Quantum stress tensor for extreme 2D Reissner-Nordström black holes

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Contrary to previous claims, it is shown that the expectation values of the quantum stress tensor for a massless scalar field propagating on a two-dimensional extreme Reissner-Nordström black hole are indeed regular on the horizon.

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Extremal black holes play an important role in gravity and string theories. They appear as soliton like objects with intrinsic parameters saturating a Bogomol'nyi bound and have zero Hawking temperature [1].

It is quite disturbing that quantum field theory (QFT hereafter) on these backgrounds seems to predict divergences [2,3] which, although being "mild" in some sense, have no clear physical explanation. These divergences are associated to "vacuum" expectation values of the stress tensor operator evaluated on the horizon. Before entering this problem, a digression on the notion of vacuum in this context is necessary [4].

In QFT in curved space-time, quantization is achieved as usual by expanding the field operator in annihilation (a_i) and creation ($a_i^\dagger$) operators according to a given set of mode solutions of the field equation. The vacuum $|0\rangle$ is the state annihilated by the a_i, i.e., $a_i|0\rangle = 0$. However, one of the most interesting outcomes of this procedure is that in the presence of a gravitational field (i.e., in a curved space-time) the notion of vacuum state becomes rather vague. Unlike Minkowski QFT, there is not a unique vacuum state. There are many (in principle, infinite) "vacuum states," no one sharing the central and unique role the Minkowski vacuum has for inertial observers. The "vacuum states" one can construct in QFT are not at all empty (at least everywhere). Furthermore, their particle content is observer dependent. According to our present understanding, these different vacua simply represent different physical situations.

In black hole space-times, one usually considers three "vacuum states." The first is the so called Boulware vacuum state $|B\rangle$ [5]. It is constructed with ingoing and outgoing modes that are positive frequency with respect to the asymptotically Minkowskian time coordinate. At infinity these modes reduce to usual plane waves and therefore there the Boulware vacuum reduces to the Minkowski vacuum ($|B\rangle \sim |0\rangle_M$). If the behavior of $|B\rangle$ at infinity seems quite reasonable, the same cannot be said at the black hole horizon. If the quantum field is in the Boulware state, an inertial observer falling across the horizon measures an infinite energy density and pressure [6]. From the physical point of view $|B\rangle$ is supposed to describe the vacuum polarization outside a static star. Being that its radius is bigger than the horizon, the above divergence is spurious.

A quantum state regular on the horizon, the so called Hartle-Hawking state $|H\rangle$ [7], can be constructed using incoming and outgoing modes that are positive frequency with respect to the affine parameters along the future and past horizons of the black hole, respectively. These "Kruskal modes" do not match at infinity the standard Minkowski plane waves. At infinity $|H\rangle$ is not empty ($|H\rangle \neq |0\rangle_M$). It describes equilibrium thermal radiation at the Hawking temperature $T = \kappa/2\pi$ (in units $\hbar = c = k_B = 1$), where κ is the surface gravity of the horizon. $|H\rangle$ has the properties of being a thermal state and is the only static state which is regular both on the future and past horizons. It is supposed to describe the thermal equilibrium of a black hole with its own quantum radiation. Equilibrium is achieved by enclosing the black hole in a reflecting box.

Finally, the Unruh state $|U\rangle$ [8] is constructed by ingoing modes that are positive frequency with respect to the asymptotic Minkowskian time, whereas the outgoing modes are positive frequency with respect to the affine parameter on the past horizon. This hybrid construction is, from the physical point of view, the most interesting, since it describes the late-time behavior of a quantum field in the space-time of a collapsing body forming a black hole. From its definition, one can show that in this state one has no particles coming in from past infinity, whereas there is a thermal flux of particles at the Hawking temperature flowing out to future infinity. $|U\rangle$ is regular on the future horizon, but on the past horizon it has the same bad behavior $|B\rangle$ had. However, for a black hole formed by the gravitational collapse of matter, the past horizon does not exist, being covered by the infalling matter, and therefore the related divergence is spurious.

