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# Photon Asymmetry in the Bremsstrahlung of Transversely Polarized Electrons

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There is a left-right asymmetry in the bremsstrahlung emitted by transversely polarized electrons analogous to the left-right asymmetry in Mott scattering. The magnitude of this effect has been calculated from the second Born approximation, where it appears in the cross terms between the matrix elements of order $Ze^3$ and those of order $Z^2e^5$. Principal contributions come from the Feynman diagram in which the photon is emitted between the Coulomb interactions with the nucleus, while the largest terms from the diagrams with the photon emitted before or after both Coulomb interactions are of relative order $v/c$ compared to the principal terms. The contributions of all three diagrams to relative order $(v/c)^2$ are included in the final result. As is the case in Mott scattering, the asymmetry in bremsstrahlung depends only on intermediate states which lie on the energy shell. The reason for this has not been investigated here. It does not seem feasible to integrate the cross section over electron momenta analytically. In order to reduce geometric complexity, the calculation is performed in detail for forward electron momentum. The matrix elements for this case are listed explicitly so that other polarization-dependent effects may be conveniently calculated. An outline of the evaluation of the integrals which appear, and a discussion of the divergences associated with the no-cutoff limit of the Coulomb field, as they apply to bremsstrahlung, are included.

## 1. INTRODUCTION

THE calculation described in this paper is concerned with determining the angular asymmetry of the photon emitted in bremsstrahlung of transversely polarized electrons. As is the case in the Coulomb scattering of transversely polarized electrons, the asymmetry arises from the interference between the two lowest order matrix elements contributing to the process.[1]

Figure 1 depicts the lowest order Feynman diagrams contributing to bremsstrahlung. These lead to the Bethe-Heitler result.[2] No angular asymmetry appears here, just as none appears in the first approximation to radiationless scattering. The second approximation to radiationless scattering does lead to an angular asymmetry when the incident particle is transversely polarized, and the effect also appears in bremsstrahlung when the diagrams of the next order are considered. These are presented in Fig. 2. They contribute terms of relative order $Z\alpha$ when compared to the diagrams of Fig. 1. The quantity $Z\alpha$ is, of course, the nuclear charge times the fine structure constant. Kacser[3] has calculated the unpolarized cross section in second Born approximation utilizing an approximation scheme to evaluate the integrals involved. The radiative corrections to the diagrams of Fig. 1 also are of relative order $\alpha$, and have been calculated elsewhere.[4] In our work the radiative corrections have been neglected so that the results are strictly valid only for large $Z$, i.e., heavy nuclei; but this is the case anyway since nuclear recoil is also neglected.

Diagrams (a) and (b) of Fig. 2 are simpler than (c), because in both (a) and (b), the electron is in a unique momentum state, although not on the mass shell, when the photon is emitted. The integrals involved then are familiar, being exactly those for radiationless scattering taken off the mass shell. In (c) the electron is not in a unique momentum state when the photon is emitted, and this leads to more complicated integrals. The largest contributions to the asymmetry come from diagram (c). Fortunately, that part of (c) which involves the most difficult integral may be neglected. There are two reasons for this. First, the effect depends on the transverse component of the initial electron's spin. This causes the Mott effect to wash out like $1-v^2/c^2$ for, example, so that not too fast electrons are involved. Secondly, only the imaginary parts of the various integrals involved contribute, and these come from regions in momentum space which are very near to the mass shell. This means that the imaginary parts come from a very restricted range of momentum transfers. These two facts make the estimation of the relative contribu-

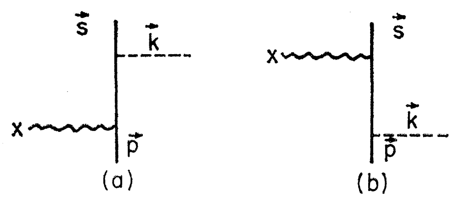


FIG. 1. Lowest order Feynman diagrams in bremsstrahlung.

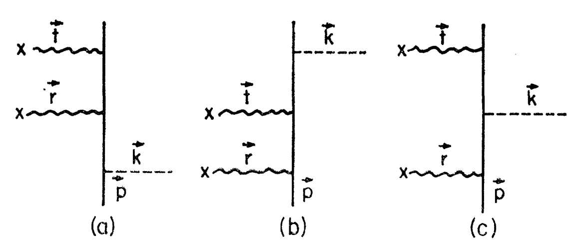


FIG. 2. Second-order Feynman diagrams in bremsstrahlung.

[1] R. H. Dalitz, Proc. Roy. Soc. (London) **A206**, 509 (1951).
[2] H. A. Bethe and W. Heitler, Proc. Roy. Soc. (London) **A146** 83 (1934).
[3] C. Kacser, Proc. Roy. Soc. (London) **A253**, 103 (1959).
[4] A. N. Mitra, P. Narayanaswamy, and L. K. Pande, **Nucl.** Phys. **10**, 629 (1959).

tions of each term of Fig. 2 quite simple. All of the points mentioned above are discussed in Sec. 2.

In order to avoid the difficulties arising from the long range of the Coulomb potential, it is cut off by the usual exponential factor. This gives rise to well-known divergences when the no cutoff, or pure Coulomb, limit is desired. Dalitz[1] has shown that these divergences lead only to an unobservable phase factor in the case of radiationless scattering, and Kacser[3] has proven that they cancel identically in bremsstrahlung, and, therefore, do not give rise to spurious interference effects. These points are briefly discussed in Sec. 2, and in Appendix C.

The matrix elements are discussed in Sec. 2, and in Appendix A. The asymmetry parameter is obtained in Sec. 3. The appendices are concerned with calculations of the matrix elements and the integrals which appear, and with a discussion of the divergences associated with the no cutoff limit of the Coulomb field as they affect bremsstrahlung.

## 2. THE MATRIX ELEMENTS

### A. Preliminaries

The natural system of units with $\hbar=c=1$ is used throughout. Four-momenta are denoted by $p_\mu$ with $p_4=ip_0=iE$. Also

$$\tilde{a}=a_\mu\gamma_\mu=\mathbf{a}\cdot\boldsymbol{\gamma}+a_4\gamma_4,$$

and we employ Hermitean $\gamma$ matrices. The electron charge is taken as $+e$, $e<0$, so that the central potential of the positively charged nucleus is

$$A(r)=-\frac{Ze}{4\pi ir}e^{-\lambda r},\quad \lambda>0. \tag{2.1}$$

The exponential cutoff has been introduced in order to avoid the difficulties associated with the long range of the Coulomb potential. The bremsstrahlung matrix element is $\langle k,s|S|p\rangle$, where $p$ and $s$ are the initial and final electron momenta, and $k$ is the photon momentum. In order to calculate the process to order $Z^2\alpha^4$, we need only consider $S^{(2)}$ and $S^{(3)}$. The matrix elements of $S^{(0)}$ will vanish because the process involves emission of radiation, not included in $S^{(0)}$, and those of $S^{(1)}$ will vanish because of energy-momentum conservation. The matrix elements for $S^{(m)}$, $m>3$, are all of higher order.

We seek the cross section $[d\sigma(\theta)/d\Omega]d\Omega$ for emission of a photon in an element of solid angle $d\Omega$, making an angle $\theta$ with the incident electron momentum $\mathbf{p}$, and in the plane perpendicular to the initial (transverse) spin of the electron. The cross section will be proportional to

$$|M^{(2)}+M^{(3)}|^2\simeq|M^{(2)}|^2+2\,\mathrm{Re}M^{(2)*}M^{(3)}. \tag{2.2}$$

where

$$M^{(2)}=M_a{}^{(2)}+M_b{}^{(2)}=\langle k,s|S_a{}^{(2)}+S_b{}^{(2)}|p\rangle, \tag{2.3}$$

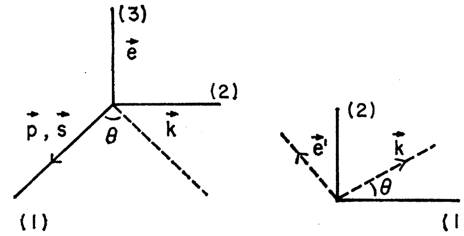


FIG. 3. Coordinate system.

and

$$M^{(3)}=M_a{}^{(3)}+M_b{}^{(3)}+M_c{}^{(3)} =\langle k,s|S_a{}^{(3)}+S_b{}^{(3)}+S_c{}^{(3)}|p\rangle. \tag{2.4}$$

