PHYSICAL REVIEW VOLUME 129, NUMBER 3 1 FEBRUARY 1963

Spin-Wave Theory for Cubic Ferromagnetics

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(Received 26 June 1961; revised manuscript received 16 August 1961)

A modification of Dyson's theory of a cubic ferromagnetic is proposed, based on a consideration of the smallness of the kinematical interaction of spin waves at low temperatures and leading to the introduction of Bloch's statistical function with Dyson's Hamiltonian or, equivalently, of the grand partition function. This approach makes possible the use of thermodynamical perturbation calculus according to Matsubara, whose formalism yields the logarithm of the grand partition function as the sum of Bloch's term and distinct parts of the connected ladder diagrams (Dyson's term). In the long spin-wave approximation (at low temperatures), Bloch's term yields a contribution to the magnetization in the form of a series of 3/2, 5/2, and 7/2 powers of the Kelvin/Curie temperature ratio. Dynamical interaction of the spin waves contributes to the magnetization at least as T^4 by means of the connected diagrams of all order. All coefficients in the magnetization series obtained are identical with the corresponding quantities in Dyson's theory.

1. INTRODUCTION

IN the present paper, a spin-wave theory is proposed which is based on the highly effective and, at the same time, simple formalism of the grand partition function and thermodynamical perturbation calculus of Matsubara.[1] Obviously, to apply this formalism to spin-wave theory would be a highly involved procedure, if it were not for the very interesting result obtained by Dyson[2] that the contribution from kinematical interaction of the spin waves to the free energy is less than $\exp(-2\pi a T_c/T)$, where a is a positive number of the order of unity, T is the absolute, and T_c the Curie temperature. Thus, at sufficiently low temperatures, the contribution from kinematical interaction to the magnetization is small. This provides the fundamental assumption for our entire investigation.

We should mention that the earliest quantum theory of the magnetization in a one-electron ferromagnetic is that of Bloch[3]; this was subsequently generalized for many electron atoms by Moeller.[4] Bloch's $T^{3/2}$ law is satisfactorily justified by the experimental results. The same law has been derived by others; for example, the semiclassical treatment of Kittel.[5] The earliest trans-Blochian contribution of order T^2 to the magnetization in poor agreement with experiment, was derived by Kramers[6] and Opechowski.[7] Later, Holstein and Primakoff[8] calculated the magnetization, taking into account dipolar interaction between the electrons. Schafroth[9] and van Kranendonk[10] derived correctional terms of order $T^{7/4}$ with different coefficients. Exact calculation of the temperature dependence of the magnetization was first carried out by Dyson[2], who proved that the trans-Blochian corrections are due to a more exact treatment of the energy spectrum (in addition to the quadratic term in the wave vector, higher power terms are taken into account) and to dynamical interaction of the spin waves. An interesting theory of retarded and advanced Green functions was proposed by Bogolyubov and Tyablikov[11] and applied by them to an isotropic ferromagnetic. It proves to be valid within the entire temperature range. Opechowski[12] perfected his theory, obtaining results which coincide with those of Dyson. Finally, recent papers by Oguchi,[13] who takes up the Holstein-Primakoff theory in stricter form, and by Keffer and Loudon,[14] who take into account spin wave interaction, lead to results almost identical with Dyson's.

2. THE GRAND PARTITION FUNCTION

We shall consider a cubic crystalline lattice of N atoms of one kind, to each of which is related the spin operator $\mathbf{S}_j$ in coordination with the spin quantum number S. As shown by Dyson,[15] this system is described by the Hamiltonian

$$\mathcal{H}=E_0+H_0+H_I, \tag{1}$$

with

$$E_0=LSN-\tfrac{1}{2}JNS^2\gamma_0, \tag{2}$$

$$H_0=\textstyle\sum_\lambda(L+\epsilon_\lambda)\alpha_\lambda{}^*\alpha_\lambda, \tag{3}$$

$$H_I=-\tfrac{1}{4}JN^{-1}\textstyle\sum_{\lambda\rho\sigma}\Gamma_{\rho\sigma}{}^\lambda\alpha_{\sigma+\lambda}{}^*\alpha_{\rho-\lambda}{}^*\alpha_\rho\alpha_\sigma, \tag{4}$$

$$\epsilon_\lambda=JS(\gamma_0-\gamma_\lambda),\quad \gamma_\lambda=\textstyle\sum_\delta\exp(i\boldsymbol{\delta}\cdot\boldsymbol{\lambda}), \tag{5}$$

$$\Gamma_{\rho\sigma}{}^\lambda=\gamma_\lambda+\gamma_{\sigma+\lambda-\rho}-\gamma_{\sigma+\lambda}-\gamma_{\rho-\lambda}, \tag{6}$$

where $L=mH/S$, and m is the magnetic moment of each spin, H, the magnetic field strength, J, the exchange integral for nearest neighbors in the lattice, and γ_0, the number of nearest neighbors. Summation

