Constraints and superspin for super Poincaré algebras in diverse dimensions

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We generalize to arbitrary dimension the construction of a covariant and supersymmetric constraint for the massless super Poincaré algebra, which was given for the 11-dimensional case in a previous work. We also contrast it with a similar construction appropriate to the massive case. Finally we show that the constraint uniquely fixes the representation of the algebra.

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I. CONSTRAINTS AND SUPERSPIN

We take as a starting point the super Poincaré algebra,[1]

$$[iJ_{\rho\sigma}, P_\mu] = \eta_{\mu\sigma}P_\rho - (\rho \leftrightarrow \sigma), \tag{1.1}$$

$$[iJ_{\rho\sigma}, J_{\mu\nu}] = [\eta_{\mu\sigma}J_{\rho\nu} - (\rho \leftrightarrow \sigma)] - (\mu \leftrightarrow \nu), \tag{1.2}$$

$$[P_\mu, P_\nu] = 0, \qquad [P_\mu, Q] = 0, \tag{1.3}$$

$$[iJ_{\rho\sigma}, Q] = -\tfrac{1}{2}\Gamma_{\rho\sigma}Q, \tag{1.4}$$

$$\{Q, \bar{Q}\} = -2i\not{P}. \tag{1.5}$$

Here, $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, \ldots, 1)$ and $\not{P} = \Gamma_\mu P^\mu$, with Γ_μ satisfying the Clifford algebra $\{\Gamma_\mu, \Gamma_\nu\} = 2\eta_{\mu\nu}$. Also, Q is a spinor of supercharge, $\bar{Q} = iQ^\dagger\Gamma^0$ and $\Gamma_{\mu\nu} = \frac{1}{2}\times[\Gamma_\mu, \Gamma_\nu]$. All spinor indices are suppressed; in particular, $Q^\dagger = (Q^*)^T$, where Q^* is the adjoint of Q and $(\cdot)^T$ indicates transposition with respect to the spinor indices. Notice also that if Q is a chiral spinor in D spacetime dimensions, then the right-hand side of (1.5) should contain a chiral projector $\frac{1\pm\Gamma}{2}$, and a convenient definition for Γ is $\Gamma = i^{D/2-1}\Gamma_0\Gamma_1\cdots\Gamma_{D-1}$. We define the supersymmetry variation of an operator $\mathcal{O}$ to be

$$\delta\mathcal{O} = \{Q, \mathcal{O}\} \quad \text{or} \quad [Q, \mathcal{O}], \tag{1.6}$$

depending on whether $\mathcal{O}$ is fermionic or bosonic.

Next, we construct two antisymmetric three-tensors, namely[2]

$$W_{\lambda\mu\nu} = P_{\langle\lambda}J_{\mu\nu\rangle} = \frac{1}{3!}\sum_{\mathrm{Perm}} \pm P_\lambda J_{\mu\nu}, \tag{1.7}$$

and

$$S_{\lambda\mu\nu} = \bar{Q}\Gamma_{\lambda\mu\nu}Q. \tag{1.8}$$

Here and in the following, the angular brackets between indices indicate a sum over all permutations of the indices, each taken with a sign and divided by the total number of permutations, as in (1.7). Furthermore, $\Gamma_{\lambda\mu\nu} = \Gamma_{\langle\lambda}\Gamma_\mu\Gamma_{\nu\rangle}$.

Using the algebra (1.1), (1.2), (1.3), (1.4), and (1.5), it is easy to compute the supersymmetry variation of W. One finds

$$\delta W_{\lambda\mu\nu} = -\frac{i}{2}P_{\langle\lambda}\Gamma_{\mu\nu\rangle}Q. \tag{1.9}$$

As for S,

$$\delta S_{\lambda\mu\nu} = \delta\bar{Q}\Gamma_{\lambda\mu\nu}Q - \bar{Q}\Gamma_{\lambda\mu\nu}\delta Q.$$