Here the subscripts refer to the Feynman diagrams of Figs. 1 and 2. The terms in $|M^{(3)}|^2$ are of $O(Ze^2)$ higher and are neglected for consistency. The coordinate system is chosen so that the 1 axis is in the direction of the incident electron, and so that the 1–2 plane is perpendicular to the incident electron spin. Figure 3 depicts the coordinate system. Polarization vectors $\mathbf{e}$ and $\mathbf{e}'$ are indicated in Fig. 3, and are defined by

$$\begin{aligned}\mathbf{e}&=\mathbf{n}_3=|k|^{-1}\mathbf{k}\times\mathbf{e}',\\ \mathbf{e}'&=-\mathbf{n}_1\sin\theta+\mathbf{n}_2\cos\theta=|k|^{-1}\mathbf{e}\times\mathbf{k},\\ \mathbf{n}_i\cdot\mathbf{n}_i&=1.\end{aligned} \tag{2.5}$$

In order to minimize geometric complexity, and since large angle scatterings of the electron are rather improbable, only the case $\mathbf{s}\|\mathbf{p}$ will be considered. Further simplifications could be obtained by considering only the high-frequency limit of the spectrum, that is the case $\mathbf{s}=0$. However, this is precisely the region in which the Born approximation is worst.[5] Since the initial and final electron momenta are parallel, the corresponding spinors may be written as[6]

$$\begin{aligned}u(\mathbf{p})&=\exp(\tfrac{1}{2}\alpha_1\chi)u(0),\\ \bar{v}(\mathbf{s})&=\bar{v}(0)\exp(-\tfrac{1}{2}\alpha_1\psi).\end{aligned} \tag{2.6}$$

Here $u(0)$ and $v(0)$ are rest spinors, $\boldsymbol{\alpha}=i\gamma_4\boldsymbol{\gamma}$, and $\psi$ and $\chi$ are the usual Lorentz transformation parameters satisfying[6]

$$\begin{aligned}\tanh\chi&=p/p_0,\\ \tanh\psi&=s/s_0.\end{aligned} \tag{2.7}$$

The choice of spinors in (2.6) means that all matrix elements are taken between positive energy rest states, and with the choice of polarization vectors in Eq. (2.5), half the matrix elements will be zero. In addition, it will be convenient to pick $u(0)$ and $v(0)$ as eigenvectors of $\sigma_3$. Then the expectation value of the transverse

[5] By considering the photoelectric effect and bremsstrahlung as inverse processes the high-frequency limit has been calculated. See K. W. McVoy and U. Fano, Phys. Rev. **116**, 1168 (1959); R. H. Pratt, *ibid.*, **120** 1717 (1960).

[6] See for example P. Stehle, Phys. Rev. **110** 1458 (1958).

component of the initial spin is proportional to

$$(m/p_0)^{1/2}u^\dagger(p)\sigma_3(m/p_0)^{1/2}u(p)=(1-v^2)^{1/2}.$$

Now the asymmetry depends essentially on this component of the spin, so we see that we should deal with electrons for which $p_0$ is not too much greater than $m$, the rest mass of the electron.

### B. The Matrix Elements

The matrix elements corresponding to the diagrams of Figs. 1 and 2 are listed below.

$$M_a^{(2)}=\frac{c_2}{\tau^2+\lambda^2}\bar{v}(s)\gamma_4\frac{\tilde{p}-\tilde{k}+im}{(p-k)^2+m^2}\tilde{e}u(p), \quad (2.8a)$$

$$M_b^{(2)}=\frac{c_2}{\tau^2+\lambda^2}\bar{v}(s)\tilde{e}\frac{\tilde{k}+\tilde{s}+im}{(k+s)^2+m^2}\gamma_4u(p), \quad (2.8b)$$

$$M_a^{(3)}=c_3\bar{v}(s)\gamma_4I(p-k\rightarrow s)\gamma_4\frac{\tilde{p}-\tilde{k}+im}{(p-k)^2+m^2}\tilde{e}u(p), \quad (2.8c)$$

$$M_b^{(3)}=c_3\bar{v}(s)\tilde{e}\frac{\tilde{k}+\tilde{s}+im}{(k+s)^2+m^2}\gamma_4I(p\rightarrow s+k)\gamma_4u(p), \quad (2.8d)$$

$$M_c^{(3)}=c_3\bar{v}(s)\gamma_4K(p,k,s)\gamma_4u(p), \quad (2.8e)$$

$$c_2=\frac{iZe^3}{(2\pi)^{7/2}}\left(\frac{m^2}{2p_0s_0k_0}\right)^{1/2}\delta(\tau_0), \quad (2.9a)$$

$$c_3=-i\frac{Ze^2}{(2\pi)^3}c_2. \quad (2.9b)$$

Here $\tau$ is the momentum transferred to the nucleus,

$$\boldsymbol{\tau}=\mathbf{p}-\mathbf{k}-\mathbf{s}=\mathbf{p}-\mathbf{l}, \quad (2.10)$$

and $\delta(\tau_0)=\delta(p_0-k_0-s_0)$ is the usual energy-conserving $\delta$ function.

$$p_0=k_0+s_0,$$
$$(\mathbf{p}^2+m^2)^{\frac{1}{2}}=k_0+(\mathbf{s}^2+m^2)^{1/2}. \quad (2.11)$$

Some of the terms in (2.8) may be simplified using the Dirac equation

$$(\tilde{p}-im)u(p)=0,\quad \bar{v}(s)(\tilde{s}-im)=0.$$

Recalling the identities

$$\tilde{a}\tilde{b}+\tilde{b}\tilde{a}=2a_\mu b_\mu,$$
$$\tilde{k}\tilde{e}+\tilde{e}\tilde{k}=0,$$

we find for example

$$(\tilde{p}-\tilde{k}+im)\tilde{e}u(p)=[\tilde{e}\tilde{k}+2p_\mu e_\mu+\tilde{e}(-\tilde{p}+im)u(p)$$
$$=(\tilde{e}\tilde{k}+2p_\mu k_\mu)u(p).$$

Further we define $I_0$, $\mathbf{I}$, $J_0$, $\mathbf{J}$, $K_0$, $\mathbf{K}$, and $K_{ij}$ from

$$I(p-k\rightarrow s)=(s_4\gamma_4+im)I_0+\boldsymbol{\gamma}\cdot\mathbf{I}, \quad (2.12)$$

$$I(p\rightarrow s+k)=(p_4\gamma_4+im)J_0+\boldsymbol{\gamma}\cdot\mathbf{J}, \quad (2.13)$$

$$K(p,k,s)$$
$$=(p_4\gamma_4+im-\tilde{k})\tilde{e}(p_4\gamma_4+im)K_0+\mathbf{K}\cdot\boldsymbol{\gamma}\tilde{e}(p_4\gamma_4+im)$$
$$+(p_4\gamma_4+im-\tilde{k})\tilde{e}\boldsymbol{\gamma}\cdot\mathbf{K}+\gamma_i\tilde{e}\gamma_jK_{ij}, \quad (2.14)$$

so that

$$(I_0,I_j)=\int d^3q\frac{(1,q_j)}{[(\mathbf{q}+\mathbf{k}-\mathbf{p})^2-\lambda^2][\mathbf{q}^2-\mathbf{s}^2-i\epsilon][(\mathbf{q}-\mathbf{s})^2+\lambda^2]}, \quad (2.15)$$

$$(J_0,J_j)=\int d^3q\frac{(1,q_j)}{[(\mathbf{q}-\mathbf{k}-\mathbf{s})^2+\lambda^2][\mathbf{q}^2-\mathbf{p}^2-i\epsilon][(\mathbf{q}-\mathbf{p})^2+\lambda^2]}, \quad (2.16)$$

$$(K_0,K_j,K_{ij})=\int d^3q\frac{(1,q_j,q_iq_j)}{[(\mathbf{q}-\mathbf{k})^2-\mathbf{s}^2-i\epsilon][\mathbf{q}^2-\mathbf{p}^2-i\epsilon][(\mathbf{q}-\mathbf{p})^2+\lambda^2][(\mathbf{q}-\mathbf{k}-\mathbf{s})^2+\lambda^2]}. \quad (2.17)$$

