THE

PHYSICAL REVIEW

A journal of experimental and theoretical physics established by E. L. Nichols in 1893

SECOND SERIES, VOL. 129, NO. 1 — 1 JANUARY 1963

Energy of a Vortex Ring in a Tube and Critical Velocities in Liquid Helium II

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(Received 30 July 1962)

Various authors have suggested that critical velocities v_c in liquid helium II may result from the formation of vortex rings according to Landau's criterion, $v_c=(E/p)_{\min}$, where E is the energy of the ring and p its impulse. In considering the possible formation of rings *inside* the channel from this point of view, however, the effect of the walls on E and p has been neglected. By solving Laplace's equation in series, we have evaluated the energy of a circular classical vortex ring with an empty streamlined core confined coaxially in a long circular tube of radius R; numerical results are presented for various core radii a and ring radii r_0. E has a maximum at $r_0\approx 0.9R$, and approaches zero as $r_0\rightarrow R$. Boundaries do not affect the impulse, so Landau's criterion applied to such a classical vortex ring gives $v_c=0$, contradicting experiment. We may conclude that for some reason vortex rings must not be formed inside the channel, unless some special mechanism prevents their formation (or their causing friction if formed) too near the walls. Numerical results are also presented for the exact solution in an unbounded fluid.

I. INTRODUCTION

THE superfluid component of liquid helium II flows without friction only at velocities smaller than a certain critical velocity v_c. This velocity has been extensively investigated in films,[1] slits,[1] and capillaries[1–6]; it changes slowly with temperature, and is about inversely proportional to the channel width in channels larger than 10^{-3} cm, but rises more slowly as the channel is made very small. Some typical values are plotted in Fig. 1.

Many attempts to account for superfluidity and its breakdown at v_c have followed the suggestion of Landau[7] that the superfluid is the ground state of the liquid and that v_c is the velocity needed to make an excitation. This velocity is found by a classical argument. If the excitation has momentum (or impulse[8]) $\mathbf{p}$ and energy E in a fluid at rest, then its creation in a fluid moving with a uniform velocity $\mathbf{v}$ would add to the energy of that fluid an amount $E+\mathbf{p}\cdot\mathbf{v}$. Since this process cannot change the total energy of the fluid, it can occur only if $E=-\mathbf{p}\cdot\mathbf{v}$. Thus, the only excitations that can be formed are those for which $E\leqslant pv$, so that v_c is the minimum value of E/p for all possible excitations.

If the excitations formed were phonons or rotons (thermal excitations), then their energy would appear directly as dissipation of the kinetic energy of flow. But Landau's argument applied to phonons and rotons gave values of v_c much higher than those actually observed. An explanation of v_c in these terms therefore requires excitations of some other kind, with smaller $(E/p)_{\min}$. Feynman[9] has suggested that for flow in channels these take the form of quantized vortex rings

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† Operated with support from the U. S. Army, Navy and Air Force.

[1] K. R. Atkins, *Liquid Helium* (Cambridge University Press, New York, 1959), p. 199. (A table of earlier measurements by various workers.)

[2] C. E. Chase, Phys. Rev. **127**, 361 (1962).

[3] D. F. Brewer and D. O. Edwards, Phil. Mag. **6**, 775 (1961).

[4] F. A. Staas and K. W. Taconis, Physica **27**, 924 (1961).

[5] J. N. Kidder and W. M. Fairbank, Phys. Rev. **127**, 987 (1962).

[6] G. Careri, F. Scaramuzzi, and J. O. Thomson, *Proceedings of the Seventh International Conference on Low-Temperature Physics* (University of Toronto Press, Toronto, 1961), p. 502.

[7] L. Landau, J. Phys. (U.S.S.R.) **5**, 71 (1941).

[8] Hereafter, we shall always refer to the impulse (the time integral of the forces on the liquid required to set up the motion from rest) rather than the momentum (the volume integral of density times velocity). The two quantities are equal if the latter is well defined, but often it is not, because the fluid is an infinite sink for momentum. If the fluid is infinite and incompressible, so that momentum is instantly transmitted to infinity, then the integral for the momentum will not even be absolutely convergent. (See reference 13). When, therefore, for mathematical convenience we consider this limiting case, it is the impulse we must work with.

[9] R. P. Feynman, in *Progress in Low-Temperature Physics*, edited by C. J. Gorter (Interscience Publishers, Inc., New York, 1955), Vol. 1, p. 19; see especially pp. 45–51.