[1] T. Matsubara, Prog. Theoret. Phys. (Kyoto) **14**, 351 (1955).
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[4] C. Moeller, Z. Physik **82**, 559 (1933).
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[12] W. Opechowski, Physica **25**, 476 (1959).
[13] T. Oguchi, Phys. Rev. **117**, 117 (1960).
[14] F. Keffer and R. Loudon, Suppl. J. Appl. Phys. **32**, 2S (1961).
[15] F. J. Dyson, Phys. Rev. **102**, 1217 (1956).

over $\boldsymbol{\delta}$ runs over all lattice points nearest to the one under consideration. $\boldsymbol{\lambda}$ is the vector of the reciprocal lattice:

$$\boldsymbol{\lambda}=2\pi N^{-1/3}\sum_{i=1}^{3}\lambda_i\mathbf{b}^i,\quad \lambda_i\leqslant\left|\frac{N^{1/3}-1}{2}\right|, \tag{7}$$

where λ_i are integers and $\mathbf{b}^i$ are three reciprocal vectors. The oscillator operators of annihilation and creation of ideal spin waves α_λ, α_λ^*, satisfy the well-known commutation rules

$$[\alpha_\lambda,\alpha_\mu]=[\alpha_\lambda^*,\alpha_\mu^*]=0,\quad [\alpha_\lambda,\alpha_\mu^*]=\delta_{\lambda,\mu}. \tag{8}$$

Invoking the fact that, at low temperatures, kinematical interaction is negligible, we shall treat the spin waves as a weakly imperfect Bose-gas with energy spectrum (5) and interactions (4). Accordingly, we write the statistical function as

$$Z=\mathrm{Spur}[\exp(-\beta\mathcal{H})]=\textstyle\sum_a(a|\exp(-\beta\mathcal{H})|a),\quad \beta=(kT)^{-1}, \tag{9}$$

with

$$|a)=\textstyle\prod_\lambda[(a_\lambda!)^{-1/2}(\alpha_\lambda^*)^{a_\lambda}]|\mathbf{0});\tag{10}$$

a_λ is a positive integer, and $|\mathbf{0})$, the ferromagnon vacuum-state vector. On the other hand, according to Matsubara[1], we have

$$\begin{aligned}\exp(-\beta\mathcal{H})&=\exp(-\beta E_0)\cdot\exp[-\beta(H_0+H_I)]\\&=\exp(-\beta E_0)\cdot\exp(-\beta H_0)\cdot S(\beta),\end{aligned}\tag{11}$$

with

$$S(\beta)=P\exp\left[-\int_0^\beta d\tau H_I(\tau)\right]=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^\beta d\tau_1\cdots\times\int_0^\beta d\tau_n P[H_I(\tau_1)\cdots H_I(\tau_n)],\tag{12}$$

where the dynamical operator

$$H_I(\tau)=\exp(\tau H_0)\cdot H_I\cdot\exp(-\tau H_0)\tag{13}$$

has been written in Matsubara's representation. P is Dyson's[16] ordering operator in arguments τ. In the derivations that follow, P is replaced by Wick's[17] ordering symbol T as, for the Bose field, we have $P=T$.

With the notation

$$Z_0=\exp(-\beta E_0)\cdot\textstyle\sum_a(a|\exp(-\beta H_0)|a),\tag{14}$$

we have, by Eqs. (9), (11), and (14),

$$\frac{Z}{Z_0}=\frac{\sum_a(a|\exp(-\beta H_0)\cdot S(\beta)|a)}{\sum_a(a|\exp(-\beta H_0)|a)}=\langle S(\beta)\rangle.\tag{15}$$

The brackets $\langle\cdots\rangle$ indicate averaging.

Moreover, from Matsubara,[1] the mean value of the operator $S(\beta)$ is given by

$$\langle S(\beta)\rangle=\exp(\sum_1^\infty\Gamma_i),\tag{16}$$

where Γ_i is the sum of all distinct parts of the connected diagram of order i. Hence,

$$Z=Z_0\cdot\exp(\sum_1^\infty\Gamma_i).\tag{17}$$

3. MATSUBARA'S FORMALISM

In computing the mean value $\langle S(\beta)\rangle$ we apply the formalism developed by Matsubara. By Eq. (3), it is easily verified that

$$\begin{aligned}\alpha_\lambda(\tau)&=\exp(\tau H_0)\alpha_\lambda\exp(-\tau H_0)\\&=\alpha_\lambda\exp[-(L+\epsilon_\lambda)\tau],\end{aligned}\tag{18}$$

$$\begin{aligned}\alpha_\lambda^*(\tau)&=\exp(\tau H_0)\alpha_\lambda^*\exp(-\tau H_0)\\&=\alpha_\lambda^*\exp[(L+\epsilon_\lambda)\tau].\end{aligned}\tag{19}$$