The matrix elements then become

$$M_a^{(2)}=\frac{c_2}{(\tau^2+\lambda^2)}\bar{v}(s)\gamma_4\frac{2e_\mu p_\mu+\tilde{e}\tilde{k}}{-2p_\mu k_\mu}u(p), \quad (2.18a)$$

$$M_b^{(2)}=\frac{c_2}{(\tau^2+\lambda^2)}\bar{v}(s)\frac{2e_\mu s_\mu+\tilde{e}\tilde{k}}{2s_\mu k_\mu}\gamma_4u(p), \quad (2.18b)$$

$$M_a^{(3)}=\frac{c_3}{-2p_\mu k_\mu}\bar{v}(s)\gamma_4[(s_4\gamma_4+im)I_0+\boldsymbol{\gamma}\cdot\mathbf{I}]$$
$$\times\gamma_4(2e_\mu p_\mu+\tilde{e}\tilde{k})\gamma_4u(p), \quad (2.18c)$$

$$M_b^{(3)}=(c_3/2s_\mu k_\mu)\bar{v}(s)(2e_\mu s_\mu+\tilde{e}\tilde{k})$$
$$\times\gamma_4[(p_4\gamma_4+im)J_0+\boldsymbol{\gamma}\cdot\mathbf{J}]\gamma_4u(p), \quad (2.18d)$$

$$M_c^{(3)}=c_3\bar{v}(s)\gamma_4\{[\mathbf{p}^2\tilde{e}-\tilde{k}\tilde{e}(p_4\gamma_4+im)]K_0$$
$$+[2(p_4\gamma_4+im)\mathbf{e}-\tilde{k}\tilde{e}\boldsymbol{\gamma}]\cdot\mathbf{K}+\gamma_i\tilde{e}\gamma_jK_{ij}\}u(p). \quad (2.18e)$$

### C. Contributions to the Asymmetry

The asymmetric part of the bremsstrahlung cross section is that part which is odd in powers of $k_2=k\sin\theta=k\sin(\mathbf{p},\mathbf{k})$. In Appendix A we show that only the imaginary parts of the integrals involved in the matrix elements contribute to the asymmetry. Using this, and the fact that we are dealing with not too fast electrons, we may consider $p_0$, $s_0\simeq m$ in order to investigate the relative magnitudes of $M_a^{(3)}$, $M_b^{(3)}$, and $M_c^{(3)}$. The key

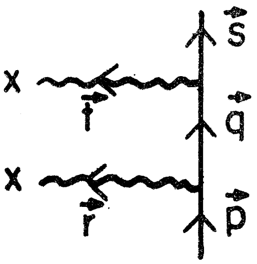


FIG. 4. Second-order Feynman diagram for Coulomb scattering.

point is that the imaginary parts of all the integrals come from momentum transfers whose magnitudes are of $O(p)$ or less. To see why this is so, consider Coulomb scattering for a moment. Here the integrals are similar, but a bit simpler and have all been discussed by Dalitz.[1] Only the imaginary parts of these integrals contribute to the asymmetry in Coulomb scattering. The pertinent Feynman diagram is that of Fig. 4.

The matrix element is

$$M\sim\bar{v}_f\gamma_4[(p_4\gamma_4+im)I_0'+\boldsymbol{\gamma}\cdot\mathbf{I}']u_i. \tag{2.19}$$

We shall only look at $I_0'$, which is the simplest and has all the typical features:

$$I_0'\sim\int d^4qd^4rd^4t\frac{\delta\text{ functions}}{[\mathbf{r}^2+\lambda^2][\mathbf{q}^2+m^2-i\epsilon][\mathbf{t}^2+\lambda^2]}$$
$$\sim\int d^3q\frac{1}{[(\mathbf{p}-\mathbf{q})^2+\lambda^2][\mathbf{q}^2-\mathbf{p}^2-i\epsilon][(\mathbf{q}-\mathbf{s})^2+\lambda^2]}. \tag{2.20}$$

Since

$$\frac{1}{\mathbf{q}^2-\mathbf{p}^2-i\epsilon}=\frac{1}{\mathbf{q}^2-\mathbf{p}^2}+\pi i\delta(\mathbf{p}^2-\mathbf{q}^2),$$

it is obvious that the imaginary part of $I_0'$ comes only from the neighborhood $q^2=p^2$. It depends on the momentum transfers $r=p-q$ and $t=q-s$, and on $\lambda$, the range parameter of the potential. Thus $I_mI_0'$ depends on values of $r$ and $t$ which are $2p$ at most. Now if either $r$ or $t$ is zero, things depend critically on $\lambda$. But Fig. 4 indicates that this corresponds simply to an external line modification, and leads at worst to an unobservable phase factor.[1] The infinite range of the Coulomb field manifests itself here in the limit $\lambda\rightarrow 0$. For bremsstrahlung, Kacser[3] has shown that these contributions actually cancel in the limit $\lambda\rightarrow 0$. In Appendix C the divergent terms of each matrix element are removed explicitly.

The comments above about $I_0'$ also apply to the integrals appearing in our matrix elements. Thus using the facts that $r, t\simeq O(p)$ and $p_0, s_0\simeq m$, we estimate below the relative magnitudes of the terms in $M^{(3)}$. It is sufficient to consider the case $r=t=\tau/2$. The momentum transfer $\boldsymbol{\tau}$ was defined in (2.10). In the following we shall use $q$ to denote the order of magnitude of all three momenta which appear, so that $2p\geqslant q$. The matrix elements are listed in (2.18). We have

$$M_a^{(3)}\sim\bar{v}\gamma_4\frac{1}{\frac{1}{4}\boldsymbol{\tau}^2+\lambda^2}\frac{\tilde{s}+\frac{1}{2}\tilde{\tau}+im}{[(s+\frac{1}{2}\tau)^2+m^2]}$$
$$\times\gamma_4\frac{1}{\frac{1}{4}\boldsymbol{\tau}^2+\lambda^2}\frac{\tilde{p}-\tilde{k}+im}{[(p-k)^2+m^2]}\tilde{e}u \tag{2.21}$$
$$\sim\frac{\bar{v}[-\mathbf{q}^2-2m^2(1+\gamma_4)+2im\gamma_4\boldsymbol{\gamma}\cdot\mathbf{q}]\tilde{e}u}{q^6(q^2+2p_0k_0)},$$

$$M_b^{(3)}\sim\frac{\bar{v}\tilde{e}[-\mathbf{q}^2-2m^2(1+\gamma_4)+2im\boldsymbol{\gamma}\cdot\mathbf{q}\gamma_4]u}{q^6(q^2-2s_0k_0)}, \tag{2.22}$$

$$M_c^{(3)}\sim-\frac{1}{q^8}\bar{v}[\boldsymbol{\gamma}\cdot\mathbf{q}\boldsymbol{\gamma}\cdot\mathbf{e}\boldsymbol{\gamma}\cdot\mathbf{q}-2im\mathbf{e}\cdot\mathbf{p}(1+\gamma_4)]u. \tag{2.23}$$

In Eq. (2.23), the $qeq$ term comes from the $K_{lm}$ term, while the rest is a combination of the $K_0$ and $\mathbf{K}$ terms. Since $u$ and $v$ are positive energy spinors we have approximately

$$\bar{v}[1,\boldsymbol{\sigma},\gamma_4]u=O(1),$$
$$\bar{v}[\boldsymbol{\alpha},\boldsymbol{\gamma}]u=O(q/m).$$

Making use of these in (2.21) and (2.22), we find

$$M_a^{(3)},\ M_b^{(3)}=O(q^{-6}),$$

and that these two matrix elements tend to cancel as far as asymmetric effects are concerned. This is borne out in the detailed results of Appendix A. See Eqs. (A3) and (A4). From (2.23) we find

$$M_c^{(3)}=O(mq^{-7}),$$

while the contribution of the $K_{lm}$ term is $O(m^{-1}q^{-5})$. Therefore, to relative order $q/m$, the major contributions to the asymmetry come from $M_c^{(3)}$, and to terms of relative order $q^2/m^2$, the $K_{lm}$ terms may be neglected. We neglect $K_{lm}$ so that our final result, Eq. (3.7), is valid only to terms of relative order $p^2/p_0^2$.

## 3. RESULTS

The results in this section refer to the case $\mathbf{p}||\mathbf{s}$ and are valid up to terms of relative order $(v/c)^2$. This latter restriction arises because the terms in $S^{(3)}$ which involve $K_{ij}$ were neglected. In Appendix B we show that all the integrals appearing may be evaluated exactly and, therefore, this restriction may be removed. Of course, the matrix elements may be calculated for arbitrary final electron momentum and the first restriction also removed. Very little further insight can be gained by carrying out the tedious program needed to remove these restrictions.