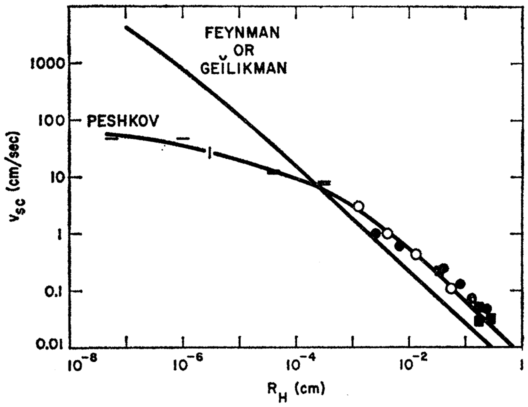

FIG. 1. Critical velocity v_c at 1.4°K in various channels plotted against hydraulic radius $R_H \equiv 2\times$(cross-sectional area)/(perimeter). This is the parameter that makes Geĭlikman's formula for narrow slits (see reference 10) most like that for circular tubes [Eq. (3)]. —, unsaturated films (reference 1); |, saturated films (reference 1); =, slits (reference 1); ○, circular tubes—mass flow (references 1, 4, and 5); ●, circular tubes—heat flow (references 2 and 3); ■, rectangular channels—heat flow (references 1, 2, and 6). Solid lines are theoretical curves; see text.

in the otherwise irrotational liquid, with strength equal to Planck's constant divided by the mass of the helium atom. The velocity field of one of these rings is supposed to be like that of a classical vortex ring except within and very near a core of about atomic radius. Energy might be removed from the steady flow of the liquid by the formation of these rings and their grow thin interaction with the walls, and then converted into heat by the interaction of the rings with the normal fluid or perhaps by their breakup into rotons.

Feynman originally devoted most of his attention to the idea that the rings responsible for resistance to flow in a channel are formed at the orifice, and in view of the complexity of the problem he undertook only qualitative comparison with experiment. But Geĭlikman[10] and Peshkov[11] have proposed more detailed explanations of the experimental dependence of v_c on channel size, based on the assumption that the rings are formed inside the channel. They used, as approximations, expressions due to Lamb[12] for the energy E_0 and impulse p of a circular classical vortex ring of strength κ in an unbounded perfect fluid with constant density ρ. In Lamb's model the vorticity is distributed uniformly through a core whose cross section is circular with radius a much smaller than the mean radius r_0 of the ring. The results are

$$E_0 = \tfrac{1}{2}\rho\kappa^2 r_0[\ln(8r_0/a) - 7/4], \tag{1}$$

$$p = \pi\rho\kappa r_0^2. \tag{2}$$

[10] B. T. Geĭlikman, J. Exptl. Theoret Phys. (U.S.S.R.) **37**, 891 (1959) [translation: Soviet Phys.—JETP **10**, 635 (1960)].

[11] V. P. Peshkov, *Proceedings of the Seventh International Conference on Low-Temperature Physics* (University of Toronto Press, Toronto, 1960), p. 555; J. Exptl. Theoret Phys. (U.S.S.R.) **40**, 379 (1961) [translation: Soviet Phys.—JETP **13**, 259 (1961)].

[12] H. Lamb, *Hydrodynamics* (Dover Publications, New York, 1945), 6th ed., Chap. 7, p. 202 (see especially Secs. 161–163).

From Eqs. (1) and (2) it is clear that E_0/p decreases monotonically as r_0 increases. Both Geĭlikman and Peshkov assume that the presence of the channel walls does not greatly affect E and p, so that the minimum value of E/p will be for rings that are as large as possible—almost as large as the channel. Therefore, according to Geĭlikman, v_c should equal the value of E_0/p for $r_0=R$, the channel radius:

$$v_c = \frac{\kappa}{2\pi R}\left(\ln\frac{8R}{a} - \frac{7}{4}\right). \tag{3}$$

This is very similar to Feynman's rough estimate for a slit, based on the idea that the liquid has to have enough kinetic energy to form vortex lines at the orifice. Since what goes on near the core is not, in fact, classically describable, a is rather an undetermined parameter of the theory than a physically measurable length; the result for v_c is fairly insensitive to the choice of a as long as $a \ll r_0$.

It will be seen in Fig. 1 that if $a = 10^{-8}$ cm then Eq. (3) gives values that are in fair agreement with experiment for $R \gtrsim 10^{-3}$ cm, but rather high for the smallest channels and for films. To remedy this, Peshkov proposed a more elaborate model involving a relaxation time τ during which a fraction α of the kinetic energy in a stretch of liquid goes into making a vortex ring. He got

$$v_c^2 R(R + v_c\tau) = (\kappa^2/\pi\alpha)\ln(R/a), \tag{4}$$

in which a, τ, and α can be adjusted in various ways to give a good fit to the data. Peshkov's curve in Fig. 1 is the one he gave for $a = 10^{-8}$ cm, $\tau = 4\times10^{-4}$ sec, and $\alpha = 0.122$.