We now split the operators $\alpha_\lambda(\tau)$, $\alpha_\lambda^*(\tau)$ into negative and positive parts:

$$\begin{aligned}\alpha_\lambda(\tau)&=\alpha_\lambda^{(-)}(\tau)+\alpha_\lambda^{(+)}(\tau),\\ \alpha_\lambda^*(\tau)&=\alpha_\lambda^{*(-)}(\tau)+\alpha_\lambda^{*(+)}(\tau),\end{aligned}\tag{20}$$

$$\begin{aligned}\alpha_\lambda^{(-)}(\tau)&=\pi_\lambda\alpha_\lambda\exp[-(L+\epsilon_\lambda)\tau],\\ \alpha_\lambda^{(+)}(\tau)&=(1-\pi_\lambda)\alpha_\lambda\exp[-(L+\epsilon_\lambda)\tau],\end{aligned}\tag{21}$$

$$\begin{aligned}\alpha_\lambda^{*(-)}(\tau)&=\omega_\lambda\alpha_\lambda^*\exp[(L+\epsilon_\lambda)\tau],\\ \alpha_\lambda^{*(+)}(\tau)&=(1-\omega_\lambda)\alpha_\lambda^*\exp[(L+\epsilon_\lambda)\tau].\end{aligned}\tag{22}$$

The coefficients π_λ and ω_λ will be computed later.

As usual, we define the normal product (N product) of a certain number of field operators as a product in which all negative operator parts stand to the right of the positive ones:

$$\begin{aligned}N[\alpha_\lambda(\tau_1)\alpha_\sigma^*(\tau_2)]&=\alpha_\lambda^{(-)}(\tau_1)\alpha_\sigma^{*(-)}(\tau_2)+\alpha_\lambda^{(+)}(\tau_1)\alpha_\sigma^{*(+)}(\tau_2)\\&\quad+\alpha_\sigma^{*(+)}(\tau_2)\alpha_\lambda^{(-)}(\tau_1)+\alpha_\lambda^{(+)}(\tau_1)\alpha_\sigma^{*(-)}(\tau_2)\\&=\alpha_\lambda(\tau_1)\alpha_\sigma^*(\tau_2)-[\alpha_\lambda^{(-)}(\tau_1),\alpha_\sigma^{*(+)}(\tau_2)]\\&=\alpha_\sigma^*(\tau_2)\alpha_\lambda(\tau_1)+[\alpha_\lambda(\tau_1),\alpha_\sigma^*(\tau_2)]\\&\qquad-[\alpha_\lambda^{(-)}(\tau_1),\alpha_\sigma^{*(+)}(\tau_2)].\end{aligned}\tag{23}$$

We define the T product as in quantum-field theory:

$$\begin{aligned}T[\alpha_\lambda(\tau_1)\alpha_\sigma^*(\tau_2)]&=\alpha_\lambda(\tau_1)\alpha_\sigma^*(\tau_2),\quad \tau_1>\tau_2,\\&=\alpha_\sigma^*(\tau_2)\alpha_\lambda(\tau_1),\quad \tau_1<\tau_2.\end{aligned}\tag{24}$$

Thus, the τ contraction of the two boson operators is of the form

$$T[\alpha_\lambda(\tau_1)\alpha_\sigma^*(\tau_2)]=N[\alpha_\lambda(\tau_1)\alpha_\sigma^*(\tau_2)]+\alpha_\lambda(\tau_1)^c\alpha_\sigma^*(\tau_2)^c.\tag{25}$$

[16] F. J. Dyson, Phys. Rev. **75**, 486 (1949).
[17] G. C. Wick, Phys. Rev. **80**, 268 (1950).

Hence, by Eqs. (20)-(24),

$$\alpha_\lambda(\tau_1)^c\alpha_\sigma{}^*(\tau_2)^c=\delta_{\lambda,\sigma}\exp[-(L+\epsilon_\lambda)\cdot(\tau_1-\tau_2)]\times\{\pi_\lambda(1-\omega_\lambda)\theta(\tau_1-\tau_2)+[\pi_\lambda(1-\omega_\lambda)-1]\theta(\tau_2-\tau_1)\},\quad(26)$$

with

$$\theta(x)=1,\quad x>0,\qquad =0,\quad x<0.\quad(27)$$

We now require that

$$\langle N[\alpha_\lambda(\tau_1)\alpha_\sigma{}^*(\tau_2)]\rangle=\frac{\sum_a(a|\exp(-\beta H_0)N[\alpha_\lambda(\tau_1)\alpha_\sigma{}^*(\tau_2)]|a)}{\sum_a(a|\exp(-\beta H_0)|a)}=0.\quad(28)$$

From Eqs. (10), (18), and (19), we have

$$\langle\alpha_\lambda\alpha_\lambda{}^*\rangle=\pi_\lambda(1-\omega_\lambda),\quad(29)$$

with

$$\langle\alpha_\lambda\alpha_\lambda{}^*\rangle=n_\lambda+1;\quad n_\lambda=\frac{\sum_{a_\lambda=0}^{\infty}a_\lambda\exp[-\beta(L+\epsilon_\lambda)a_\lambda]}{\sum_{a_\lambda=0}^{\infty}\exp[-\beta(L+\epsilon_\lambda)a_\lambda]}=\{\exp[\beta(L+\epsilon_\lambda)]-1\}^{-1}.\quad(30)$$