The first term in the expression above can be computed using (1.5). The second term is different from zero only if Q is a Majorana spinor. If it is a Majorana spinor, δQ can again be computed from (1.5) using the Majorana condition[3]. The details of the computation vary depending on the spacetime dimension and also, for even dimensions, on whether Q is a chiral or a Dirac spinor. But the final result is that, if the second term is nonzero, then it is exactly equal to the first term. Hence, the supersymmetry variation of S is twice as large when Q is a Majorana spinor. In particular, we find

$$\delta S_{\lambda\mu\nu} = \begin{cases} -2i\not{P}\Gamma_{\lambda\mu\nu}Q, & \text{if } Q \text{ is not a Majorana spinor,} \\ -4i\not{P}\Gamma_{\lambda\mu\nu}Q, & \text{if } Q \text{ is a Majorana spinor.} \end{cases} \tag{1.10}$$

Now, if we are interested in massless representations of the super Poincaré algebra, it is convenient to rewrite the variation of S as the sum of two terms, using the identity

$$\not{P}\Gamma_{\lambda\mu\nu} = 6P_{\langle\lambda}\Gamma_{\mu\nu\rangle} - \Gamma_{\lambda\mu\nu}\not{P}. \tag{1.11}$$

The first term has the same form as the variation of W (1.9), and can be used to cancel it if we take an appro-

[1] Notice that the conventions we use in this paper differ slightly from those employed in [1].

[2] In four dimensions, (1.7) is the dual of the Pauli-Lubański vector, so that W should be thought of as its generalization to higher dimensions.

[3] The Majorana condition takes the form $Q = BQ^*$ where B is a matrix chosen in such a way that $B^{-1}\Gamma_{\mu\nu}B = \Gamma^*_{\mu\nu}$. The Majorana condition can be imposed consistently in $D = 2, 3, 4, 8, 9 \bmod 8$ dimensions. In $D = 2 \bmod 8$, it can be imposed directly on a chiral spinor.

priate linear combination of W and S. The second term, instead, yields a variation proportional to $\not{P}Q$, and $\not{P}Q = 0$ in a massless representation. We denote the relative coefficient between W and S by κ and their linear combination by Δ,

$$\Delta_{\lambda\mu\nu} \equiv W_{\lambda\mu\nu} - \kappa S_{\lambda\mu\nu}. \tag{1.12}$$

It should be clear from the discussion above that the value of κ depends only on whether Q is or is not a Majorana spinor and, in particular,

$$\kappa = \begin{cases} \frac{1}{24} & \text{if } Q \text{ is not a Majorana spinor,} \\ \frac{1}{48} & \text{if } Q \text{ is a Majorana spinor.} \end{cases} \tag{1.13}$$

With the value of κ as above, we find

$$\delta\Delta_{\lambda\mu\nu} = -\frac{i}{12}\Gamma_{\lambda\mu\nu}\not{P}Q, \tag{1.14}$$

so that it is possible to impose the constraints

$$P^2 = 0, \qquad \not{P}Q = 0, \qquad \Delta_{\lambda\mu\nu} = 0 \tag{1.15}$$

consistently with the full super Poincaré algebra[4]. The constraints (1.15) were found in 11-dimensional spacetime in the course of the off-shell quantization of the superparticle [1], with the appropriate value $\kappa = \frac{1}{48}$ for the relative coefficient (Q is a Majorana spinor in 11 dimensions). We will show in the next section that the constraint Δ actually fixes completely the representation and that in 11 dimensions it fixes it to be the supergravity multiplet.

We make two remarks. First, in the case of extended supersymmetry one can construct a tensor analogous to $\Delta_{\lambda\mu\nu}$ in a straightforward way. Namely, if there are $\mathcal{N}$ supercharges Q_I, then

$$\Delta_{\lambda\mu\nu} \equiv W_{\lambda\mu\nu} - \sum_{I=1}^{\mathcal{N}} \kappa_I \bar{Q}_I \Gamma_{\lambda\mu\nu} Q_I, \tag{1.16}$$

where each of the κ_I is given by (1.13). Then,

$$\delta_I \Delta_{\lambda\mu\nu} \equiv \{Q_I, \Delta_{\lambda\mu\nu}\} = -\frac{i}{12}\Gamma_{\lambda\mu\nu}\not{P}Q_I, \tag{1.17}$$

and again Δ can be set to zero consistently. Second, in four spacetime dimensions the Δ tensor is only one of a continuous class of supercovariant objects that can be constructed. Indeed, if one defines

$$\Delta^{(\chi)}_{\lambda\mu\nu} \equiv \Delta_{\lambda\mu\nu} - \tfrac{1}{3}\chi P^\alpha \epsilon_{\alpha\lambda\mu\nu}, \tag{1.18}$$

where χ is an arbitrary real number and ϵ is the completely antisymmetric tensor with $\epsilon_{0123} = +1$, the supersymmetry variation of $\Delta^{(\chi)}$ is the same as that of Δ. Hence, $\Delta^{(\chi)} = 0$ is also a good constraint, compatible with the full super Poincaré algebra. We will elaborate on this point in the next section.