The above arguments depend, however, on the assumption that the energy and impulse of a vortex ring of given strength, ring radius, and core radius are not much affected by the presence of walls; otherwise, Lamb's formulas may not apply, since they were derived for an unbounded fluid. And indeed, there is reason to suspect that they do not. Most of the kinetic energy in the velocity field of a vortex ring is in the fluid fairly close to the core, so we should expect that the energy of a ring whose distance from the walls is small compared to its radius will be asymptotically equal to that of the same length of straight vortex line at the same distance from a plane wall. The energy of this configuration goes to zero as the core approaches the wall. Therefore, the assumption that Lamb's formulas are applicable seems most unlikely for rings near the wall, that is to say for just those rings that, according to the above theories, are responsible for the critical velocity. Hence, it is worthwhile to look more carefully at the actual behavior of the energy and impulse when the ring is in a channel.

Now, the impulse does not, in fact, depend on the size or shape of the channel so long as it is simply connected.

This is shown by Lin,[13] who notes that Lamb's derivation does not use the assumption that the fluid is unbounded. Only for the energy, therefore, is a special computation needed. This will be carried out in the following section.

II. ENERGY COMPUTATION

A. Assumptions

For simplicity in computation, we shall consider a circular vortex ring of strength κ in an incompressible inviscous fluid, confined coaxially in a very long tube of radius R (Fig. 2). The velocity field outside the core is assumed to be that of a classical ring with an infinitesimally thin core of radius r_0, which we shall call the *source circle*. The core of the actual ring is taken to be empty, and to have a cross section bounded by a streamline. The core radius a is then defined as the distance from the source circle to the streamline bounding the core, measured radially toward the tube axis. If, as will usually be true, the core is very small compared to its distances from the walls and the axis of the tube, then it will be very nearly circular with radius a as in the case considered by Lamb.

It should be noted that the above model was chosen for mathematical convenience only, and is not meant as a physical proposal for the core of a real vortex ring in liquid helium. Rather, one hopes that the detailed assumptions about the core, which are physically meaningless, will have little influence on the result of the calculation. To see that this is so, and in order to be able to check the results against those of Lamb and of common sense, we shall consider briefly some effects of these assumptions.

Making the core empty eliminates the kinetic energy of the fluid inside it. If $a \ll r_0$ and the vorticity is uniformly distributed through the core, as Lamb supposes, then the velocity field inside the core is like that inside the core of a straight vortex line. This velocity field can easily be shown to be $\kappa x/2\pi a^2$ in magnitude, where x is the distance from the axis of the core. Hence, the kinetic energy per unit length is

$$\int_0^a (\rho\kappa^2 x^2/8\pi^2 a^4)2\pi x dx = \rho\kappa^2/16\pi$$

and the total kinetic energy in the core is $\frac{1}{8}\rho\kappa^2 r_0$. This is about 1.4% of Eq. (1) if $r_0/a = 10^7$.

Suppose instead, following Feynman's[9] heuristic argument, we include the surface energy of the empty core, and assume that the core size is such as to minimize the total of surface and kinetic energy. Then if σ is the surface tension, and $a \ll r_0$ as before, it turns out that $a = \rho\kappa^2/8\pi^2\sigma$ and the surface energy is $\frac{1}{2}\rho\kappa^2 r_0$, or four times the kinetic energy of a full core.

The effect of allowing the core to depart from circularity is small unless either r_0 or $R-r_0$ is of the order of a, in which case a classical ring is no longer a good approximation anyway. This is discussed in the appendix.

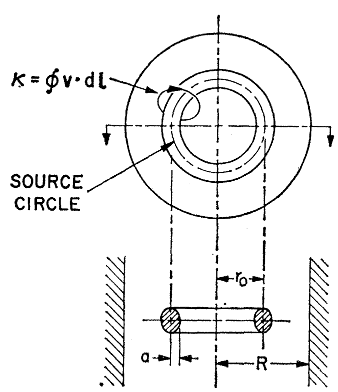

FIG. 2. Configuration considered.

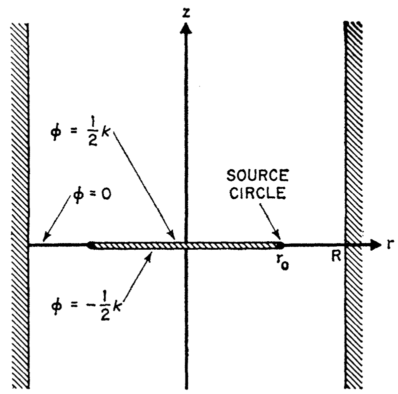

FIG. 3. Coordinates and boundary conditions for solution of Laplace's equation.