The asymmetry parameter $\delta$ is defined by (3.1)

$$\delta=\frac{|\langle k,s|S^{(2)}+S^{(3)}|p\rangle|^2_{\text{asym}}}{|\langle k,s|S^{(2}|p\rangle|^2}. \tag{3.1}$$

The denominator is just the Bethe-Heitler amplitude squared for an initial beam of transversely polarized electrons with $\mathbf{p}\|\mathbf{s}$. The numerator is the asymmetric contribution to the scattering calculated in Appendix A. See Eq. (A5). Since we are interested in the limit $\lambda\rightarrow 0$, the $\lambda$ dependence of the integrals has been removed as per the discussion in Appendix C. Denoting the finite imaginary parts of these integrals by a caret,

$$\hat{I}_0 = ImI_0 - ImI_0(\lambda),\quad \text{etc.,} \tag{3.2}$$

the numerator becomes

$$2(k/\tau^2)\sin\theta|c_2c_3|\{a\hat{I}_0+b\hat{J}_0+c\hat{K}_0+d\hat{K}_1+e\hat{K}_2\}. \tag{3.3}$$

The quantities $a$, $b$, $c$, $d$, $e$ are defined in (A6), while $c_2$ and $c_3$ are defined in (2.9). For the case under consideration ($\mathbf{p}\|\mathbf{s}$), the denominator is

$$\frac{|c_2k_4|^2}{m^2\tau^4\Gamma^4\rho^4}[\alpha+\beta\cos\theta+\gamma\cos^2\theta], \tag{3.4}$$

where

$$\begin{aligned}\rho^2&=-2p_\mu k_\mu=(\mathbf{p}-\mathbf{k})^2-\mathbf{s}^2,\\ \Gamma^2&=-2s_\mu k_\mu=\mathbf{p}^2-(\mathbf{s}+\mathbf{k})^2,\end{aligned} \tag{3.5}$$

$$\begin{aligned}\alpha&=2[\nu+s_0p_0(\Gamma^4+\rho^4)-2m^2\Gamma^2\rho^2],\\ \beta&=2(ps_0-sp_0)(\rho^4-\Gamma^4),\\ \gamma&=-2[\nu+ps(\Gamma^4+\rho^4)],\\ \nu&=(p/k\Gamma^2-s/k\rho^2)\{(s_0p_0+sp+m^2)\\ &\quad\times(p/k\Gamma^2-s/k\rho^2)-(sp_0+ps_0)(\Gamma^2+\rho^2)\}.\end{aligned} \tag{3.6}$$

Thus the asymmetry parameter becomes

$$\delta=\frac{2Ze^2m^2\tau^2\Gamma^4\rho^4}{(2\pi)^3k}\times\sin\theta\left\{\frac{aI_0+bJ_0+cK_0+dK_1+eK_2}{\alpha+\beta\cos\theta+\gamma\cos^2\theta}\right\}. \tag{3.7}$$

This result is complicated but numerical calculations should facilitate its analysis. Preliminary work indicates that the effect is of the order of 10% for electrons with $v/c=0.95$ on Pb at 15°.[7]

It should be pointed out that if only the no-spin-flip terms contributed, the radiation would be more left than right i.e., $\delta\geq 0$. This is true qualitatively since the "force" responsible for the asymmetry $\boldsymbol{\mu}\times\mathbf{v}\cdot\nabla\mathbf{E}$ is always in the same direction. However, "when the spin flips," this force is in the opposite direction and can cause $\delta$ to become negative when the spin-flip terms dominate. This is true simply because the axes of the multipoles responsible for the radiation may be oriented at different angles to the left and right of $\mathbf{p}$ for the two cases.

[7] J. C. Strauss, M.S. thesis, University of Pittsburgh, 1962 (unpublished).

**ACKNOWLEDGMENTS**

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## APPENDIX A

The matrix elements of (2.18) have been calculated[8] between the positive-energy rest spinors of Eq. (2.6), for the photon polarizations of (2.5), and with initial and final electron momenta colinear. In Sec. 2 we showed that the contribution of the $K_{ij}$ terms was of order $q^2/m^2$ smaller than the largest terms in $M^{(3)}$. The momentum $q$ is typical of the system so that $q\ll m$, and $q=2p$ at worst. Compared to the largest terms in $M^{(3)}$, we retain only contributions of relative order 1 and $q/m$. This means that $\mathbf{I}$ and $\mathbf{J}$ may be neglected, as well as sundry terms involving $I_0$, $J_0$, $K_0$, and $\mathbf{K}$. If the photon is polarized in the direction perpendicular to the $pk$ plane, polarization direction $\mathbf{e}=\mathbf{n}_3$, only the spin-flip matrix elements contribute. For photon polarization in the plane of emission only the no-spin-flip matrix elements contribute. The matrix elements are listed in Table I.

In Table I, $L=\cosh\frac{1}{2}(\chi+\psi)$, $N=\cosh\frac{1}{2}(\chi-\psi)$, $P=\sinh\frac{1}{2}(\chi-\psi)$, $R=\sinh\frac{1}{2}(\chi+\psi)$, and $K_1$, $K_2$, and $K_3$ are the components of $\mathbf{K}$.

In order to find the asymmetric part of the cross

TABLE I. Matrix elements.

Polarization $\mathbf{e}\|\mathbf{p}\times\mathbf{k}$. Only spin flip contributes.

$${}_eM_a^{(2)}=(c_2k_4/\tau^2\rho^2)[iN\cos\theta-L\sin\theta-iP]$$
$${}_eM_b^{(2)}=-(c_2k_4/\tau^2\Gamma^2)[iN\cos\theta-L\sin\theta+iP]$$
$${}_eM_a^{(3)}=-(c_3k_0/\rho^2)I_0[s_0(iN\cos\theta-L\sin\theta)+m(iL\cos\theta-N\sin\theta)]$$
$${}_eM_b^{(3)}=(c_3k_0/\Gamma^2)J_0[p_0(iN\cos\theta-L\sin\theta)+m(iL\cos\theta-N\sin\theta)]$$
$${}_eM_c^{(3)}=c_3k_0\{K_0[p_0(iN\cos\theta-L\sin\theta-iP)+m(iL\cos\theta-L\sin\theta+iR)]+iNK_1-LK_2\}$$

Polarization $\mathbf{e}'\perp\mathbf{p}\times\mathbf{k}$. Only no spin flip contributes.

$${}_{e'}M_a^{(2)}=-(c_2/\tau^2\rho^2)[2Lp\sin\theta+k_4(N+iR\sin\theta-P\cos\theta)]$$
$${}_{e'}M_b^{(2)}=(c_2/\tau^2\Gamma^2)[2Ls\sin\theta+k_4(N-iR\sin\theta+P\cos\theta)]$$
$${}_{e'}M_a^{(3)}=-(ic_3/\rho^2)I_0[s_0(2Lp\sin\theta+k_4N)+m(2Np\sin\theta+k_4L)]$$
$${}_{e'}M_b^{(3)}=(ic_3/\Gamma^2)J_0[p_0(2Ls\sin\theta+k_4N)+m(2Ns\sin\theta+k_4L)]$$
$${}_{e'}M_c^{(3)}=c_3\{K_0[p_0k_0(P\cos\theta-N-iR\sin\theta)-mk_0(R\cos\theta+L-iP\sin\theta)]+i(2p_0L+2mN-k_0L)\times(K_2\cos\theta-K_1\sin\theta)-k_0N(K_1\cos\theta+K_2\sin\theta)\}.$$

[8] E. Sobolak, thesis, University of Pittsburgh, 1961 (unpublished).

section, the odd part of $2\,\mathrm{Re}M^{(2)*}M^{(3)}$ is required.

$$\begin{aligned}2\,\mathrm{Re}M^{(2)*}M^{(3)} &= 2\,\mathrm{Re}M^{(2)}\,\mathrm{Re}M^{(3)}+2\,\mathrm{Im}M^{(2)}\,\mathrm{Im}M^{(3)}\\ &= (2/\tau^2)[\mathrm{Re}c_2Q^{(2)}\,\mathrm{Re}c_3Q^{(3)}+\mathrm{Im}c_2Q^{(2)}\,\mathrm{Im}c_3Q^{(3)}]\\ &= (2|c_2|c_3/\tau^2)[\mathrm{Re}Q^{(2)}\,\mathrm{Im}Q^{(3)}-\mathrm{Re}Q^{(3)}\,\mathrm{Im}Q^{(2)}].\end{aligned} \tag{A1}$$