It is instructive to compare the Δ tensor to a similar construction which is useful for massive representations. We could rewrite the variation (1.10) of S using the identity

$$\not{P}\Gamma_{\lambda\mu\nu} = 3P_{\langle\lambda}\Gamma_{\mu\nu\rangle} + \tfrac{1}{2}[\not{P}, \Gamma_{\lambda\mu\nu}], \tag{1.19}$$

instead of (1.11). Again the first term can be used to cancel the variation of W if a suitable relative coefficient between W and S is chosen. Let us emphasize that the coefficient needed differs from κ by a factor of 2. We call the linear combination $C_{\lambda\mu\nu}$ and the relative coefficient ρ. Hence,

$$C_{\lambda\mu\nu} \equiv W_{\lambda\mu\nu} - \rho S_{\lambda\mu\nu}, \tag{1.20}$$

with

$$\rho = \begin{cases} \frac{1}{12} & \text{if } Q \text{ is not a Majorana spinor,} \\ \frac{1}{24} & \text{if } Q \text{ is a Majorana spinor.} \end{cases} \tag{1.21}$$

Then,

$$\delta C_{\lambda\mu\nu} = -\frac{i}{12}[\Gamma_{\lambda\mu\nu}, \not{P}]Q, \tag{1.22}$$

and because of the identity $[P^\lambda \Gamma_{\lambda\mu\nu}, \not{P}] = 0$, the antisymmetric tensor

$$C_{\mu\nu} \equiv P^\lambda C_{\lambda\mu\nu} \tag{1.23}$$

is invariant under supersymmetry transformations. Then, the scalar

$$C \equiv C_{\mu\nu}C^{\mu\nu} \tag{1.24}$$

is a Casimir of the full super Poincaré algebra and can be used to label its massive representations. For massless representations, on the other hand, it is possible to show that C vanishes identically[5]. In that case Δ is a more useful quantity to consider. Notice that C generalizes to arbitrary dimension a four-dimensional Casimir constructed in [2–5], where the eigenvalues of that Casimir were termed "superspin."

II. NATURE OF THE CONSTRAINT Δ

To investigate how $\Delta = 0$ constrains massless representations of the super Poincaré algebra, we choose a frame in which $P = (E, E, 0, \ldots, 0)$ (light-cone frame). In D spacetime dimensions, this choice breaks SO($D-1, 1$) down to the "little group," ISO($D-2$), namely, the group of rotations and translations in $D-2$ dimensions. For convenience, we introduce Latin indices of two types, $a, b, c = 0, 1$ and $i, j, k = 2, \ldots, D-1$, so that we can

[4]Actually, to impose consistently (1.15), one also needs $\delta\not{P}Q \propto P^2$, which is true.

[5]More precisely, one can show that in the frame (2.1), $C \propto A_i A^i$ where A_i is given in (2.3). As explained below, A_i must vanish in a physically sensible representation.

express the choice of frame with

$$P^a = E, \qquad P^i = 0. \tag{2.1}$$

Then the components of W are as follows:

$$W_{abc} = W_{ijk} = 0, \tag{2.2}$$

$$W_{abi} = \epsilon_{ab}\frac{E}{3}(J_{i0} - J_{i1}) \equiv \epsilon_{ab}\frac{E}{3}A_i, \tag{2.3}$$

$$W_{aij} = \pm\frac{E}{3}J_{ij}, \tag{2.4}$$

where ϵ_{ab} is the antisymmetric tensor in two dimensions with $\epsilon_{01} = +1$, and the upper sign in (2.4) holds when $a = 1$, the lower when $a = 0$; similarly in (2.9) and (2.12) below. Note that $A_i = (J_{i0} - J_{i1})$ are precisely the generators of the translations of ISO$(D-2)$.