B. Computation

For the numerical solution of the problem, we use cylindrical coordinates r, θ, z, with the source circle in the $z=0$ plane (Fig. 3). Since the fluid is incompressible ($\nabla\cdot\mathbf{v}=0$) and the motion is curl free ($\nabla\times\mathbf{v}=0$) except for the singularity at the source circle, the velocity field $\mathbf{v}$ is the gradient of a potential ϕ that is a solution of Laplace's equation $\nabla^2\phi=0$ in any simply connected region away from the source circle. We take for this region the whole interior of the tube except for a barrier consisting of the disk bounded by the source circle. On crossing this barrier, ϕ changes discontinuously by an amount equal to the line integral of $\mathbf{v}$ along a path that goes from just below the barrier to just above it without crossing it; that is, by just the circulation κ through the ring. By symmetry, the flow across the barrier is normal to it, so that ϕ is constant over the barrier. Therefore, choosing the arbitrary constant in ϕ agreeably to the symmetry of the problem, we must have for the boundary condition at $z=0$:

$$\begin{aligned}\phi &= \pm\tfrac{1}{2}\kappa, \quad 0 \leqslant r < r_0 \\ &= 0, \quad\quad r_0 < r \leqslant R\end{aligned}, \quad z \to \pm 0. \tag{5}$$

[13] C. C. Lin, lecture notes, Enrico Fermi International School of Physics, Varenna, Italy, 1961, (unpublished), Chap. 1 (see especially pp. 10–17).

The velocity at the wall must be tangential, so the boundary condition at $r=R$ is

$$\partial\phi/\partial r=0. \tag{6}$$

Since the problem has cylindrical symmetry, the solution has the form

$$\phi=\sum_{n=0}^{\infty} A_n J_0(k_n r)e^{-k_n|z|}, \tag{7}$$

where the A_n's and k_n's are constants. Equation (6) then gives

$$k_n=x_n/R, \tag{8}$$

where x_n is the nth root of $J_1(x)=0$. Expanding Eq. (5) in the orthogonal functions $J_0(x_n r/R)$ and equating term by term to Eq. (7) with $z=0$, we obtain

$$A_n=\frac{\kappa r_0 J_1(x_n r_0/R)}{x_n R[J_0(x_n)]^2}. \tag{9}$$

The kinetic energy E equals the integral of $\frac{1}{2}\rho(\nabla\phi)^2$ over the volume of the tube outside the core. To evaluate it we integrate by parts, converting the volume integral to a surface integral by applying the divergence theorem to the vector field $\phi\nabla\phi$. The result[14] is

$$E=\tfrac{1}{2}\rho\int\int\phi\nabla\phi\cdot d\mathbf{S}, \tag{10}$$

where the integral is over the boundaries of the simply connested region in which ϕ is defined and continuous. Since the velocity is tangential at the walls and at the boundaries of the core, the only contribution to Eq. (10) is from the flux through the barrier, and we have for the energy

$$E=-\frac{1}{2}\kappa\rho\int_0^{r_0-a}(\partial\phi/\partial z)_{z=0}2\pi r dr$$

$$=\pi\rho\kappa^2\frac{r_0^2}{R}\sum_{n=0}^{\infty}\frac{J_1(x_n r_0/R)J_1[x_n(r_0-a)/R]}{x_n[J_0(x_n)]^2}. \tag{11}$$

The convergence of this series is extremely slow for interesting values of the parameters, the number of terms required being of order greater than R/a. Therefore, we sum to a practical value of n, say N, and integrate the summand as a continuous function of n from $N+\frac{1}{2}$ to ∞, using trigonometric approximations for the Bessel functions of large argument. This gives for the summation in Eq. (11)

$$\sum_{n=0}^{\infty}=-\frac{R}{2\pi r_0}\,\mathrm{Ci}\left[\left(N+\frac{1}{2}\right)\frac{\pi a}{R}\right]+\sum_{n=0}^{N}, \tag{12}$$

[14] See, e.g., J. Serrin, in *Handbuch der Physik*, edited by S. Flügge, (Springer-Verlag, Berlin 1959), Vol. VIII, Part I, p. 159. The procedure is perfectly analogous to finding the magnetic field energy of a current loop by an inductance calculation.