All the features of the "vacuum states" we have presented can be easily seen in a two-dimensional space-

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time context where the quantum field is a massless minimally coupled scalar. For this case, exact analytical results can be found. The space-time we shall consider is the 2D section of the Reissner-Nordström one,

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} = -f(r)dudv, \tag{1}$$

where $f(r) = 1 - 2M/r + Q^2/r^2$, M is the mass and Q the charge of the black hole ($M > |Q|$). u and v are, respectively, the retarded and advanced Eddington-Finkelstein coordinates

$$u = t - r^*, \qquad v = t + r^*, \tag{2}$$

where $r^* = \int dr/f(r)$. The horizon is located at $r_+ = M + \sqrt{M^2 - Q^2}$. The field equation for the scalar field is

$$\Box\psi = 0 \Leftrightarrow \partial_{\bar{u}}\partial_{\bar{v}}\psi = 0, \tag{3}$$

where $\{\bar{u}, \bar{v}\}$ are null coordinates related to $\{u, v\}$ by a generic conformal coordinate transformation $u \to \bar{u}$, $v \to \bar{v}$. The normal modes of Eq. (3) are simply plane waves $\{e^{-iw\bar{u}}, e^{-iw\bar{v}}\}$. Expanding ψ in these modes one constructs the $|\bar{u}, \bar{v}\rangle$ vacuum state. The expectation values of the quantum stress tensor operator for the ψ field in this state are [9]

$$\begin{aligned} \langle \bar{u}, \bar{v}|T_{uu}|\bar{u}, \bar{v}\rangle &= -(12\pi)^{-1} f^{\frac{1}{2}}(f^{-\frac{1}{2}})_{,uu} + \Delta(u, \bar{u}), \\ \langle \bar{u}, \bar{v}|T_{vv}|\bar{u}, \bar{v}\rangle &= -(12\pi)^{-1} f^{\frac{1}{2}}(f^{-\frac{1}{2}})_{,vv} + \Delta(v, \bar{v}), \\ \langle T^a_a\rangle &= (24\pi)^{-1}R = (6\pi)^{-1} f^{-1}(\ln f)_{,uv}, \end{aligned} \tag{4}$$

where

$$-(12\pi)^{-1} f^{\frac{1}{2}}(f^{-\frac{1}{2}})_{,uu} = -(12\pi)^{-1} f^{\frac{1}{2}}(f^{-\frac{1}{2}})_{,vv} = \tag{5}$$

$$\equiv H(r) = (24\pi)^{-1}\left(-\frac{M}{r^3} + \frac{3}{2}\frac{M^2 + Q^2}{r^4} - \frac{3MQ^2}{r^5} + \frac{Q^4}{r^6}\right), (6\pi)^{-1} f^{-1}(\ln f)_{,uv}$$

$$= (6\pi)^{-1}\left(\frac{M}{r^3} - \frac{3}{2}\frac{Q^2}{r^4}\right) \tag{6}$$

and

$$\Delta(u, \bar{u}) = (24\pi)^{-1}\left(\frac{F''}{F} - \frac{1}{2}\frac{F'^2}{F^2}\right) \tag{7}$$

is the Schwarzian derivative associated to the transformation $u \to \bar{u}$ with $F = du/d\bar{u}$ and a prime means differentiation with respect to u. Similarly for $\Delta(v, \bar{v})$ with F replaced by $G = dv/d\bar{v}$.

The last equation in (4) is the well known trace anomaly, where R is the Ricci scalar. This expression holds in every conformal vacuum state, and this explains the omission of the specification of the quantum state.

