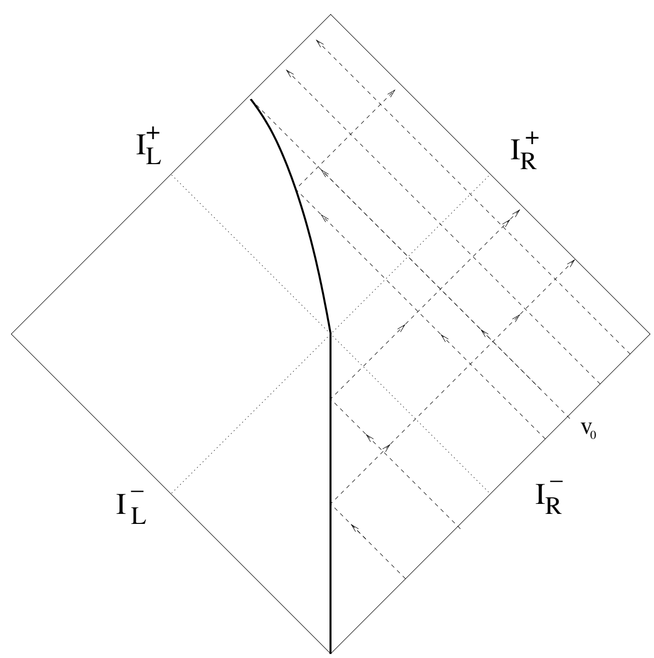

FIG. 2. Hyperbolic mirror accelerating from rest.

$$\langle 0_{\rm in}|\partial_{x^-}\phi(x_1^-)\partial_{x^-}\phi(x_2^-)|0_{\rm in}\rangle = \begin{cases} -\frac{1}{4\pi}\frac{1}{(x_1^- - x_2^-)^2} & \text{if } x_1^-, x_2^- > 0 \text{ or } x_1^-, x_2^- < 0 \\ 0 & \text{if } x_1^- < 0, x_2^- > 0 \text{ or } x_1^- > 0, x_2^- < 0. \end{cases} \tag{63}$$

The particle production is then given by the expression

$$\langle 0_{\rm in}|N_k|0_{\rm in}\rangle = -\frac{1}{\pi}\int_{-\infty}^{0} dx_1^- \int_0^{\infty} dx_2^- [u_k^{\rm out}(x_1^-)u_k^{\rm out *}(x_2^-) + u_k^{\rm out}(x_2^-)u_k^{\rm out *}(x_1^-)]\left[-\frac{1}{(x_1^- - x_2^-)^2}\right]. \tag{64}$$

We observe that the nonvanishing contribution comes from the OUT vacuum correlations between positive and negative points, since they cannot be canceled out by the correlations of the IN vacuum state, as it happens instead for the pair of points $x_1^-, x_2^- > 0$, or $x_1^-, x_2^- < 0$. Because of this there is a divergent flux of energy concentrated at the point $x^- = 0$, dividing the two uncorrelated regions with respect to the IN vacuum state. This divergence is then even more drastic than that found in the previous example, for which the correlations between points $x_1^- > 0$ and $x_2^- < 0$ are diminished with respect to the OUT vacuum, but are nonzero. We can also infer the same type of conclusions for the production of quanta. At late, or early times, we will never detect quanta, since the required modes are those with support covering the point $x^- = 0$ where the energy flux is concentrated. This result agrees with that obtained in terms of particle detectors [4].

Let us now compare this analysis with the interpretation of [1,3] carried out employing naively the Bogoliubov coefficients. As we have already remarked, the Bogoliubov coefficients associated to special conformal transformations for plane waves involve ill-defined integrals. Restriction of the special conformal transformation to one single branch still produces ill-defined expressions for the Bogoliubov coefficients:

$$\alpha_{ww'} = \frac{1}{2\pi}\sqrt{\frac{w}{w'}}\int_0^{+\infty} dy^- e^{-iwy^- - iw'/a^2y^-},$$
$$\beta_{ww'} = -\frac{1}{2\pi}\sqrt{\frac{w}{w'}}\int_0^{+\infty} dy^- e^{-iwy^- + iw'/a^2y^-}. \tag{65}$$

Following the original work on the moving-mirror system [3], one can evaluate these integrals using a Wick rotation (i.e., integrating along the imaginary axis).[3] The results are [1,3,11]

$$\alpha_{ww'} = \frac{1}{a\pi}K_1(2i\sqrt{ww'/a^2}),$$
$$\beta_{ww'} = \frac{i}{a\pi}K_1(2\sqrt{ww'/a^2}), \tag{66}$$

where K_1 is a modified Bessel function.[4]. The nonvanishing of the $\beta_{ww'}$ coefficients was interpreted [1,3] as an indication of the existence of particle production even in the absence of energy fluxes. However, due to the term $1/\sqrt{ww'}$ in the asymptotic form of K_1 for small frequencies, the quantity

$$\langle 0_{\rm in}|N_w|0_{\rm in}\rangle = \int_0^{+\infty} dw' |\beta_{ww'}|^2 \tag{67}$$

diverges. As usual, this should be cured introducing wave packet modes, but, as far as we know, an explicit calculation has never been performed (see also [8]). It is therefore difficult to find a clear physical picture for the time distribution of the emitted quanta and draw definite conclusions within the approach of Bogoliubov coefficients.