Thus, Eq. (26) becomes

$$\alpha_\lambda(\tau_1)^c\alpha_\sigma{}^*(\tau_2)^c=\delta_{\lambda,\sigma}\exp[-(L+\epsilon_\lambda)\cdot(\tau_1-\tau_2)]\times[\theta(\tau_1-\tau_2)\cdot(n_\lambda+1)+\theta(\tau_2-\tau_1)n_\lambda].\quad(31)$$

Similarly, we obtain

$$\alpha_\lambda{}^*(\tau_1)^c\alpha_\sigma(\tau_2)^c=\delta_{\lambda,\sigma}\exp[(L+\epsilon_\lambda)\cdot(\tau_1-\tau_2)]\times[\theta(\tau_1-\tau_2)n_\lambda+\theta(\tau_2-\tau_1)\cdot(n_\lambda+1)]\quad(32)$$

and

$$-\omega_\lambda(1-\pi_\lambda)=n_\lambda.\quad(33)$$

In the special case of coinciding arguments τ, we have

$$\alpha_\lambda(\tau)^c\alpha_\sigma{}^*(\tau)^c=\delta_{\lambda,\sigma}(n_\lambda+1)\quad(34)$$

and

$$\alpha_\lambda{}^*(\tau)^c\alpha_\sigma(\tau)^c=\delta_{\lambda\,\sigma}n_\lambda.\quad(35)$$

From Eqs. (29), (30), and (33), we obtain

$$\pi_\lambda=n_\lambda+1\pm[n_\lambda(n_\lambda+1)]^{1/2},\qquad \omega_\lambda=-n_\lambda\pm[n_\lambda(n_\lambda+1)]^{1/2}.\quad(36)$$

By a theorem derived by Thouless,[18] we have quite generally

$$\langle N[\alpha_{\lambda_1}{}^*(\tau_1)\alpha_{\lambda_2}{}^*(\tau_2)\cdots\alpha_{\lambda_n}{}^*(\tau_n)\alpha_{\sigma_1}(\tau_{n+1})\alpha_{\sigma_2}(\tau_{n+2})\cdots\times\alpha_{\sigma_n}(\tau_{2n})]\rangle=0,\quad n=1,2,3,\cdots,\infty.\quad(37)$$

By the last equation, the mean value of the T product is obtained as the sum of all possible contractions. Since these are c numbers, they can be put outside the operation of averaging.

4. GRAPHICAL INTERPRETATION[19]

Returning to the mean value of $S(\beta)$, we have, by Eqs. (4) and (12),

$$\langle S(\beta)\rangle=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^\beta d\tau_1\cdots\int_0^\beta d\tau_n\langle T[H_I(\tau_1)\cdots H_I(\tau_n)]\rangle=1-\int_0^\beta d\tau\langle T[H_I(\tau)]\rangle+\frac{1}{2!}\int_0^\beta d\tau_1\int_0^\beta d\tau_2\langle T[H_I(\tau_1)H_I(\tau_2)]\rangle-\frac{1}{3!}\int_0^\beta d\tau_1\int_0^\beta d\tau_2\int_0^\beta d\tau_3\langle T[H_I(\tau_1)H_I(\tau_2)H_I(\tau_3)]\rangle+\cdots,\quad(38)$$

where

$$H_I(\tau)=-\tfrac{1}{4}JN^{-1}\sum_{\lambda\rho\sigma}\Gamma_{\rho\sigma}{}^\lambda\alpha_{\sigma+\lambda}{}^*(\tau)\alpha_{\rho-\lambda}{}^*(\tau)\alpha_\rho(\tau)\alpha_\sigma(\tau).\quad(39)$$

Examples of distinct parts of connected ladder diagrams up to the fourth order inclusively are illustrated in Fig. 1. Let us compute the diagram Γ_1:

$$\Gamma_1=-\int_0^\beta d\tau\langle T[H_I(\tau)]\rangle=\tfrac{1}{4}JN^{-1}\sum_{\lambda\rho\sigma}\Gamma_{\rho\sigma}{}^\lambda\int_0^\beta d\tau\langle T[\alpha_{\sigma+\lambda}{}^*(\tau)\alpha_{\rho-\lambda}{}^*(\tau)\alpha_\rho(\tau)\alpha_\sigma(\tau)]\rangle=\tfrac{1}{4}JN^{-1}\sum_{\lambda\rho\sigma}\Gamma_{\rho\sigma}{}^\lambda$$
$$\times\int_0^\beta d\tau[\alpha_{\sigma+\lambda}{}^*(\tau)^c\alpha_{\rho-\lambda}{}^*(\tau)^{cc}\alpha_\rho(\tau)^c\alpha_\sigma(\tau)^{cc}+\alpha_{\sigma+\lambda}{}^*(\tau)^c\alpha_{\rho-\lambda}{}^*(\tau)^{cc}\alpha_\rho(\tau)^{cc}\alpha_\sigma(\tau)]=\tfrac{1}{4}\beta JN^{-1}\sum_{\lambda\rho\sigma}\Gamma_{\rho\sigma}{}^\lambda$$
$$\times[\delta(\boldsymbol{\sigma}+\boldsymbol{\lambda}-\boldsymbol{\rho})\delta(\boldsymbol{\rho}-\boldsymbol{\lambda}-\boldsymbol{\sigma})n_\rho n_\sigma+\delta(\boldsymbol{\sigma}+\boldsymbol{\lambda}-\boldsymbol{\rho})\delta(\boldsymbol{\rho}-\boldsymbol{\lambda}-\boldsymbol{\rho})n_\rho n_\sigma]=\tfrac{1}{4}\beta JN^{-1}\sum_{\rho\sigma}[\Gamma_{\rho\sigma}{}^{\rho-\sigma}+\Gamma_{\rho\sigma}{}^0]n_\rho n_\sigma.\quad(40)$$

The terms in the square brackets are equal. In order to prove this, we must take into consideration that, by Eq. (6):

$$\Gamma_{\rho\sigma}{}^{\rho-\sigma}=\gamma_{\rho-\sigma}+\gamma_0-\gamma_{-\sigma}-\gamma_\rho=\Gamma_{\rho\sigma}{}^0,\quad(41)$$

since $\gamma_\lambda=\gamma_{-\lambda}$ as a result of lattice symmetry. Hence,

$$\Gamma_1=\tfrac{1}{2}\beta JN^{-1}\sum_{\rho\sigma}\Gamma_{\rho\sigma}{}^0n_\rho n_\sigma.\quad(42)$$

[18] D. J. Thouless, Phys. Rev. **107**, 1162 (1957).
[19] R. P. Feynman, Phys. Rev. **76**, 769 (1949).

5. BLOCH'S PART IN THE MAGNETIZATION SERIES

Returning to Eq. (17), we now compute the Z_0 term in the grand partition function. By (14), we have

$$Z_0=\exp(-\beta E_0)\sum_a(a|\exp(-\beta H_0)|a)$$
$$=\exp(-\beta E_0)\prod_\lambda \sum_{a_\lambda=0}^{\infty}\exp[-\beta(L+\epsilon_\lambda)a_\lambda]$$
$$=\exp(-\beta E_0)\prod_\lambda\{1-\exp[-\beta(L+\epsilon_\lambda)]\}^{-1}. \quad (43)$$

The mean spontaneous magnetization per atom is given by the relation

$$M(T)=(1/N\beta)(\partial\ \ln Z/\partial H)_{H=0}$$
$$=(m/NS\beta)(\partial\ \ln Z/\partial L)_{L=0}, \quad (44)$$

as $L=mH/S$.

From Eqs. (17) and (43),

$$\ln Z=-\beta E_0-\sum_\lambda \ln\{1-\exp[-\beta(L+\epsilon_\lambda)]\}+\sum_{i=1}^{\infty}\Gamma_i. \quad (45)$$

We shall call the first two terms in (45) Bloch's part and the third term Dyson's part.

Moreover,

$$\left(\frac{\partial\ \ln Z}{\partial L}\right)_{L=0}$$
$$=\beta NS-\beta\sum_\lambda[\exp(\beta\epsilon_\lambda)-1]^{-1}+\sum_{i=1}^{\infty}\left(\frac{\partial\Gamma_i}{\partial L}\right)_{L=0}. \quad (46)$$

We now prove that Bloch's part in the magnetization series consists of a constant term (saturation magnetization) and of a series with powers 3/2, 5/2, and 7/2 of the temperature (the corrections proportional to 5/2 and 7/2 were originally derived by Dyson[2]). Dyson's part begins at T^4.

As we are interested in the case of low temperatures, i.e., long spin waves, we have, by Eq. (5),

$$\epsilon_\lambda=JS(\gamma_0-\gamma_\lambda)=JS[\gamma_0-\sum_\delta\exp(i\boldsymbol{\delta}\cdot\boldsymbol{\lambda})]$$
$$=JS[\mu x_2(\boldsymbol{\lambda})-\mu^2x_4(\boldsymbol{\lambda})+\mu^3x_4(\boldsymbol{\lambda})-\cdots],$$
$$\mu^s x_{2s}(\boldsymbol{\lambda})=[(2s)!]^{-1}\sum_\delta(\boldsymbol{\delta}\cdot\boldsymbol{\lambda})^{2s},\quad |\boldsymbol{\delta}\cdot\boldsymbol{\lambda}|\ll 1. \quad (47)$$