Now for the Boulware state $|B\rangle$ the modes are given by $u = \bar{u}$, $v = \bar{v}$, i.e., $\Delta_B(u, \bar{u}) = 0 = \Delta_B(v, \bar{v})$ yielding

$$\langle B|T_{uu}|B\rangle = \langle B|T_{vv}|B\rangle = H(r). \tag{8}$$

For the Hartle-Hawking state $|H\rangle$, $\bar{u} = U$, $\bar{v} = V$, where $\{U, V\}$ are the Kruskal coordinates

$$U = -\frac{1}{\kappa}e^{-\kappa u}, \qquad V = \frac{1}{\kappa}e^{\kappa v}. \tag{9}$$

κ is the surface gravity at the horizon

$$\kappa = \frac{\sqrt{M^2 - Q^2}}{r_+^2}. \tag{10}$$

This gives

$$\Delta_H(u, U) = \Delta_H(v, V) = (48\pi)^{-1}\kappa^2 \tag{11}$$

and

$$\langle H|T_{uu}|H\rangle = \langle H|T_{vv}|H\rangle = \langle B|T_{uu}|B\rangle + (48\pi)^{-1}\kappa^2. \tag{12}$$

Finally for the Unruh state $|U\rangle$, $\bar{u} = U$, $\bar{v} = v$ and

$$\begin{aligned} \langle U|T_{uu}|U\rangle &= \langle H|T_{uu}|H\rangle \\ &= \langle B|T_{uu}|B\rangle + (48\pi)^{-1}\kappa^2, \\ \langle U|T_{vv}|U\rangle &= \langle B|T_{vv}|B\rangle. \end{aligned} \tag{13}$$

This form of the stress tensor is physically quite interesting, since it is obtained as the late-time behavior in the case of an arbitrary collapse in two dimensions. As shown, for example, in Birrell and Davies [4], the effect of the collapse is to increase the vacuum polarization part (i.e., $\langle B|T_{ab}|B\rangle$) with an outgoing (retarded) flux or radiation that is constant along u rays and that asymptotically approaches $(48\pi)^{-1}\kappa^2$.

From the previous expressions, it is easy to see that the expectation values in the three states differ just by conserved traceless radiation at the Hawking temperature $T_H = \kappa/2\pi$. Now regularity of the stress tensor (in a regular frame) on the future horizon is achieved as $r \to r_+$ if [6]

$$f^{-2}T_{uu} < \infty, T_{vv} < \infty, \qquad f^{-1}T_{uv} < \infty. \tag{14}$$

Regularity on the past horizon is given by analogous requirements with just u and v interchanged. Using these relations, it is easy to verify the statements made previously concerning the behavior of the three quantum states on the horizon.

Let us now consider in detail what happens for the extreme Reissner-Nordström black hole, for which $M = |Q|$, since things become now trickier. This kind of black hole has zero surface gravity (see Eq. (10)), hence zero Hawking temperature. This is often stated to imply that the Boulware, Unruh, and Hartle-Hawking states all coincide; for $\kappa = 0$ there is no difference between the Eqs. (8), (12), and (13), namely

$$\langle |T_{uu}|\rangle = \langle |T_{vv}|\rangle = -(24\pi)^{-1}\frac{M}{r^3}\left(1-\frac{M}{r}\right)^3 \equiv H^{\rm extr}(r), \tag{15}$$

$$\langle T^a_a\rangle = (24\pi)^{-1}R = (6\pi)^{-1}f^{-1}(\ln f)_{,uv} = (6\pi)^{-1}\frac{M}{r^3}\left(1-\frac{3}{2}\frac{M}{r}\right), \tag{16}$$

for all three states. Given this, let us check the regularity conditions on the horizon. It is rather disappointing to see that the first condition in Eqs. (14) is not satisfied since

$$\lim_{r\to r_+} f^{-2}\langle T_{uu}\rangle = \lim_{r\to M}\left(1-\frac{M}{r}\right)^{-4}H^{\rm extr}(r) = -(24\pi)^{-1}\frac{M}{r^3}\left(1-\frac{M}{r}\right)^{-1} \tag{17}$$

diverges. A similar divergence is found in the past horizon. An observer crossing the horizon measures therefore an unbounded energy density and pressure. It must be noted, however, that this singularity is regarded to be sufficiently mild, since it leads to finite tidal distortions and finite curvature [2]. In any case, before reaching any definitive conclusion one should take the extreme black hole limit with care, since the Kruskal coordinates transformation given by Eq. (9) makes no sense for $\kappa = 0$ and hence the expression Eq. (15), obtained by calculating the Schwarzian derivative and taking the limit $\kappa = 0$ at the end, becomes rather doubtful.