TABLE I. Energy E, in units of $\rho\kappa^2 r_0$, of a circular vortex ring of strength κ with an empty streamlined core, confined coaxially in a long tube full of an incompressible fluid of density ρ, for various ratios of the ring radius r_0 and the core radius a to the tube radius R. This is R/r_0 times the quantity plotted in Fig. 4.

r_0/R	$E/\rho\kappa^2 r_0$					
	$a/R=10^{-7}$	10^{-6}	10^{-5}	10^{-4}	10^{-3}	10^{-2}
0.10	6.949	5.798	4.646	3.496	2.350	1.244
0.20	7.291	6.140	4.989	3.838	2.690	1.568
0.30	7.482	6.331	5.179	4.028	2.879	1.749
0.40	7.601	6.449	5.298	4.147	2.997	1.863
0.50	7.668	6.516	5.365	4.214	3.065	1.928
0.60	7.686	6.535	5.384	4.233	3.083	1.946
0.70	7.648	6.497	5.345	4.194	3.045	1.908
0.80	7.527	6.376	5.225	4.074	2.924	1.789
0.85	7.414	6.263	5.112	3.960	2.811	1.680
0.90	7.235	6.083	4.932	3.781	2.633	1.508
0.95	6.903	5.752	4.600	3.449	2.303	1.198
0.98	6.449	5.298	4.146	2.996	1.857	0.806
0.99	6.103	4.952	3.801	2.652	1.523	0.552
1.00	0	0	0	0	0	0

where $\mathrm{Ci}\ x\equiv-\int_x^\infty t^{-1}\cos t dt$ is the cosine-integral function,[15] and we have neglected two small oscillatory terms. As N is increased, the right-hand side of Eq. (12) eventually oscillates about the true value of the sum with decreasing amplitude and with period R/r_0 (if $r_0\leqslant\frac{1}{2}R$) or $R/(R-r_0)$ (if $r_0\geqslant\frac{1}{2}R$). An IBM 7090 computer was programmed to carry out the calculation for each N at least up to $N=50$ (to insure the validity of the approximations used) and as far beyond as necessary to make the amplitude of the oscillations in the estimate for $E/\rho\kappa^2 r_0$ less than 0.05. The midpoint of the oscillations could of course be estimated to within a considerably smaller range. The results, which are presented in Table I, are thus always good to at least two decimal places, and the third decimal place is significant or certain when a/R is very small and r_0/R is not too close to 1, so that the series converges rapidly. It should be recognized, however, that this accuracy obtains only for vortex rings whose core has the particular form assumed. As was shown in Sec. IIA, the differences due to making other assumptions about the core can be much greater than the estimated errors in the calculation.

The results are also plotted in Fig. 4, normalized using R rather than r_0 so that the dependence of E on ring radius for a fixed tube radius can be seen.

C. Limiting Cases

If r_0/R is small enough, the walls will not affect the energy much, and as long as r_0/a remains very large, the results should be given by Lamb's formula [Eq. (1)] corrected for the difference in assumptions about the core, that is,

$$E_0=\tfrac{1}{2}\rho\kappa^2 r_0[\ln(8r_0/a)-2]. \tag{13}$$

[15] E. Jahnke and F. Emde, *Tables of Functions with Formulae and Curves* (Dover Publications, New York, 1945), 4th ed., Chap. 1, p. 1.

In fact, e.g., if $a/R=10^{-7}$, then Eq. (11) differs from Eq. (13) by less than 1% for r_0 as large as $0.5R$. When a/R is larger, E falls off from E_0 more rapidly.

If the ring is almost as large as the tube, so that $R-r_0 \ll r_0$, then the argument in Sec. I tells us that E should be the energy of a straight vortex line of length $2\pi r_0$ at a distance $s=R-r_0$ from a plane wall. The velocity field of this configuration can be got from that of a line in an unbounded fluid by superposing the velocity field of an opposing image line at a distance s behind the walls. (See Fig. 5.) The fluid velocity in the plane of the source lines is $(\kappa/2\pi)[(x-s)^{-1}-(x+s)^{-1}]$, where x is the distance from the wall. Therefore, using the same method as for the ring in Sec. B, we get for the energy

$$E_1=\left(\frac{\rho\kappa^2}{4\pi}\right)\int_{s+a}^{\infty}[(x-s)^{-1}-(x+s)^{-1}]dx\cdot 2\pi r_0$$
$$=\frac{\rho\kappa^2 r_0}{2}\ln\left(2\frac{R-r_0}{a}+1\right). \tag{14}$$