Here, μ is an auxiliary parameter introduced for convenience; its power multiplied by 2 indicates the order in the expansion of a given quantity in a power series in $(\boldsymbol{\delta}\cdot\boldsymbol{\lambda})$. In order to compute the second term in (46), we expand the denominator:

$$[\exp(\beta\epsilon_\lambda)-1]^{-1}$$
$$=\sum_{n=0}^{\infty}\exp[-(n+1)\beta\epsilon_\lambda]=\sum_{n=0}^{\infty}\exp[-\alpha_{n+1}\mu x_2(\boldsymbol{\lambda})]$$
$$\times[1+\alpha_{n+1}\mu^2x_4(\boldsymbol{\lambda})-\alpha_{n+1}\mu^3x_6(\boldsymbol{\lambda})$$
$$+\mu^4[\tfrac{1}{2}\alpha_{n+1}{}^2x_4{}^2(\boldsymbol{\lambda})+\alpha_{n+1}x_8(\boldsymbol{\lambda})]+O(\mu_5)], \quad (48)$$

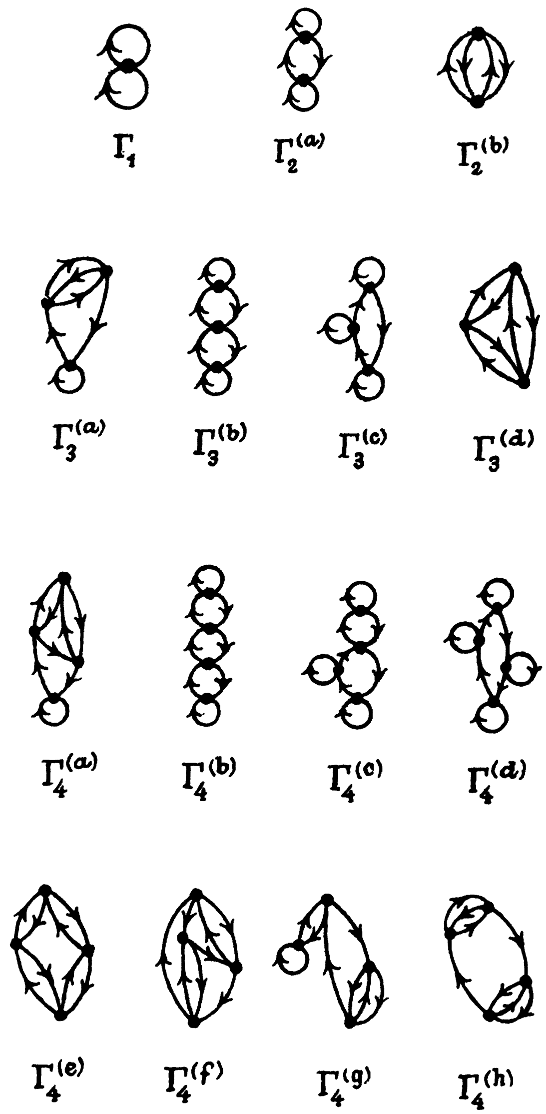

FIG. 1. Examples of distinct parts of the connected ladder diagrams.

where, for brevity,

$$\alpha_{n+1}=\beta JS(n+1). \quad (49)$$

Passing from the sum to the integral, we obtain

$$\sum_\lambda[\exp(\beta\epsilon_\lambda)-1]^{-1}$$
$$=(2\pi)^{-3}V\sum_{n=0}^{\infty}\int d\boldsymbol{\lambda}\,\exp[-\alpha_{n+1}\mu x_2(\boldsymbol{\lambda})]$$
$$\times[1+\alpha_{n+1}\mu^2x_4(\boldsymbol{\lambda})-\alpha_{n+1}\mu^3x_6(\boldsymbol{\lambda})$$
$$+\mu^4(\tfrac{1}{2}\alpha_{n+1}{}^2x_4{}^2(\boldsymbol{\lambda})+\alpha_{n+1}x_8(\boldsymbol{\lambda})+O(\mu^5)], \quad (50)$$

and, carrying out the summation over nearest neighbors and the integration over $\boldsymbol{\lambda}$,

$$\sum_\lambda[\exp(\beta\epsilon_\lambda)-1]^{-1}=N[\zeta(3/2)\theta^{3/2}+\tfrac{3}{4}\pi\nu\zeta(5/2)\theta^{5/2}+\omega\pi^2\nu^2\zeta(7/2)\theta^{7/2}+O(\theta^{9/2})], \quad (51)$$

with Riemann's ζ function of indices 3/2, 5/2, 7/2, and

$$\theta=3kT/2\pi JS\gamma_0\nu=(2\pi)^{-1}T/T_c, \quad (52)$$

$$\omega=33/32,\ 15/16,\ 281/288, \quad \nu=1,\ 2^{1/3},\ 3\times2^{-4/3}, \quad \gamma_0=6,\ 12,\ 8, \quad (53)$$

for the simple, face-centered, and body-centered cubic lattices, respectively. The part (51) of the magnetization series is identical with that of Dyson. From the derivation procedure of relation (51), the trans-Blochian corrections proportional to $T^{5/2}$ and $T^{7/2}$ result from our having gone beyond quadratic wave vector approximation of the energy; equivalently, they can be said to have arisen through the stricter calculation of the effective mass of the spin wave.