Before evaluating the components of S, we need to discuss how the frame choice (2.1) affects the supercharges Q, which, in a massless representation, are subject to the constraint $\not{P}Q = 0$. The answer is that some components are projected out. Indeed

$$\pi_+ Q = Q, \qquad \pi_- Q = 0, \tag{2.5}$$

where π_+ and π_- are complementary projectors given by

$$\pi_\pm = \frac{1 \pm \Gamma^1\Gamma^0}{2}. \tag{2.6}$$

Using (2.5) and performing some algebra[6], we see that the components of S are

$$S_{abc} = S_{ijk} = 0, \tag{2.7}$$

$$S_{abi} = 0, \tag{2.8}$$

$$S_{aij} = \mp i Q^\dagger \Gamma_{ij} Q. \tag{2.9}$$

Therefore, the components of Δ are

$$\Delta_{abc} = \Delta_{ijk} = 0, \tag{2.10}$$

$$\Delta_{abi} = \epsilon_{ab}\frac{E}{3}A_i, \tag{2.11}$$

$$\Delta_{aij} = \pm\frac{E}{3}\left(J_{ij} + 3i\kappa\frac{Q^\dagger\Gamma_{ij}Q}{E}\right), \tag{2.12}$$

with κ given by (1.13). We see that setting $\Delta = 0$ is equivalent to imposing the pair of conditions

$$A_i = 0, \tag{2.13}$$

$$J_{ij} = -3i\kappa\frac{Q^\dagger\Gamma_{ij}Q}{E}. \tag{2.14}$$

The first condition requires that the translations of the little group be represented trivially. This is desirable on physical grounds since a nontrivial representation would lead to unwanted continuous degrees of freedom, by a standard field theory argument. The second condition, on the other hand, restricts the eigenvalues of J_{ij} to be those of the quadratic operator to the right of (2.14). Those eigenvalues can be computed explicitly in any dimension. They are of course independent of the values of i and j, because the frame choice (2.1) does not break the rotational invariance in the i and j indices. The eigenvalues are quantized as a result of the supersymmetry algebra (1.5). In this frame, the algebra can also be written as[7]

$$\{Q, Q^\dagger\} = 4E\pi_+, \tag{2.15}$$

from which it follows that the nonzero components of Q are proportional to fermionic oscillators. How many oscillators exactly will depend on the spacetime dimension and on what kind of spinor Q is (for instance, a chiral or a Majorana condition will each reduce by half the number of independent oscillators). In the end, the right-hand side of (2.14) can be written as a simple function of several fermionic number operators. Hence, the eigenvalues of J and their multiplicities can be easily computed and from that a representation can be inferred uniquely.

Shortly, we will give the form of that function of number operators for some interesting dimensions, including the 11-dimensional case of [1]. But first two remarks are in order. When extended supersymmetry is present (but with no central charges), all that we have done can be repeated with only minor changes. The principal difference is that (2.14) is replaced by

$$J_{ij} = -3i\sum_{I=1}^{\mathcal{N}}\kappa_I\frac{Q_I^\dagger\Gamma_{ij}Q_I}{E}, \tag{2.16}$$

which in turn can be written as a function of $\mathcal{N}$ sets of number operators. The second remark concerns the case of four spacetime dimensions, where a continuous class of constraints $\Delta^{(\chi)}$ exists, as mentioned in the previous section. In four dimensions the little group is ISO(2) and it consists of the helicity and of two translations. With our choice of frame (2.1), the generators are, respectively, J_{23}, A_2, and A_3. Now, setting $\Delta^{(\chi)} = 0$ adds a shift to the eigenvalues of J_{23}, namely, to the helicities of the representation, while, interestingly, the constraints $A_2 = 0$ and $A_3 = 0$ are unaffected,

$$A_2 = A_3 = 0, \qquad J_{23} = -3i\kappa\frac{Q^\dagger\Gamma_{23}Q}{E} + \chi. \tag{2.17}$$

[6]As an example of the kind of derivations involved in computing (2.7), (2.8), and (2.9), we give a proof of (2.8).

$$S_{abi} = \bar{Q}\Gamma_{abi}Q = \overline{(\pi_+Q)}\Gamma_{abi}\pi_+Q = \bar{Q}\pi_-\Gamma_{abi}\pi_+Q$$
$$= \bar{Q}\Gamma_{abi}\pi_-\pi_+Q = 0.$$

[7]Again if the spinor Q is chiral, one must add a chiral projector to the right-hand side of Eq. (2.15).