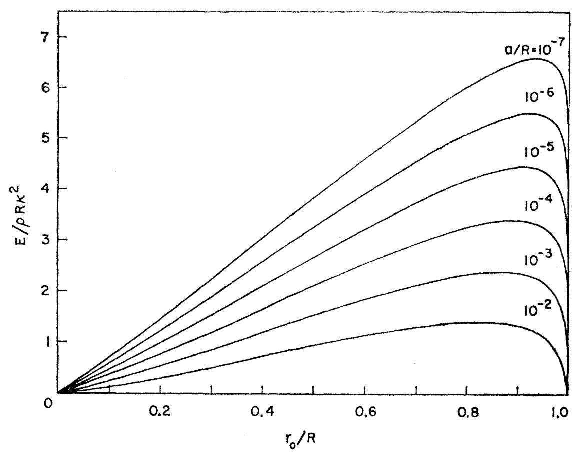

FIG. 4. Energy, in units of $\rho\kappa^2 R$, of a circular vortex ring of strength κ confined coaxially in a long tube of radius R full of an incompressible fluid of density ρ, as a function of the ring radius r_0 for various radii a of the empty streamlined core, each being given in terms of R [Eq. (11)].

This is in excellent agreement with the result of Eq. (11) for r_0 near R, differing from it, e.g., by less than 1% if $a/R=10^{-7}$ and $r_0>0.9R$.

The way the actual value of E deviates from these two limiting cases is shown in Fig. 6 for two values of a/R.

III. DISCUSSION

It is clear from the foregoing [see Fig. 4 and Eq. (14)] that $E\rightarrow 0$ as $r_0\rightarrow R$. Thus, the minimum value of E/p for a classical ring confined in a tube does indeed occur for the largest possible rings; but this value is zero for all tube radii. The classical computation is presumably not a good approximation to the actual quantum-mechanical problem when the core gets

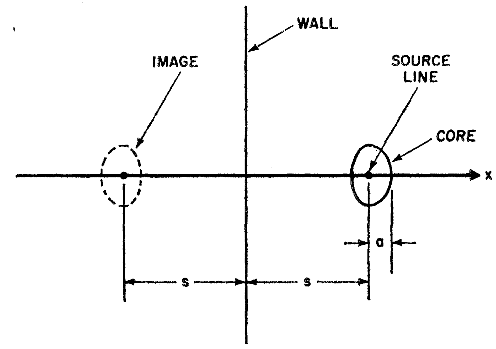

FIG. 5. Straight vortex line and image line equivalent to a vortex ring almost as large as the tube. Here $s=R-r_0$.

within a few core radii of the wall, but before this happens the energy is already considerably smaller than that of the unenclosed ring. For example, if $R-r_0=5a$, then E_0/E is 6.75 for $r_0/a=10^7$, and 3.87 for $r_0/a=10^4$. Therefore, critical velocities as large as those actually observed cannot be accounted for on the hypothesis that vortex rings are formed and cause friction wherever $v \geqslant (E/p)_{\min}$. Indeed, if this hypothesis were true, and the classical formula were valid, superfluidity would never be observed at all. Thus, Landau's condition alone, applied to vortex rings, cannot be taken as sufficient for the appearance of friction, though it may still be necessary.

The smallest effective change one could make in the model is to suppose that rings very near the walls do not cause friction for some reason, even though they may be formed at very low velocities. (More specifically, one could argue that rings larger than that of maximum energy cannot get away from the walls, since to do so they would have to become smaller and

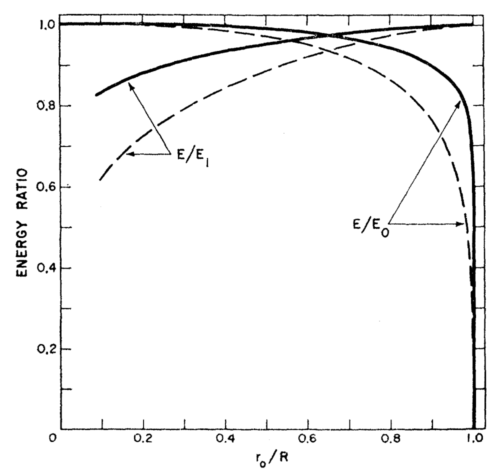

FIG. 6. Energy of a vortex ring in a tube in terms of the energies of two limiting configurations: an unenclosed ring of the same radius r_0 [E_0, Eq. (13) or (18)], and the same length of straight vortex line at the same distance from a plane wall [E_1, Eq. (14)]. ——, $a/R=10^{-7}$; - - - -, $a/R=10^{-3}$; a=core radius; R=tube radius.

their energy would have to increase. It seems implausible, however, to imagine that such rings would be stable against dimpling and tangling.) Alternatively, it might be that rings that are too large cannot be formed at all. In either of these cases, the critical velocity will be the value of E/p at some value of r_0 fairly near R, but not so near that the effect of the walls on E is very large. This velocity will not be very different from the value for an unenclosed ring with $r_0=R$ as given in Eq. (3). Thus, such a model could be made to fit the experimental results about as well as Geĭlikman's. We have not, however, been able to make a convincing argument along the above lines.