However, we can easily match with our conclusions from the expression (64) if we make the following reasoning. The IN vacuum at I_R^- can be expanded in terms of correlated Rindler particles between both sides of the line $x^+ = 0$. For this we can introduce the unconstrained coordinates $\kappa y^+ = -\ln(-\kappa x^+)$, for $x^+ < 0$, and $\kappa z^+ = \ln\kappa x^+$, for $x^+ > 0$. κ is an arbitrary positive constant which plays only an auxiliary role in the discussion. In a parallel way we can introduce the coordinates $\kappa y^- = \ln\kappa x^-$, for $x^- > 0$, and $\kappa z^- = -\ln(-\kappa x^-)$, for $x^- < 0$. The Rindler modes e^{-iwy^+} and e^{-iwy^-} are related, due to

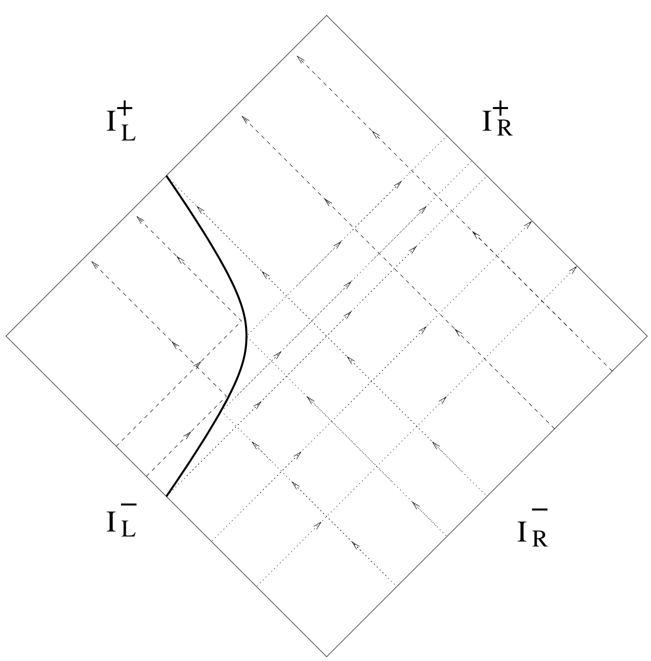

FIG. 3. A single-branch hyperbolic mirror.

[3]This has been recently criticized in [8].

[4]Notice that if the integral is from $-\infty$ to $+\infty$ (i.e., the two branches of the special conformal transformation) there is cancellation between both branches and the final result is $\beta_{ww'} = 0$

the reflection, by means of the special conformal transformation. But in the new coordinates the special conformal transformation (62) is ''inertial'':

$$\kappa y^+ = -\ln\frac{\kappa^2}{a^2} + \kappa y^-. \tag{68}$$

The corresponding β coefficients are then clearly zero. The restriction of the IN vacuum to the segment $x^+ < 0$ is a mixed state. Moreover, in terms of the Rindler coordinate y^+, this mixed state takes the form of a thermal state (of Rindler particles) with temperature $T = \kappa/2\pi$. The OUT vacuum state is also mixed when restricted to $x^- > 0$, and a thermal state with respect to the Rindler coordinate y^-. Because of the vanishing of the β coefficients the thermal description of the IN vacuum, when restricted to $x^+ < 0$, is reflected without distortion by the mirror, producing the same thermal state at I_R^+. This thermal state is supported in the region $x^- > 0$, and has no correlations with the region $x^- < 0$. As we have already noted, this thermal state is perceived, for the inertial observer at I_R^+ using the coordinate x^-, the same as the Minkowski OUT vacuum for measurements restricted to the region $x^- > 0$. Therefore, in the region $x^- > 0$ neither particle production nor energy flux can be detected.

VII. CONCLUSIONS

In this paper we have analyzed the particle production due to conformal transformations or, equivalently, due to reflections on moving mirrors, based on a viewpoint different from the standard approach. We were motivated by the fact that, under special conformal transformations, the vanishing of the β coefficients is not trivial, despite the fact that the invariance of the vacuum under Mobius transformations is one of the ''postulates'' of CFT. The expression for the particle production that we analyze here [Eq. (26)] immediately clarifies this aspect. This is so because it emphasizes that the deviation of the two-point correlation function from that of the vacuum, weighted by wave packets of a definite mode, is the source of the production of quanta of the corresponding mode. We have first shown that the proposed expression for the particle production works nicely in recovering, in a simple way, the standard thermal radiation in Rindler space. We have also revised, from this point of view, the moving-mirror systems with different examples of hyperbolic trajectories. We have pointed out the close relation between the production of quanta and energy, which contrasts early conclusions based on ill-defined expressions for the Bogoliubov coefficients. Finally, we remark that we are not criticizing the Bogoliubov approach in favor of the approach presented here. Both approaches are different ways of measuring the same physical quantity.

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Note added.–After completion of this work we were informed that in [12] particle production is also investigated without making use of Bogoliubov coefficients in a cosmological scenario.

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