In summary, all possible representations with $A_2 = A_3 = 0$ are recovered as χ varies. It should not come as a surprise that χ appears to be a continuous variable, because our construction is purely algebraic, whereas the quantization of the helicities in four dimensions is a consequence of the topology of the little group[8].

As promised, we now give an expression for J_{ij} in several interesting dimensions, being understood that A_i is always zero.

Two and three spacetime dimensions are not interesting for our purposes since the little groups are either trivial or consist only of a translation. In four dimensions, when Q is taken to be a chiral spinor or equivalently a Majorana spinor, there is only one independent fermionic oscillator a and the helicity operator is given by

$$J_{23}[4D] = \begin{cases} \chi \mp \frac{1}{2}a^*a, & \text{if } Q \text{ is left or right chiral respectively,} \\ \chi + \frac{1}{4} - \frac{1}{2}a^*a, & \text{if } Q \text{ is Majorana.} \end{cases} \tag{2.18}$$

As claimed, (2.18) determines completely the representation of the super Poincaré algebra in terms of χ. It is given by the massless $\mathcal{N} = 1$ supermultiplet in which the helicities are $(\chi \mp \frac{1}{2}, \chi)$ for a left- or a right-handed chiral spinor, and $(\chi + \frac{1}{4}, \chi - \frac{1}{4})$ for a Majorana spinor. Now, in four dimensions it is a matter of convention whether $\mathcal{N} = 1$ supersymmetry is implemented with a left chiral, a right chiral, or a Majorana supercharge[9]. Correspondingly, the way χ enters in the expression for the helicities depends on that conventional choice, but what matters is that in all cases each possible supermultiplet is recovered for an appropriate value of χ. For that value, $\Delta^{(\chi)} = 0$ determines the representation in a fully supercovariant way. It is intriguing that a supercovariant constraint for which $A_i \neq 0$ does not seem to exist. The generalization to the case of extended supersymmetry is straightforward and again one recovers all possible representations for appropriate values of χ.

In five dimensions, for $\mathcal{N} = 1$, there are two independent fermionic oscillators $a_{1,2}$ and

$$J_{ij}[5D] = \tfrac{1}{2}(a_2^*a_2 - a_1^*a_1), \tag{2.19}$$

which yields the supermultiplet with eigenvalues $(\frac{1}{2}, 0, 0, -\frac{1}{2})$, namely, one spin-$\frac{1}{2}$ and two spin zero particles, all three complex (indeed it is not possible to impose a Majorana condition on a spinor in five dimensions). Under dimensional reduction, one obtains the $\mathcal{N} = 2$ massless hypermultiplet in four dimensions. It would be interesting to find supercovariant constraints that characterize the other representations of the super Poincaré algebra in five or higher dimensions, as we were able to do in four dimensions.

Next, we consider ten dimensions. If we take Q to be a Majorana-Weyl spinor, we find that there are four independent fermionic oscillators, $a_{1,\ldots,4}$ and that

$$J_{ij}[10D] = \frac{1}{2}\sum_{n=1}^{4} a_n^*a_n - 1. \tag{2.20}$$

Then, the eigenvalues for J_{ij} are $(1, \frac{1}{2}, 0, -\frac{1}{2}, -1)$ with multiplicities $(1, 4, 6, 4, 1)$ and the multiplet must be the gauge supermultiplet, which consists of an SO(8) vector and an SO(8) chiral spinor for a total of eight fermionic and eight bosonic degrees of freedom. The generalization to extended supersymmetry is, once again, straightforward. In particular for $\mathcal{N} = 2$, we find the supergravity multiplets of type IIA or type IIB depending on the chiralities of the two supercharges, as was to be expected.

Finally, in 11 dimensions with Q Majorana, there are eight oscillators and

$$J_{ij}[11D] = \frac{1}{2}\sum_{n=1}^{8} a_n^*a_n - 2. \tag{2.21}$$

The eigenvalues of J_{ij} are $(\pm 2, \pm\frac{3}{2}, \pm 1, \pm\frac{1}{2}, 0)$ with multiplicities $(1, 8, 28, 56, 70)$ for a total of 128 fermionic and 128 bosonic states, pointing inequivocably to the 11-dimensional supergravity multiplet.

III. THE MASSIVE CASE

For completeness, we present in this section a discussion of the tensor $C_{\mu\nu}$ of (1.23). We proceed along the lines of the discussion of Δ. What follows is a generalization to generic spacetime dimensions of similar arguments that can be found in [2–5] for the four-dimensional case.