Another possibility is to go back to Feynman's original picture in which the tube "blows smoke rings" at the orifices. In this approach the production of vorticity is not deduced from the Landau criterion, but is regarded as a quasiclassical phenomenon due to the abrupt acceleration required for irrotational flow around the corner of the orifice. The quantum nature of the process appears only in the relation it imposes between energy and vorticity, and perhaps in the details of the subsequent propagation of vorticity along the tube.

Finally, the breakdown of the superfluid regime may, at least in some situations, have nothing to do with quantized vorticity at all. This is suggested by the observations of Meservey,[16] Chase,[2] and Staas *et al.*[17] that v_c can sometimes be described in terms of a Reynolds number. Also, in an elegant recent experiment, Peshkov and Tkachenko[18] have shown that turbulence in a long tube may nucleate at either end and spread down the tube toward the other, or (at somewhat higher velocities) start within the tube and spread toward both ends. Although they give a qualitative explanation in terms of vortex rings, the reported behavior somewhat resembles that of classical turbulence, which nucleates on boundary irregularities at a flow velocity dependent on their size.

It thus appears that there is no straightforward way to account for v_c in terms of the production of vortex rings in the region of uniform flow according to Landau's criterion. However, those who wish to consider new versions of this notion may find the above numerical results useful.

ACKNOWLEDGMENTS

We are grateful to Mrs. N. Rawson for programming the computation of the values of E; to Mrs. A. N. Carpenter for calculating the values of E_0 given in the appendix; and to Dr. E. Maxwell for very helpful discussions.

[16] R. Meservey, Bull, Am. Phys. Soc. **6**, 64 (1961); and Phys. Rev. **127**, 995 (1962).

[17] F. A. Staas, K. W. Taconis, and W. M. Van Alphen, Physica **27**, 893 (1961).

[18] V. P. Peshkov and V. K. Tkachenko, Zhur. Eksp. i Teoret. Fiz. **41**, 1427 (1961) [translation: Soviet. Phys.—JETP **14**, 1019 (1962)].

APPENDIX. ENERGY OF A VORTEX RING IN AN UNBOUNDED FLUID

The limit E_0 of E in Eq. (11) as $R\rightarrow\infty$ cannot be obtained directly because the summation converges more and more slowly as R increases. It can, however, be got by an independent computation letting the sum go over into an integral, and this is worth doing as a check on Eq. (11) for small r_0/R and to find the limits of applicability of Lamb's approximation [Eq. (1)].

We choose the configuration and define the parameters just as in Sec. IIA, except that the walls are removed. The solution of Laplace's equation then takes the form

$$\phi=\int_0^\infty A(k)J_0(kr)e^{-k|z|}dk. \tag{15}$$

Using the orthonormality of the J_0's over the interval from zero to infinity[19] we express $\phi_{z=0}$ as an integral with kernel $rJ_0(kr)$. Comparison with Eq. (15) then gives

$$A(k)=\tfrac{1}{2}\kappa r_0 J_1(kr_0). \tag{16}$$

Therefore, using Eq. (10) we get

$$E_0=\pi\rho\kappa^2 r_0(r_0-a)\int_0^\infty J_1(kr_0)J_1[k(r_0-a)]dk. \tag{17}$$

The integral is of a kind treated by Watson[20]; the result is

$$E_0=\tfrac{1}{4}\pi\rho\kappa^2 r_0\gamma_2\, F_1(\tfrac{3}{2},\tfrac{1}{2};2;\gamma), \tag{18}$$

where $\gamma\equiv(1-a/r_0)^2$ and the hypergeometric function[21]

$${}_2F_1(\tfrac{3}{2},\tfrac{1}{2};2;\gamma)=\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(n+\frac{1}{2})[\Gamma(n+\frac{1}{2})]^2}{(n+1)[\Gamma(n+1)]^2}\gamma^n. \tag{19}$$

If $a\ll r_0$, then $\gamma\approx 1$ and the series converges very slowly. We may again, however, derive an approximate expression for the tail after the Nth term by integrating Eq. (19) as a continuous function of n from $N+\frac{1}{2}$ to ∞. To facilitate the integration, we use Stirling's approximation for the gamma functions and simplify the resulting expression by the additional approximation $(1-1/m)^m\approx e^{-1}(1-1/2m)$, where $m=2n-2$. This gives for the summation in Eq. (19):

$$\sum_{n=0}^{\infty}\approx-\tfrac{5}{8}\,\mathrm{Ei}[(N+\tfrac{1}{2})\ln\gamma]-\gamma(\tfrac{3}{8}-\tfrac{1}{8}\ln\gamma)\,\mathrm{Ei}[(N-\tfrac{1}{2})\ln\gamma]-\frac{\gamma^{N-\frac{1}{2}}}{4(N+2)}+\sum_{n=0}^{N}, \tag{20}$$

[19] See, e.g., P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 1, p. 766.