In (A1) we have used the fact that $c_2$ is pure imaginary and $c_3$ is real. Now since each final electron state involves only one photon polarization, the sum over final electron states is equivalent to summing $2\,\mathrm{Re}M^{(2)*}M^{(3)}$ for each polarization. Further

$$\left.\begin{array}{l}\mathrm{Re}Q^{(2)}\text{ is even}\\ \mathrm{Im}Q^{(2)}\text{ is odd}\end{array}\right\}\text{for polarization }\mathbf{e},$$

$$\left.\begin{array}{l}\mathrm{Re}Q^{(2)}\text{ is odd}\\ \mathrm{Im}Q^{(2)}\text{ is even}\end{array}\right\}\text{for polarization }\mathbf{e}',$$

so that, denoting even and odd parts by $E$ and $O$, respectively,

$$O(2\,\mathrm{Re}M^{(2)*}M^{(3)})_e = 2|c_2|c_3\tau^{-2}[\mathrm{Re}Q^{(2)}O(\mathrm{Im}Q^{(3)}) - \mathrm{Im}Q^{(2)}E(\mathrm{Re}Q^{(3)})], \tag{A2a}$$

$$O(2\,\mathrm{Re}M^{(2)*}M^{(3)})_{e'} = 2|c_2|c_3\tau^{-2}[\mathrm{Re}Q^{(2)}E(\mathrm{Im}Q^{(3)}) - \mathrm{Im}Q^{(2)}O(\mathrm{Re}Q^{(3)})]. \tag{A2b}$$

Then since $K_0$ and $K_1$ are even and $K_2$ is odd for $\mathbf{p}\|\mathbf{s}$, we find

$$\begin{aligned}O(2\,\mathrm{Re}M^{(2)*}M^{(3)})_e &= 2|c_2|c_3\tau^{-2}k_0^2\sin\theta\Bigg\{\left(\frac{N\cos\theta+P}{\Gamma^2}-\frac{N\cos\theta-P}{\rho^2}\right)\Bigg[\frac{s_0L+mN}{\rho^2}\,\mathrm{Im}I_0-\frac{p_0L+mN}{\Gamma^2}\,\mathrm{Im}J_0\\ &\quad -L(p_0+m)\,\mathrm{Im}K_0-L\csc\theta\,\mathrm{Im}K_2\Bigg]+\left(\frac{L}{\rho^2}-\frac{L}{\Gamma^2}\right)\Bigg[\frac{s_0N+mL}{\rho^2}\cos\theta\,\mathrm{Im}I_0\\ &\quad -\frac{p_0N+mL}{\Gamma^2}\cos\theta\,\mathrm{Im}J_0+(p_0P-mR-p_0N\cos\theta-mL\cos\theta)\,\mathrm{Im}K_0-N\,\mathrm{Im}K_1\Bigg]\Bigg\}.\end{aligned} \tag{A3}$$

$$\begin{aligned}O(2\,\mathrm{Re}M^{(2)*}M^{(3)})_{e'} &= 2|c_2|c_3\tau^{-2}k_0^2\sin\theta\Bigg\{\left(\frac{2sL+k_0R}{\Gamma^2}-\frac{2pL-k_0R}{\rho^2}\right)\Bigg[\frac{s_0N+mL}{\rho^2}\,\mathrm{Im}I_0-\frac{p_0N+mL}{\Gamma^2}\,\mathrm{Im}J_0\\ &\quad +(p_0P\cos\theta-p_0N-mR\cos-mL)\,\mathrm{Im}K_0-N(\mathrm{Im}K_1\cos\theta+\mathrm{Im}K_2\sin\theta)\Bigg]\\ &\quad +\left(\frac{N-P\cos\theta}{\rho^2}-\frac{N+P\cos\theta}{\Gamma^2}\right)\Bigg[\frac{2s_0pL+2mpN}{\rho^2}\,\mathrm{Im}I_0-\frac{2p_0sL+2msN}{\Gamma^2}\,\mathrm{Im}J_0\\ &\quad +(p_0k_0R-mk_0P)\,\mathrm{Im}K_0+(2p_0L+2mN-k_0L)(\mathrm{Im}K_1-\mathrm{Im}K_2\cot\theta)\Bigg]\Bigg\}.\end{aligned} \tag{A4}$$

Here we see in detail that only the imaginary parts of the various integrals are involved, and that the contributions from $M_a^{(3)}$ and $M_b^{(3)}$ really do tend to cancel as was indicated in the qualitative argument of Sec. 2. In these cross terms there are still some terms which would yield contributions of the same order as the $K_{lm}$ terms. These have been weeded out in the expressions (A5) and (A6), which give the asymmetric part of the transition rate as imaginary parts of the integrals multiplied by coefficients. The lowest contribution of the $K_{lm}$ terms would be $O(1/m^2p^3)$. So we keep only terms of $O(1/mp^4)=(m/p)O(K_{lm})$ and of $O(1/p^5)=(m/p)^2O(K_{lm})$. We find then that

$$|\langle k,s|S^{(2)}+S^{(3)}|p\rangle|_{\mathrm{asym}}^2 = 2|c_2|c_3\tau^{-2}k_0\sin\theta\{a\,\mathrm{Im}I_0+b\,\mathrm{Im}J_0 + c\,\mathrm{Im}K_0+d\,\mathrm{Im}K_1+e\,\mathrm{Im}K_2\}, \tag{A5}$$

where

$$\begin{aligned}a&=(m\rho^4\Gamma^2)^{-1}\{(\Gamma^2-\rho^2)p[s_0^2+2s_0p_0+m^2-2p+k_0s\cos\theta]\\ &\quad +(s\rho^2-p\Gamma^2)[s_0^2+2s_0p_0+m^2+sp]-(\Gamma^2+\rho^2)[sk_0^2+p(2s_0p-ss_0-sp_0)\cos\theta]\},\\ b&=(m\Gamma^4\rho^2)^{-1}\{(\rho^2-\Gamma^2)s[p_0^2+2s_0p_0+m^2-sp+k_0s\cos\theta]\\ &\quad +(p\Gamma^2-s\rho^2)[p_0^2+2s_0p_0+m^2+sp]-(\Gamma^2+\rho^2)[pk_0(p_0+s_0)+s(2sp_0-pp_0-ps_0)\cos\theta]\},\\ c&=(2m\rho^2\Gamma^2)^{-1}\{(\Gamma^2-\rho^2)k_0[((m-s_0)(m-p_0)+sp)\cos\theta-2pk_0]\\ &\quad -(\Gamma^2+\rho^2)[k_0(p_0+m)(p-s)+2sp_0+pp_0+ps_0+2ps\cos\theta]\\ &\quad -2(sp^2-p\Gamma^2)[m^2+s_0p_0+sp_0+ps_0+sp+(2sp_0-pk_0)\cos\theta]\}.\\ d&=(m\Gamma^2\rho^2)^{-1}\{(\Gamma^2-\rho^2)[s_0^2+2p_0s_0+m^2-sp]+(p\Gamma^2-sp^2)(p_0+s_0)\cos\theta-(\Gamma^2+\rho^2)(pp_0+ps_0+sk_0-2sp_0)\cos\theta\},\\ e&=(m\Gamma^2\rho^2)^{-1}\{(\rho^2-\Gamma^2)[s_0^2+2p_0s_0+m^2-sp]\cot\theta+(p\Gamma^2-sp^2)(p_0+s_0)\sin\theta\\ &\quad -\tfrac{1}{2}(\Gamma^2+\rho^2)[k_0(p-s)\csc\theta+k_0(p+s)\sin\theta-(pp_0+ss_0+3ps_0-3sp_0)\cos\theta\cot\theta]\}.\end{aligned} \tag{A6}$$

## APPENDIX B

### Exact Evaluation of $K_0$, $\mathbf{K}$, and $K_{ij}$

Since integrals of the form of $I_0$ and $\mathbf{I}$ have been evaluated elsewhere,[1,9] only $\mathrm{Im}K_0$ and $\mathrm{Im}\mathbf{K}$ are required for the results of this paper. As we shall see, $\mathbf{K}$ and $K_{ij}$ may be written in terms of $K_0$ and simpler integrals. Since $\mathrm{Re}K_0$ may also be evaluated exactly by standard techniques,[7] or to a good approximation by Kacser's[3] method, the only unknown quantity is $\mathrm{Im}K_0$.