[8]Namely, it follows from the fact that a 4π rotation around the direction of the momentum can be continuously deformed into the identity.

[9]Indeed, in four dimensions the complex conjugate of a left chiral spinor is linearly related to a right chiral spinor and vice versa. Similarly a Majorana spinor can be constructed in terms of a chiral spinor and vice versa.

To begin, we choose a frame in which $P = (m, 0, \ldots, 0)$ (rest frame). The little group is $SO(D-1)$ and it is generated by J_{ij} where $i, j = 1, \ldots, D-1$. In the rest frame, $\not{P} = m\Gamma_0$ and the supersymmetry algebra (1.5) becomes[10] $\{Q, Q^\dagger\} = 2m$. If we rescale the supercharge Q, by defining $a \equiv \frac{Q}{\sqrt{2m}}$, then a satisfies

$$\{a, a^\dagger\} = 1, \tag{3.1}$$

$$[iJ_{ij}, a] = -\tfrac{1}{2}\Gamma_{ij}a, \tag{3.2}$$

$$[iJ_{ij}, a^\dagger] = +\tfrac{1}{2}a^\dagger\Gamma_{ij}, \tag{3.3}$$

where the latter two equations follow from (1.4). In the rest frame the components of $C_{\mu\nu}$ are

$$C_{0\mu} = 0, \qquad C_{ij} = -\frac{m^2}{3}[J_{ij} + 6i\rho a^\dagger\Gamma_{ij}a], \tag{3.4}$$

with ρ given by (1.21).

We now define the tensors

$$T_{ij} = -6i\rho a^\dagger\Gamma_{ij}a, \tag{3.5}$$

and

$$Y_{ij} = -\frac{3C_{ij}}{m^2}. \tag{3.6}$$

The point of those definitions is that, as we will show presently, T_{ij} and Y_{ij} are angular momentum operators, in the sense that they satisfy each the commutation relations of the generators of the little group $SO(D-1)$, exactly as J_{ij} does. Furthermore, T and Y commute with one another. Equation (3.4) becomes

$$Y_{ij} = J_{ij} - T_{ij}, \qquad \text{or equivalently } J_{ij} = Y_{ij} + T_{ij}, \tag{3.7}$$

and therefore we can conclude that J is the composition of two independent angular momentum operators, T and Y.

To clarify the last statement, we should perhaps emphasize the following. For the massive case, giving a representation of Y_{ij} amounts to fixing a particular representation of the super Poincaré algebra. The reason is that the massive representations of the super Poincaré algebra are labeled by $C = C_{\mu\nu}C^{\mu\nu}$, and, in the rest frame, C is proportional to the quadratic Casimir associated with Y_{ij}, $C \propto Y_{ij}Y^{ij}$. Hence, in reading Eq. (3.7), we should keep in mind that the representation of Y_{ij} is given. Furthermore, we will argue below that the representation of T_{ij} is fixed by its expression in terms of a, and that it depends on the spacetime dimension and on what kind of spinor Q is. At any rate, it will be a reducible representation of $SO(D-1)$. Then Eq. (3.7) states precisely that J_{ij} belongs to the tensor product of the given representation of Y_{ij} and the fixed representation of T_{ij}. The irreducible representations of $SO(D-1)$ contained in that tensor product can be computed, and together they form the supermultiplet associated with the given representation of the super Poincaré algebra. We will illustrate this with an explicit example.

To find the commutation relations of Y and T, one can proceed as follows. The commutation relations of T with J can be computed using (3.2) and (3.3). One finds that T transforms as a tensor under J, namely

$$[iJ_{ij}, T_{kl}] = [\eta_{kj}T_{il} - (i \leftrightarrow j)] - (k \leftrightarrow l). \tag{3.8}$$

The same is true for $Y = J - T$, since J also transforms as a tensor, by (1.2),

$$[iJ_{ij}, Y_{kl}] = [\eta_{kj}Y_{il} - (i \leftrightarrow j)] - (k \leftrightarrow l). \tag{3.9}$$