6. DYSON'S TERM IN THE MAGNETIZATION EXPANSION

We now compute the coefficient of T^4. The first contribution to the term $\propto\theta^4$ is from Γ_1 diagram, Eq. [42],

$$\Gamma_1=\tfrac{1}{2}\beta JN^{-1}\sum_{\rho\sigma}\Gamma_{\rho\sigma}{}^0 n_\rho n_\sigma.$$

Since

$$(\partial\Gamma_1/\partial L)_{L=0}=\beta JN^{-1}\sum_{\rho\sigma}\Gamma_{\rho\sigma}{}^0(n_\rho\partial n_\sigma/\partial L)_{L=0}=-\beta^2JN^{-1}\sum_{\rho\sigma}\Gamma_{\rho\sigma}{}^0[\exp(\beta\epsilon_\rho)-1]^{-1}\exp(\beta\epsilon_\sigma)[\exp(\beta\epsilon_\sigma)-1]^{-2}, \quad (54)$$

$$\Gamma_{\rho\sigma}{}^0=\mu[x_2(\boldsymbol{\rho})+x_2(\boldsymbol{\sigma})-x_2(\boldsymbol{\sigma}-\boldsymbol{\rho})]-\mu^2[x_4(\boldsymbol{\rho})+x_4(\boldsymbol{\sigma})-x_4(\boldsymbol{\sigma}-\boldsymbol{\rho})]+O(\mu^3), \quad (55)$$

and also

$$[\exp(\beta\epsilon_\rho)-1]^{-1}=\sum_{n=0}^\infty\exp[-\alpha_{n+1}\mu x_2(\boldsymbol{\rho})]\cdot[1+\alpha_{n+1}\mu^2x_4(\boldsymbol{\rho})+O(\mu^3)], \quad (56)$$

$$\exp(\beta\epsilon_\sigma)[\exp(\beta\epsilon_\sigma)-1]^{-2}=\sum_{p=0}^\infty(p+1)\exp[-\alpha_{p+1}\mu x_2(\boldsymbol{\sigma})]\cdot[1+\alpha_{p+1}\mu^2x_4(\boldsymbol{\sigma})+O(\mu^3)], \quad (57)$$

with

$$\alpha_{n+1}=\beta JS(n+1), \quad \alpha_{p+1}=\beta JS(p+1), \quad (58)$$

we obtain

$$\beta^{-1}\left(\frac{\partial\Gamma_1}{\partial L}\right)_{L=0}=-(\beta JS)(NS)^{-1}\left[\frac{V}{(2\pi)^3}\right]^2\sum_{n,p=0}^{\infty}(p+1)\int d\boldsymbol{\rho}\int d\boldsymbol{\sigma}\exp[-\alpha_{n+1}\mu x_2(\boldsymbol{\rho})]\times\exp[-\alpha_{p+1}\mu x_2(\boldsymbol{\sigma})]\{\mu[x_2(\boldsymbol{\rho})+x_2(\boldsymbol{\sigma})-x_2(\boldsymbol{\sigma}-\boldsymbol{\rho})]-\mu^2[x_4(\boldsymbol{\rho})+x_4(\boldsymbol{\sigma})-x_4(\boldsymbol{\sigma}-\boldsymbol{\rho})]+O(\mu^3)\}. \quad (59)$$

The μ term in the braces vanishes owing to lattice symmetry; the μ^2 term yields the coefficient of T^4, whereas the higher order terms $O(\mu^3)$ yield the contribution $\propto\theta^5$ and higher powers. Carrying out the summation over $\boldsymbol{\delta}$ and the integrations over $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$, we have

$$\beta(\partial\Gamma_1/\partial L)_{L=0}=-\tfrac{3}{2}\pi\nu NS^{-1}\zeta(3/2)\zeta(5/2)\theta^4+O(\theta^5), \quad (60)$$

where θ is given by Eq. (52).

We now prove that the contribution to the magnetization from the second-order diagram is the same as that calculated by Dyson. We have

$$\Gamma_2=\frac{1}{2!}(\tfrac{1}{4}JN^{-1})^2\sum_{\lambda\rho\sigma}\sum_{\kappa\mu\nu}\Gamma_{\rho\sigma}{}^\lambda\Gamma_{\mu\nu}{}^\kappa\int_0^\beta d\tau_1\int_0^\beta d\tau_2\langle T[\alpha_{\sigma+\lambda}{}^*(\tau_1)\alpha_{\rho-\lambda}{}^*(\tau_1)\alpha_\rho(\tau_1)\alpha_0(\tau_1)\alpha_{\nu+\kappa}{}^*(\tau_2)\alpha_{\mu-\kappa}{}^*(\tau_2)\alpha_\mu(\tau_2)\alpha_\nu(\tau_2)]\rangle. \quad (61)$$