On the other hand, despite the mathematical inconsistency of its derivation, the stress tensor whose components are given by Eq. (15) is the only conserved tensor in the extreme 2D Reissner-Nordström space-time with the correct trace anomaly which is static (i.e., has the time translation invariance of the underlying manifold) and vanishes asymptotically as a zero temperature equilibrium state should do. The solidity of this argument seems to leave no way out concerning the singular behavior at the horizon.

The critical point is whether the state whose stress tensor is given by Eq. (15) has any physical significance, i.e., it can be realized by some physical process.

To examine this point, let us consider the formation of an extremal black hole and the corresponding stress tensor for a massless scalar field propagating in this geometry.

For simplicity, let us model the collapsing body forming the black hole by an ingoing null shell. The space-time is depicted in Fig. 1 An incoming null shell at $v = v_0$ creates an extreme Reissner-Nordström black hole whose metric is given by:

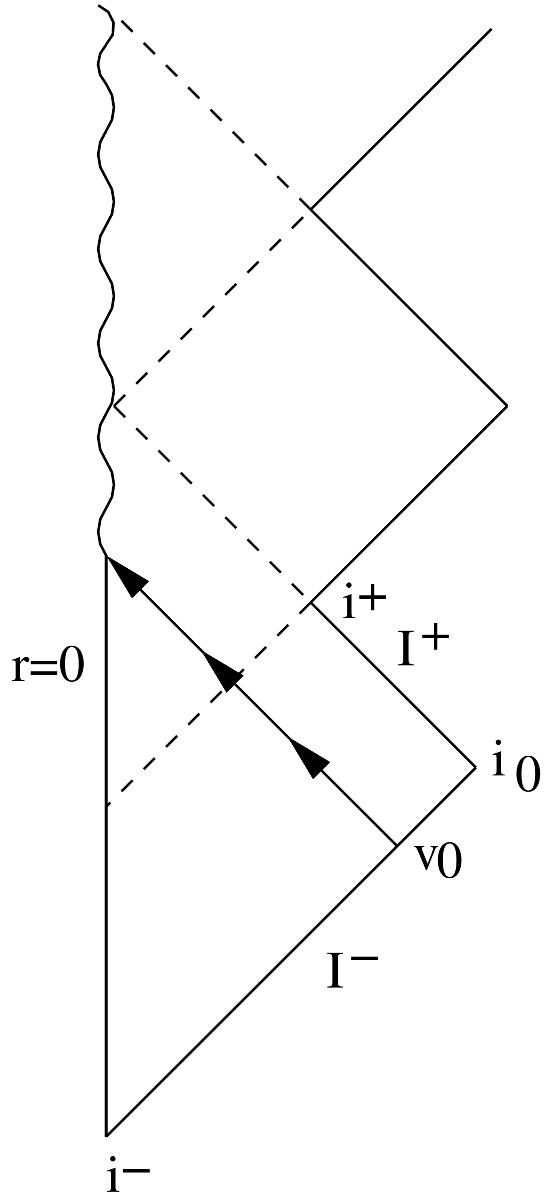

FIG. 1. Penrose diagram describing the formation of an extreme Reissner-Nordström black hole.

$$ds^2 = -\left(1-\frac{M}{r}\right)^2 dt^2 + \left(1-\frac{M}{r}\right)^{-2}dr^2 = -\left(1-\frac{M}{r}\right)^2 du dv, \tag{18}$$

where, as usual, $u = t - r^*$, $v = t + r^*$, but now:

$$r^* = \int \frac{dr}{(1-\frac{M}{r})^2} = r + 2M\ln\left(\frac{r}{M}-1\right) - \frac{M^2}{(r-M)}. \tag{19}$$

In the past of the shell, the space-time is Minkowski, with metric

$$ds^2 = -dT^2 + dr^2 = -d\bar{u}d\bar{v} \tag{20}$$

and

$$\bar{u} = T - r, \bar{v} = T + r. \tag{21}$$

Asymptotic flatness in the past ($\mathcal{I}^-$ exists) implies $v = \bar{v}$.