Next, the commutator of T with itself can be computed in terms of anticommutators of a and a^*. More precisely there will be terms containing the anticommutator of a with a^* and terms containing the anticommutator of a with a or of a^* with a^*. The first kind of terms, those involving anticommutators of a with a^*, can be readily computed using (3.1). The second kind of terms will vanish unless Q, and therefore a, is a Majorana spinor. If Q and a are Majorana spinors, the anticommutators of a with a and a^* with a^* can be computed using (3.1) together with the Majorana condition (see the footnote in the first section) and the result can be used to evaluate the terms of the second kind. The details of the computation depend on the spacetime dimensions and on whether Q is chiral or not, but the end result is, once again, that the second kind of terms gives a contribution exactly equal to that of the first kind of terms. Therefore, we find

$$[iT_{ij}, T_{kl}] = \begin{cases} (12\rho)[\eta_{kj}T_{il} - (i \leftrightarrow j)] - (k \leftrightarrow l) & \text{if } Q \text{ is not a Majorana spinor,} \\ (24\rho)[\eta_{kj}T_{il} - (i \leftrightarrow j)] - (k \leftrightarrow l) & \text{if } Q \text{ is a Majorana spinor,} \end{cases} \tag{3.10}$$

[10]As usual, a chiral projector on the right-hand side is required if Q is chiral. The same for (3.1).

and since ρ is given by (1.21) we find, as promised,

$$[iT_{ij}, T_{kl}] = [\eta_{kj}T_{il} - (i \leftrightarrow j)] - (k \leftrightarrow l). \quad (3.11)$$

Finally, it is easy to see that Y and T commute, by combining (3.8) and (3.11). Then the commutators of Y with itself follows from (3.9) and the fact that Y and T commute,

$$[Y_{ij}, T_{kl}] = 0,$$
$$[iY_{ij}, Y_{kl}] = [\eta_{kj}Y_{il} - (i \leftrightarrow j)] - (k \leftrightarrow l). \quad (3.12)$$

This proves the claim that Y_{ij} and T_{ij} are two commuting angular momentum operators.

To complete the discussion, we need only to present an argument that the representation of T_{ij} is fixed. The reasoning follows closely that given in the preceding section for J_{ij} in the massless case. Indeed, T is quadratic in a and a^* and the components of a, a^* are fermionic oscillators as a result of (3.1). Therefore the eigenvalues of T_{ij} can be obtained, together with their multiplicities, by rewriting T_{ij} as a function of fermionic number operators. That in turn fixes the representation uniquely. To illustrate this point, we consider one example, namely $D = 4$ and Q a chiral spinor. The analysis is analogous to the five-dimensional massless case: The little group is SO(3) and there are two independent fermionic oscillators a_1, a_2. One finds

$$T_{12} = \frac{1}{2}(a_2^* a_2 - a_1^* a_1), \quad (3.13)$$

with similar expressions for the other components of T. This fixes the eigenvalues of T_{ij} to be $(\frac{1}{2}, 0, 0, -\frac{1}{2})$. Hence, T is determined to be a generator of the representation $\mathbf{0} \oplus \mathbf{0} \oplus \mathbf{\frac{1}{2}}$. Then, if Y_{ij} is chosen to be in a spin-Y representation of SO(3), J will be in $(\mathbf{0} \oplus \mathbf{0} \oplus \mathbf{\frac{1}{2}}) \otimes \mathbf{Y} = (\mathbf{Y} - \mathbf{\frac{1}{2}}) \oplus \mathbf{Y} \oplus \mathbf{Y} \oplus (\mathbf{Y} + \mathbf{\frac{1}{2}})$, namely, in the $\mathcal{N} = 1$ massive supermultiplet where the highest spin is $Y + \frac{1}{2}$.

IV. CONCLUSION

We introduced a covariant tensor Δ which, in the case of a massless representation of the super Poincaré algebra, is also supersymmetric. Imposing $\Delta = 0$ is a supercovariant way to fix the representation completely, including the generators of the translations in the little group. In particular, the translations are represented trivially, as required on physical grounds.

For the case of nonvanishing mass, we have constructed an angular momentum operator Y_{ij} and a Casimir C. The latter generalizes to higher dimensions the superspin operator. We have also shown how Y_{ij} can be used to construct representations of the super Poincaré algebra.

The present investigation originated in work [1] intended to define a superstar product appropriate for a consistent formulation of supersymmetric string field theory. The results described here are interesting by themselves. They show that (super)Lie algebras can admit constraints amenable to exact treatment and that statements about their representations can be worked out.

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