[20] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, New York, 1952), p. 401.

[21] Reference 20, p. 100.

TABLE II. Energy, E_0, in units of $\rho\kappa^2 r_0$, of a circular vortex ring of strength κ with an empty streamlined core in an unbounded fluid, for various ratios of the ring radius r_0 to the core radius a.

r_0/a	$E_0/\rho\kappa^2 r_0$	r_0/a	$E_0/\rho\kappa^2 r_0$	r_0/a	$E_0/\rho\kappa^2 r_0$
10	1.109	100	2.328	1 000	3.492
20	1.487	200	2.681	2 000	3.840
30	1.703	300	2.887	3 000	4.043
40	1.855	400	3.032	4 000	4.187
50	1.971	500	3.144	5 000	4.298
60	2.066	600	3.236	6 000	4.389
70	2.145	700	3.313	7 000	4.466
80	2.214	800	3.381	8 000	4.533
90	2.274	900	3.440	9 000	4.592
				10 000	4.645
				$>10\ 000$	$\frac{1}{2}\ln(8r_0/a)-1$

where the exponential integral Ei is a tabulated function[15] defined by $-\mathrm{Ei}(-x)\equiv\int_x^\infty t^{-1}e^{-t}dt$. Values of $E_0/\rho r_0\kappa^2$, which are given in Table II, were computed on a desk calculator using Eq. (20) with $N=20$. They are accurate to within 0.1%.

The difference between these values and those given by Eq. (13) is indeed small for reasonable a, being less than 1% if $r_0/a>100$, and less than 0.1% if $r_0/a>500$. As is to be expected, the results are also very close to those of Eq. (11) if a/r_0 and r_0/R are both very small. Hence, as is shown in Fig. 6, the ratio of the energy of an enclosed ring to that of an unenclosed ring with the same r_0 and a goes to unity as r_0 becomes small, as long as r_0 does not become comparable with a. If this last condition is not satisfied, the energy of the enclosed ring actually becomes a little higher than that of the unenclosed ring, owing to distortion of the core. The effect is about 1∞ for $10a=r_0=0.1R$. It is clearly irrelevant to the present application of this calculation.

PHYSICAL REVIEW VOLUME 129, NUMBER 1 1 JANUARY 1963

Energy Distribution of Electrons Ejected from Tungsten by He⁺ †

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(Received 16 July 1962; revised manuscript received 3 October 1962)

The results of a calculation of the energy distribution of electrons ejected from tungsten by low-energy He^+ are presented. The calculation is based on a mechanistic model of the process in which the ejected electrons are divided into two groups: (1) the electrons excited in the primary process that can escape directly; and (2) the electrons that escape because of interactions between the primary electrons and those of the band structure of the solid. Secondary electron data are used to predict the portion due to this second mechanism.

I. INTRODUCTION

THE potential ejection of electrons from solid surfaces by low-energy ions has been studied extensively both experimentally and theoretically.[1–6] Since the phenomenon is sensitive to the surface structure, both the experimental and theoretical treatments are quite complicated. In this paper we give the preliminary results of a calculation (based on a mechanistic model of the process) of the energy distribution of electrons ejected from tungsten by He^+. We attempt to take into account the interactions of the electrons excited in the primary Auger process with those of the band structure of the solid. In the case under discussion, these interactions appear to give rise to about 50% of the total measured yield.

In order to calculate the energy distribution of electrons ejected by ions, we must know: (1) the distribution in energy and angle, $N(E,\Omega)$, of the electrons excited inside the metal in the primary process; (2) the escape probability, $F(E,\Omega)$, of the electrons; and (3) the effect of interactions between the primary electrons and the electrons of the solid. These items are treated in the following sections.

II. ENERGY DISTRIBUTION OF PRIMARY ELECTRONS

Figure 1 shows a sketch commonly used to describe the situation that exists when an ion approaches a solid surface. One electron falls into the vacant atomic level. The energy released in the transition is then absorbed by a second electron from the solid. We can look at the process in two ways. First, we can assume that the Coulomb interaction between the two participating

† The research reported in this paper was made possible by support extended to the University of Illinois, Coordinated Science Laboratory, jointly by the Department of the Army (Signal Corps and Ordnance Corps), Department of the Navy (Office of Naval Research), and the Department of the Air Force (Office of Scientific Research, Air Research and Development Command) under Signal Corps Contract DA-36-039-SC-85122.

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