$$K_0=\lim_{\epsilon\to 0+}\int d^3q\frac{1}{[(\mathbf{q}-\mathbf{k})^2-\mathbf{s}^2-i\epsilon][\mathbf{q}^2-\mathbf{p}^2-i\epsilon][(\mathbf{q}-\mathbf{p})^2+\lambda^2][(\mathbf{q}-\mathbf{k}-\mathbf{s})^2+\lambda^2]}$$

$$=\int\left\{\frac{d^3q}{[(\mathbf{q}-\mathbf{p})^2+\lambda^2][(\mathbf{q}-\mathbf{k}-\mathbf{s})^2+\lambda^2]}\left[P\frac{1}{\mathbf{q}^2-\mathbf{p}^2}+\pi i\delta(\mathbf{q}^2-\mathbf{p}^2)\right]\left[P\frac{1}{(\mathbf{q}-\mathbf{k})^2-\mathbf{s}^2}+\pi i\delta[(\mathbf{q}-\mathbf{k})^2-\mathbf{s}^2]\right]\right\},\quad\text{(B1)}$$

whence

$$I_mK_0=\pi\int\left\{\frac{d^3q}{[(\mathbf{q}-\mathbf{p})^2+\lambda^2][(\mathbf{q}-\mathbf{k}-\mathbf{s})^2+\lambda^2]}\left[P\frac{\delta(\mathbf{q}^2-\mathbf{p}^2)}{(\mathbf{q}-\mathbf{k})^2-\mathbf{s}^2}+P\frac{\delta[(\mathbf{q}-\mathbf{k})^2-\mathbf{s}^2]}{\mathbf{q}^2-\mathbf{p}^2}\right]\right\}.\quad\text{(B2)}$$

The "principal value" is superfluous in (B2). We see this as follows. Consider the first principal value term:

$$(\mathbf{q}-\mathbf{k})^2-\mathbf{s}^2=\mathbf{q}^2+\mathbf{k}^2-2\mathbf{q}\cdot\mathbf{k}-\mathbf{s}^2$$
$$=\mathbf{p}^2+\mathbf{k}^2-\mathbf{s}^2-2pk\cos(\mathbf{q},\mathbf{k}),$$

where we have used the fact the $|\mathbf{q}|$ integration involves $\delta(\mathbf{q}^2-\mathbf{p}^2)$. Now

$$p^2+k^2-s^2-2pk\cos(\mathbf{q},\mathbf{k})\geqslant(p-k)^2-s^2\quad\text{(no vectors).}$$

But conservation of energy tells us that

$$(p^2+m^2)^{1/2}=(s^2+m^2)^{1/2}+k,$$
$$p^2+k^2-2k(p^2+m^2)^{1/2}=s^2,$$
$$(p-k)^2-s^2>0.$$

Therefore, the term $[(\mathbf{q}-\mathbf{k})^2-\mathbf{s}^2]^{-1}$ will involve no poles in the angular integration, and the principal value designation may be omitted. The second term is similar with the roles of $\mathbf{q}^2-\mathbf{p}^2$ and $(\mathbf{q}-\mathbf{k})^2-\mathbf{s}^2$ interchanged. We have then

$$\mathrm{Im}K_0=\pi(X+Y),$$

where

$$X=\int\frac{d^3q\,\delta(\mathbf{q}^2-\mathbf{p}^2)}{[(\mathbf{q}-\mathbf{p})^2+\lambda^2][(\mathbf{q}-\mathbf{k}-\mathbf{s})^2+\lambda^2][(\mathbf{q}-\mathbf{k})^2-\mathbf{s}^2]},$$

$$Y=\int\frac{d^3q\,\delta(\mathbf{q}^2-\mathbf{s}^2)}{[(\mathbf{q}-\mathbf{s})^2+\lambda^2][(\mathbf{q}+\mathbf{k}-\mathbf{p})^2+\lambda^2][(\mathbf{q}+\mathbf{k})^2-\mathbf{p}^2]}.\quad\text{(B3)}$$

The replacements $\mathbf{p}\to\mathbf{s}$, $\mathbf{k}\to-\mathbf{k}$ yield $X\to Y$. Performing the $\delta$ function integration we find

$$X=\tfrac{1}{2}p\int_{-1}^{1}d\mu\int_0^{2\pi}d\phi\,\{[2p^2(1-\mu)+\lambda^2]^{-1}\{p^2+k^2-s^2-2pk[\mu\cos\theta+(1-\mu^2)^{\frac{1}{2}}\sin\theta\cos\phi]\}^{-1}$$
$$\times\{p^2+l^2+\lambda^2-2pl[\mu\cos(\mathbf{p},\mathbf{l})+(1-\mu^2)^{\frac{1}{2}}\sin(\mathbf{p},\mathbf{l})\cos\phi]\}^{-1}\}.\quad\text{(B4)}$$

Recall $\theta$ is the angle between $\mathbf{p}$ and $\mathbf{k}$. Equation (B4) may be written as

$$X=\tfrac{1}{2}p\int_{-1}^{1}d\mu\int_0^{2\pi}d\phi\,[2p^2(1-\mu)+\lambda^2]^{-1}$$
$$\times[A-B\cos\phi]^{-1}[C-D\cos\phi]^{-1}.\quad\text{(B5)}$$

We've seen above that $A-B\cos\phi>0$ always, and, therefore, $A^2>B^2$. Conservation of energy also says that $|\mathbf{p}|>|\mathbf{k}+\mathbf{s}|=|\mathbf{l}|$. Thus, $C-D\cos\phi>0$, and $C^2>D^2$.

The $\phi$ integration then yields[10]

$$X=p\pi\int_{-1}^{1}d\mu\frac{1}{[2p^2(1-\mu)+\lambda^2][BC-AD]}$$
$$\times\left[\frac{B}{(A^2-B^2)^{1/2}}-\frac{D}{(C^2-D^2)^{1/2}}\right].\quad\text{(B6)}$$

Since

$$A^2-B^2=\Lambda^4-4p^2k^2\sin^2\theta-4\Lambda^2pk\mu\cos\theta+4p^2k^2\mu^2\quad\text{(B7)}$$
$$=r+n\mu+t\mu^2,$$

[9] H. Mitter and P. Urban, Acta Phys. Austriaca **7**, 311 (1953); M. Gavrila, Phys. Rev. **113**, 514 (1959).

[10] B. O. Pierce, *A Short Table of Integrals* (Ginn and Company, Boston, 1929), Eq. (298).

and

$$C^2-D^2=(p^2+l^2)^2-4p^2l^2\sin^2(\mathbf{p},\mathbf{l})-4(p^2+l^2)pl\mu\cos(\mathbf{p},\mathbf{l})+4p^2l^2\mu^2=e+f\mu+g\mu^2. \tag{B8}$$

We have

$$X=p\pi\int_{-1}^{1}d\mu\frac{1}{(a-b\mu)(c-d\mu)}\times\left(\frac{pk\sin\theta}{(r+n\mu+t\mu^2)^{1/2}}-\frac{pl\sin(\mathbf{p},\mathbf{l})}{(e+f\mu+g\mu^2)^{1/2}}\right), \tag{B9}$$

with

$$\begin{aligned}\Lambda^2&=p\ +k^2-s^2,\\ a&=2p^2+\lambda^2,\\ b&=2p^2,\\ c&=(p^2+l^2)pk\sin\theta-\Lambda^2 pl\sin(\mathbf{p},\mathbf{l}),\\ d&=2[\mathbf{p}\cdot\mathbf{l}pk\sin\theta-\mathbf{p}\cdot\mathbf{k}pl\sin(\mathbf{p},\mathbf{l}).\end{aligned} \tag{B10}$$

The $\mu$ integration is straightforward and results in[11]

$$\begin{aligned}X=\frac{p\pi}{bc-ad}\Bigg\{&\frac{bpk\sin\theta}{\alpha}\ln\left[\left(\frac{a+b}{a-b}\right)\left(\frac{2\alpha^2-(a-b)(bn+2at)+2b\alpha(r+n+t)^{1/2}}{2\alpha^2-(a+b)(bn+2at)+2b\alpha(r-n+t)^{1/2}}\right)\right]\\ &-\frac{bpl\sin(\mathbf{p},\mathbf{l})}{\beta}\ln\left[\left(\frac{a+b}{a-b}\right)\left(\frac{2\beta^2-(a-b)(bf+2ag)+2b\beta(e+f+g)^{1/2}}{2\beta^2-(a+b)(bf+2ag)+2b\beta(e-f+g)^{1/2}}\right)\right]\\ &-\frac{dpk\sin\theta}{\gamma}\ln\left[\left(\frac{c+d}{c-d}\right)\left(\frac{2\gamma^2-(c-d)(dn+2ct)+2d\gamma(r+n+t)^{1/2}}{2\gamma^2-(c+d)(dn+2ct)+2d\gamma(r-n+t)^{1/2}}\right)\right]\\ &+\frac{dpl\sin(\mathbf{p},\mathbf{l})}{\delta}\ln\left[\left(\frac{c+d}{c-d}\right)\left(\frac{2\delta^2-(c-d)(df+2cg)+2d\delta(e+f+g)^{1/2}}{2\delta^2-(c+d)(df+2cg)+2d\delta(e-f+g)^{1/2}}\right)\right]\Bigg\},\end{aligned} \tag{B11}$$

where

$$\begin{aligned}\alpha^2&=rb^2+abn+a^2t,\\ \beta^2&=eb^2+abf+a^2g,\\ \gamma^2&=rd^2+cdn+c^2t,\\ \delta^2&=ed^2+cdf+c^2g.\end{aligned} \tag{B12}$$