An incoming $\bar{v}$ mode from past infinity is reflected from the regular Minkowski origin $r = 0$ (i.e.,. $\bar{u} = \bar{v}$) and becomes an outgoing $\bar{u}$ mode. Matching across the shell at $v = \bar{v} = v_0$ yields

$$u = \bar{u} - 4M\left[\ln\left(\frac{v_0 - \bar{u}}{2M} - 1\right) - \frac{1}{2(\frac{v_0-\bar{u}}{2M} - 1)}\right]. \tag{22}$$

Being the horizon ($r = M$) located at $\bar{u} = v_0 - 2M$, i.e,. $u \rightarrow +\infty$, it is easy to see that a positive frequency $\bar{v}$ mode on $\mathcal{I}^-$ becomes at late advanced time ($u \rightarrow +\infty$) a positive frequency U mode, where

$$u = -4M\left[\ln\left(-\frac{U}{M}\right) + \frac{M}{2U}\right], \tag{23}$$

and we have redefined $\bar{u} \rightarrow 2U - 2M + v_0$ so that the horizon lies now at $U = 0$.

The relation (23) has been proposed by Liberati *et al.* [10] as generalization of the Kruskal transformation to the extreme Reissner-Nordström black hole. Gao [11] has later shown that Eq. (23) defines indeed a smooth extension across the horizon. To get this result, the presence of the subleading (as $U \rightarrow 0$) logarithmic term is critical.

One can now evaluate in the future of the shell the stress tensor using the general expression of Eq. (4). The late-time behavior defines the Unruh state ($\bar{u} = U, \bar{v} = v$) for the extreme Reissner-Nordström black hole

$$\begin{aligned} \langle U|T_{uu}|U\rangle &= H^{\mathrm{extr}}(r) + \Delta(u, U), \\ \langle U|T_{vv}|U\rangle &= H^{\mathrm{extr}}(r), \end{aligned} \tag{24}$$

$$\langle T^a_a\rangle = (24\pi)^{-1}R = (6\pi)^{-1}f^{-1}(\ln f)_{,uv}.$$

The Schwarzian derivative $\Delta(u, U)$ calculated from Eq. (23) yields

$$\Delta(u, U) = (24\pi)^{-1}\frac{U^3(U - 2M)}{2M^2(2U - M)^4}, \tag{25}$$

so that

$$\begin{aligned} \langle U|T_{uu}|U\rangle &= -(24\pi)^{-1}\frac{M}{r^3}\left(1 - \frac{M}{r}\right)^3 + (24\pi)^{-1} \\ &\quad\times\frac{U^3(U - 2M)}{2M^2(2U - M)^4}, \\ \langle U|T_{vv}|U\rangle &= -(24\pi)^{-1}\frac{M}{r^3}\left(1 - \frac{M}{r}\right)^3, \\ \langle T^a_a\rangle &= (6\pi)^{-1}\frac{M}{r^3}\left(1 - \frac{3}{2}\frac{M}{r}\right), \end{aligned} \tag{26}$$

which, when compared with the expression obtained by naively taking the limit $\kappa = 0$ (Eq. (15)), shows a striking difference in the T_{uu} component. Note that the tensor of Eq. (26) is conserved, has the correct trace but is not time independent. So there is no contradiction with our previous remark on the unicity of the expression Eq. (15).

The most remarkable feature of the stress tensor (26) is that it is regular on the future event horizon. Using $U = -(r - M)$, one can easily check that

$$\begin{aligned} \lim_{r\rightarrow M} f^{-2}\langle U|T_{uu}|U\rangle &= \lim_{r\rightarrow M} -\frac{1}{24\pi}\left[\frac{M}{r^3}\left(1 - \frac{M}{r}\right)^{-1} - \frac{1}{f^2}\Delta(u, U)\right] \\ &= -\lim_{r\rightarrow M}\frac{1}{24\pi}\left[\frac{M}{r^2}\frac{1}{r - M} - \frac{1}{M(r - M)} + \text{finite}\right] \\ &= -\frac{1}{24\pi}\left(\frac{3}{2M^2}\right) < \infty. \end{aligned} \tag{27}$$

The divergence in $f^{-2}\langle U|T_{uu}|U\rangle$ coming from the vacuum polarization part (i.e., $H^{\mathrm{extr}}(r)$) is exactly canceled by the divergent term $f^{-2}\Delta(u, U)$.

One should appreciate the fundamental role played by the subleading logarithmic term in the relation between u and U (Eq. (23)). Omission of this term (which corresponds to the extension proposed by Lake [12]) yields an identically vanishing Schwarzian derivative, and the resulting stress tensor would reduce again to the static one of Eq. (15) with the associated divergence on the horizon. However, as already stressed, the logarithmic term is necessary to have a smooth extension across horizon, and this explains the regular behavior of the stress tensor (Eq. (26)) on the horizon which emerges from our analysis. This result is not a peculiar feature of the simple collapse model (null shell) we have used. The asymptotic relation Eq. (23) is completely general in two dimensions [13] and its validity can be extended to the physical space-time (i.e., four dimensions) because of the propagation of outgoing rays near the horizon according to geometric optics [11]. The late-time radiation is independent of the details of the collapse which affects the $O(U^4)$ term but not the $O(U^3)$.

The stress tensor of Eq. (26) is explicitly time dependent. Asymptotically (i.e., $r \to \infty$) there is no incoming radiation on $\mathcal{I}^-$ whereas on $\mathcal{I}^+$ there is an outgoing flux given by $\Delta(u, U)$ vanishing at late advanced time ($U \to 0, u \to +\infty$). However, unlike the $\kappa \neq 0$ case, one can not simply discard this radiation term to get the late-time behavior, since this procedure would lead to the incorrect result of Eq. (15). Also, the vacuum polarization part is vanishing for $U \to 0$ and a careful consideration of both terms is required, as we have shown to have regularity on the future horizon.

Finally, one can consider both the future and past extensions across the horizon, namely, introducing Kruskal like coordinates (U, V) [10],

$$u = -4M\left[\ln\left(-\frac{U}{M}\right) + \frac{M}{2U}\right], \quad v = 4M\left[\ln\left(\frac{V}{M}\right) - \frac{M}{2V}\right], \tag{28}$$

and define a state which is regular both on the future and past horizons

$$\begin{aligned}\langle H|T_{uu}|H\rangle &= H^{\mathrm{extr}}(r) + \Delta(u, U) = \langle U|T_{uu}|U\rangle, \\ \langle H|T_{vv}|H\rangle &= H^{\mathrm{extr}}(r) + \Delta(v, V) \\ &= \langle U|T_{vv}|U\rangle + (24\pi)^{-1}\frac{V^3(V + 2M)}{2M^2(2V + M)},\end{aligned} \tag{29}$$

$$\begin{aligned}\langle|T^a_a|\rangle &= (24\pi)^{-1}R = (6\pi)^{-1}f^{-1}(\ln f)_{,uv} \\ &= (6\pi)^{-1}\frac{M}{r^3}\left(1 - \frac{3}{2}\frac{M}{r}\right).\end{aligned} \tag{30}$$

This state is by no way unique. Any smooth extension of the coordinates (U, V) will lead to the same U^3 (V^3) behavior in the Schwarzian derivative on the horizon responsible for the divergence cancellation, the difference being of order U^4 (V^4).

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