The replacements $\mathbf{p}\leftrightarrow\mathbf{s}$, $\mathbf{k}\leftrightarrow-\mathbf{k}$ in the parameters defined by (B7), (B8), (B10) and (B12) give $Y$. From (B10) and (B11) we see directly that if $\mathbf{s}$ is not parallel to $\mathbf{p}$, $\mathrm{Im}K_0$ involves terms which are odd in $\sin\theta$. These arise from the $\mathbf{k}\cdot\mathbf{s}$ term in (B1). However, if $\mathbf{p}\|\mathbf{s}$, then (B1) or (B10) and (B11) tell us that $\mathrm{Im}K_0$ is even. Note in connection with this that $\sin(\mathbf{p},\mathbf{l})=(k/l)\sin\theta$ for $\mathbf{p}\|\mathbf{s}$, and $\sin(\mathbf{p},\mathbf{l})=k\sin(\mathbf{p},\mathbf{k})+s\sin(\mathbf{p},\mathbf{s})/l$ for $\mathbf{p}$ and $\mathbf{s}$ not collinear. Of course, if we could integrate the result over all final electron momenta, the $\mathbf{k}\cdot\mathbf{s}$ dependence of $K_0$ would disappear leaving only contributions which are even in $\sin\theta$. For our case, $X$ assumes the simpler form below.

$$X=\frac{\pi}{2p\tau^2\rho^2}\ln\frac{4p^2}{\lambda^2}+\frac{\pi}{4p\mathbf{s}\cdot\boldsymbol{\tau}}\left\{\frac{1}{\tau^2}\ln\frac{\tau^4}{(p^2-l^2)^2}-\frac{1}{\rho^2}\ln\frac{\rho^4}{\Lambda^4-2p^2k^2}+\frac{1}{\Gamma^2}\ln\left[\left(\frac{\mathbf{p}\cdot\mathbf{v}}{\mathbf{p}\cdot\boldsymbol{\tau}}\right)\left(\frac{\Gamma^2(ps+s^2)+2k^2\mathbf{s}\cdot\mathbf{l}-\Lambda^2\mathbf{s}\cdot\mathbf{k}}{\Gamma^2(ps-s^2)-2k^2\mathbf{s}\cdot\mathbf{l}+\Lambda^2\mathbf{s}\cdot\mathbf{k}}\right)\right]\right\}, \tag{B13}$$

where

$$\mathbf{v}=\mathbf{p}+\mathbf{l}=\mathbf{p}+\mathbf{k}+\mathbf{s}. \tag{B14}$$

Of course, $Y$ follows from the substitutions mentioned previously.

The next task is to evaluate $\mathrm{Im}\mathbf{K}$. We note that in general $\mathbf{K}$ may be expanded in terms of the linearly independent set of momentum vectors $\mathbf{p}$, $\mathbf{k}$, $\mathbf{s}$,[8]

$$\begin{aligned}\mathbf{K}&=\int d^3q\mathbf{q}[(\mathbf{q}-\mathbf{k})^2-s^2-i\epsilon]^{-1}[\mathbf{q}^2-\mathbf{p}^2-i\epsilon]^{-1}\times[(\mathbf{q}-\mathbf{p})^2+\lambda^2]^{-1}[(\mathbf{q}-\mathbf{k}-\mathbf{s})^2+\lambda^2]^{-1}\\ &=\int\frac{\mathbf{q}d^3q}{(1)(2)(3)(4)}\\ &=A\mathbf{p}+B\mathbf{k}+C\mathbf{s}.\end{aligned} \tag{B15}$$

We are interested in the case $\mathbf{p}\|\mathbf{s}$ so that the expansion above may not be valid, but the one below certainly will be.

$$\mathbf{K}=A\mathbf{p}+B\mathbf{k}+C'\mathbf{p}\times\mathbf{k}. \tag{B16}$$

As might be expected, $C'=0$ for $\mathbf{p}\|\mathbf{s}$. We see this as as follows.

$$\mathbf{K}\cdot\mathbf{p}\times\mathbf{k}=C'(\mathbf{p}\times\mathbf{k})^2=\int d^2q\int_{-\infty}^{+\infty}\frac{q_{\mathbf{p}\times\mathbf{k}}dq_{\mathbf{p}\times\mathbf{k}}}{(1)(2)(3)(4)}|\mathbf{p}\times\mathbf{k}|.$$

But the integrand is an odd function of $q_{\mathbf{p}\times\mathbf{k}}$, q.e.d. Since we choose $\mathbf{p}$ along the 1 axis, $\mathbf{k}$ in the 1–2 plane,

[11] B. O. Pierce, reference 9, Eq. (195).

this means $K_3=0$, a fact we have used in Appendix A. Similar arguments hold for **I**, **J**.

We evaluate $A$ and $B$ next. Noting that

$$\begin{aligned}2\mathbf{p}\cdot\mathbf{q}&=q^2+p^2+\lambda^2-(3)\\&=(2)-(3)+2p^2+\lambda^2+i\epsilon\\&=(2)-(3)+2p^2,\end{aligned}$$

since eventually both $\lambda$ and $\epsilon\rightarrow 0$. Also,

$$2\mathbf{k}\cdot\mathbf{q}=(2)-(1)+\Lambda^2.$$

We have then

$$\begin{aligned}\mathbf{p}\cdot\mathbf{K}=M&=\frac{1}{2}\int d^3q\frac{[(2)-(3)+2p^2]}{(1)(2)(3)(4)},\\ \mathbf{k}\cdot\mathbf{K}=N&=\frac{1}{2}\int d^3q\frac{[(2)-(1)+\Lambda^2]}{(1)(2)(3)(4)},\end{aligned}\tag{B17}$$

or

$$\mathbf{K}=\frac{Mk^2-N\mathbf{p}\cdot\mathbf{k}}{p^2k^2-(\mathbf{p}\cdot\mathbf{k})^2}\mathbf{p}+\frac{Np^2-M\mathbf{p}\cdot\mathbf{k}}{p^2k^2-(\mathbf{p}\cdot\mathbf{k})^2}\mathbf{k},\tag{B18}$$

where

$$\begin{aligned}M&=p^2K_0+\tfrac{1}{2}I_0-\tfrac{1}{2}J_0',\\ N&=\tfrac{1}{2}\Lambda^2K_0+\tfrac{1}{2}I_0-\tfrac{1}{2}J_0,\end{aligned}\tag{B19}$$

$$J_0'=\int d^3q[(\mathbf{q}-\mathbf{k})^2-\mathbf{s}^2-i\epsilon]^{-1}[\mathbf{q}^2-\mathbf{p}^2-i\varepsilon]^{-1}\times[(\mathbf{q}-\mathbf{k}-\mathbf{s})^2+\lambda^2]^{-1}.\tag{B20}$$

Thus we see that **K** may be written in terms of $K_0$ and simpler integrals.

For completeness we also indicate how $K_{ij}$ may be evaluated.

$$\mathsf{K}=\int d^3q\frac{\mathbf{qq}}{(1)(2)(3)(4)}.\tag{B21}$$

This is simply a dyadic constructed from the linearly independent vectors **p**, **k** and **p**×**k**.

$$\mathsf{K}=(a\mathbf{p}+b\mathbf{k}+c\mathbf{p}\times\mathbf{k})(\alpha\mathbf{p}+\beta\mathbf{k}+\gamma\mathbf{p}\times\mathbf{k}).\tag{B22}$$

We have then

$$\mathbf{p}\cdot\mathsf{K}\cdot\mathbf{p}=(ap^2+b\mathbf{p}\cdot\mathbf{k})(\alpha p^2+\beta\mathbf{p}\cdot\mathbf{k})=\int d^3q\frac{(p\cdot q)^2}{(1)(2)(3)(4)}.\tag{B23}$$

Since

$$\mathbf{p}\cdot\mathbf{q}=\tfrac{1}{2}[(2)-(3)+2p^2],$$

we find

$$\mathbf{p}\cdot\mathsf{K}\cdot\mathbf{p}=\frac{1}{4}\bigg\{p^4K_0+3p^2I_0-3p^2J_0'-2\mathbf{p}\cdot\mathbf{J}_0'+\int\frac{q^2d^3q}{(1)(3)(4)}+\int\frac{q^2d^3q}{(1)(2)(4)}-2\int\frac{d^3q}{(1)(4)}\bigg\}.\tag{B24}$$

Constructing the other products of **p**, **k**, and **p**×**k** with $\mathsf{K}$ then enables us to obtain the coefficients in (B22) in terms of $K_0$ and simpler integrals.

We conclude this appendix by listing the results for $\mathrm{Im}J_0$, neglecting terms of $O(\lambda)$ and higher.[8]

$$\mathrm{Im}J_0=\frac{\pi^2}{2p(\mathbf{p}-\mathbf{l})^2}\ln\left[\left(\frac{4p^2}{\lambda^2}\right)\frac{\tau^4}{(p^2-l^2)^2}\right].$$

The usual substitutions relate $J_0$ to $I_0$.

## APPENDIX C. THE DIVERGENCES

In order to avoid the difficulties associated with the infinite range of the Coulomb potential we have introduced the usual exponential cutoff factor $e^{-\lambda r}$ in Eq. (2.1), and pointed out that we would be interested eventually in the limit $\lambda\rightarrow 0$. However, in Appendix B we saw that all of the integrals involved contain a term of order $\log\lambda$, which, of course, diverges in this limit. In Sec. 2 it was pointed out that these divergences correspond to zero-momentum insertions into external electron lines, and, therefore, lead to some phase modification. Physically, it would be unfortunate if interference effects depending on $\lambda$ should survive. For the case of Coulomb scattering, Dalitz[1] has shown that these phase modifications are actually unobservable, at least through the third Born approximation. Kacser[3] has proven that the $\lambda$-dependent terms cancel in bremsstrahlung to the order which we calculate here. Since we have neglected certain terms in the calculation, it expedient to separate off explicitly the divergent part of each term we retain. In order to do this we shall find it convenient to repeat Kacser's proof.

As mentioned above, the divergences arise from zero-momentum transfers to the nucleus, which correspond to the neighborhoods $\mathbf{q}=\mathbf{p}$, $\mathbf{s}$, or $\mathbf{l}$ in the integrals of $M^{(3)}$. The treatment of $J_0$ below is typical. From Eq. (B5), we have

$$\mathrm{Im}J_0=\frac{\pi p}{2}\int_0^{2\pi}d\phi\int_{-1}^{1}d\mu[2p^2(1-\mu)+\lambda^2]^{-1}\{p^2+l^2+\lambda^2-2pl[\mu\cos(\mathbf{p},\mathbf{l})+(1-\mu^2)^{1/2}\sin(\mathbf{p},\mathbf{l})\cos\phi]\}^{-1}.$$

The divergent part arises from the neighborhood of $\mu=1$. We can separate off a region of the $\mu$ integration, say from $1-\delta$ to 1, where $p\gg\delta$, $p\gg\lambda$. Then we have

$$\mathrm{div}J_0=\frac{\pi ip}{2}\int_0^{2\pi}d\phi\int_{1-\delta}^{1}d\mu[2p^2(1-\mu)+\lambda^2]^{-1}\times[p^2+l^2+\lambda^2-2\mathbf{p}\cdot\mathbf{l}\mu]^{-1},$$

or

$$\mathrm{div}J_0=\frac{\pi^2 i}{2p\tau^2}\ln\frac{4p^2}{\lambda^2}+O(\delta)$$

The $O(\delta)$ terms are independent of the limit $\lambda\rightarrow 0$, so that we have the expected result

$$\operatorname{div} J_0 = \frac{\pi^2 i}{2p\tau^2} \ln \frac{4p^2}{\lambda^2}. \quad \text{(C1)}$$

This term corresponds to $\mathbf{r}=0$ in Fig. 2(b), so that the insertion is into an electron line of momentum $p$, and $\lambda$ is compared to $p$. The form (C1) is that for which the cancellations occur, and is the most convenient for the subtractions. Similarly we find

$$\begin{aligned} \operatorname{div} I_0 &= (\pi^2 i/2s\tau^2) \ln(4s^2/\lambda^2), \\ \operatorname{div} \mathbf{I} &= \mathbf{s} \operatorname{div} I_0, \\ \operatorname{div} \mathbf{J} &= \mathbf{p} \operatorname{div} J_0, \\ \operatorname{div} K_0 &= (1/\rho^2) \operatorname{div} J_0 - (1/\Gamma^2) \operatorname{div} I_0. \end{aligned} \quad \text{(C2)}$$

Thus for the divergent part of $M_a^{(3)}$, we have

$$\begin{aligned} \operatorname{div} M_a^{(3)} &= c_3 \bar{v}(s)[(s_4\gamma_4 + im) \operatorname{div} I_0 - \boldsymbol{\gamma}\cdot\mathbf{s} \operatorname{div} I_0] \\ &\quad \times [2p_\mu e_\mu + \tilde{e}\tilde{k}]\rho^{-2} u(p) \quad \text{(C3)} \\ &= 2s_4\tau^2 c_2^{-1} c_3 M_a^{(2)} \operatorname{div} I_0. \end{aligned}$$

Similarly,

$$\operatorname{div} M_b^{(3)} = 2p_4 c_2^{-1} c_3 M_b^{(2)} \operatorname{div} J_0 \quad \text{(C4)}$$

In $M_c^{(3)}$ we have two divergent terms from $\mathbf{q}=\mathbf{p}$ and $\mathbf{q}=\mathbf{l}$, respectively.

$$\begin{aligned} \operatorname{div} M_c^{(3)} &= c_3\rho^{-2}\bar{v}\gamma_4(\tilde{p} - \tilde{k} + im)\tilde{e}(\tilde{p} + im)\gamma_4 u \operatorname{div} J_0 \\ &\quad - c_3\Gamma^{-2}\bar{v}\gamma_4(\mathbf{l}\cdot\boldsymbol{\gamma} - \tilde{k} + p_4\gamma_4 + im) \\ &\quad \times \tilde{e}(\mathbf{l}\cdot\boldsymbol{\gamma} + p_4\gamma_4 + im)\gamma_4 u \operatorname{div} I_0. \end{aligned}$$

As above, we find

$$\begin{aligned} \operatorname{div} M_c^{(3)} &= 2c_3 p_4 \rho^{-2} \bar{v}(s)\gamma_4(2\mathbf{p}\cdot\mathbf{e} + \tilde{e}\tilde{k})u(p) \operatorname{div} J_0 \\ &\quad - 2c_3 s_4 \Gamma^{-2} \bar{v}(s)\gamma_4(2\mathbf{s}\cdot\mathbf{e} + \tilde{e}\tilde{k})u(p) \operatorname{div} I_0. \quad \text{(C5)} \end{aligned}$$

Whence,

$$\operatorname{div} M_c^{(3)} = 2c_3 c_2^{-1} \tau^2 [p_4 M_a^{(2)} \operatorname{div} J_0 + s_4 M_b^{(2)} \operatorname{div} I_0]. \quad \text{(C6)}$$

Collecting these results we have

$$\operatorname{div} M^{(3)} = 2c_3 c_2^{-1} \tau^2 M^{(2)} [s_4 \operatorname{div} I_0 + p_4 \operatorname{div} J_0]. \quad \text{(C7)}$$

Here the square bracket is real, and $c_3 c_2^{-1}$ is imaginary. Therefore,

$$\begin{aligned} &\operatorname{div} 2\operatorname{Re} M^{(2)*} M^{(3)} \\ &= \operatorname{div}[M^{(2)*} M^{(3)} + M^{(2)} M^{(3)*}] \\ &= 2\tau^2 c_3 c_2^{-1} (s_4 \operatorname{div} I_0 + p_4 \operatorname{div} J_0) \quad \text{(C8)} \\ &\qquad \times [M^{(2)*} M^{(2)} - M^{(2)} M^{(2)*}] \\ &= 0. \end{aligned}$$

It is worth mentioning that even if only the $\ln\lambda$ terms are kept in (C7), as opposed to $\ln(\lambda^2/\text{momenta})$ terms, that $\operatorname{div}|M^{(3)}|^2$ survives. The conjecture is that those $\ln\lambda$ terms surviving in $|M^{(3)}|^2$ are cancelled by the cross term $2\operatorname{Re} M^{(2)*} M^{(4)}$.

We also see that our expression for the asymmetric part of the cross section may be written in a divergence free form by omitting the terms in $\ln(\lambda^2/\text{momenta})$, as they stand, in the expression for the integrals in Appendix B.