Applying Wick's theorem, we obtain the two distinct parts of the connected diagram:

$$\Gamma_2{}^{(a)}=\frac{16}{2!}(\tfrac{1}{4}JN^{-1})^2\sum_{\lambda\rho\sigma}\sum_{\kappa\mu\nu}\Gamma_{\rho\sigma}{}^\lambda\Gamma_{\mu\nu}{}^\kappa\int_0^\beta d\tau_1\int_0^\beta d\tau_2\,\alpha_{\sigma+\lambda}{}^*(\tau_1)^c\alpha_{\rho-\lambda}{}^*(\tau_1)^{cc}\alpha_\rho(\tau_1)^{ccc}\alpha_\sigma(\tau_1)^c\times\alpha_{\nu+\kappa}{}^*(\tau_2)^{cccc}\alpha_{\mu-\kappa}{}^*(\tau_2)^{ccc}\alpha_\mu(\tau_2)^{cc}\alpha_\nu(\tau_2)^{cccc} \quad (62)$$

and

$$\Gamma_2{}^{(b)}=\frac{4}{2!}(\tfrac{1}{4}JN^{-1})^2\sum_{\lambda\rho\sigma}\sum_{\kappa\mu\nu}\Gamma_{\rho\sigma}{}^\lambda\Gamma_{\mu\nu}{}^\kappa\int_0^\beta d\tau_1\int_0^\beta d\tau_2\,\alpha_{\sigma+\lambda}{}^*(\tau_1)^c\alpha_{\rho-\lambda}{}^*(\tau_1)^{cc}\alpha_\rho(\tau_1)^{ccc}\alpha_\sigma(\tau_1)^{cccc}\times\alpha_{\nu+\kappa}{}^*(\tau_2)^{cccc}\alpha_{\mu-\kappa}{}^*(\tau_2)^{ccc}\alpha_\mu(\tau_2)^{cc}\alpha_\nu(\tau_2)^c. \quad (63)$$

By Eqs. (31), (32), (34), and (35), and integrating over the arguments τ_1 and τ_2, we obtain

$$\Gamma_2{}^{(a)}=\tfrac{1}{2}\beta^2(JN^{-1})^2[V/(2\pi)^3]^3\int d\boldsymbol{\lambda}\int d\boldsymbol{\rho}\int d\boldsymbol{\sigma}\Gamma_{\lambda\rho}{}^0\Gamma_{\rho\sigma}{}^0n_\lambda n_\rho(n_\rho+1)n_\sigma \quad (64)$$

and

$$\Gamma_2^{(b)}=\tfrac{1}{8}\beta(JN^{-1})^2[V/(2\pi)^3]^2\sum_\lambda\int d\boldsymbol{\varrho}\int d\boldsymbol{\sigma}\Gamma_{\rho\sigma}{}^{\lambda}\Gamma_{\sigma+\lambda,\rho-\lambda}{}^{\lambda}(\epsilon_\rho+\epsilon_\sigma-\epsilon_{\sigma+\lambda}-\epsilon_{\rho-\lambda})^{-1}\times[(n_\rho+1)(n_\sigma+1)n_{\sigma+\lambda}n_{\rho-\lambda}-n_\rho n_\sigma(n_{\sigma+\lambda}+1)(n_{\rho-\lambda}+1)]. \quad (65)$$

Introducing new variables

$$\begin{aligned}\delta(\beta JS\kappa)^{1/2}\boldsymbol{\lambda}&=\boldsymbol{\lambda}', & d\boldsymbol{\lambda}&=\delta^{-3}(\beta JS\kappa)^{-3/2}d\boldsymbol{\lambda}',\\ \delta(\beta JS\kappa)^{1/2}\boldsymbol{\varrho}&=\boldsymbol{\varrho}', & d\boldsymbol{\varrho}&=\delta^{-3}(\beta JS\kappa)^{-3/2}d\boldsymbol{\varrho}',\\ \delta(\beta JS\kappa)^{1/2}\boldsymbol{\sigma}&=\boldsymbol{\sigma}', & d\boldsymbol{\sigma}&=\delta^{-3}(\beta JS\kappa)^{-3/2}d\boldsymbol{\sigma}',\end{aligned} \quad (66)$$

where

$$\kappa=1,\ 2,\ 4/3, \quad (67)$$

for the simple, face-centered, and body-centered cubic lattices, respectively, we see that, by Eq. (55), in the lowest order of the expansion in $(\boldsymbol{\delta}\cdot\boldsymbol{\lambda})$, $\Gamma_{\rho\sigma}{}^0$ is proportional to $(\beta JS)^{-1}$. We obtain $(\beta JS)^{-9/2}$ from transformation of the variables of integration. Thus,

$$\Gamma_2^{(a)}\propto NS^{-2}(\beta JS)^2\cdot(\beta JS)^{-2}\cdot(\beta JS)^{-9/2}\propto\theta^{9/2}. \quad (68)$$

To evaluate diagram $\Gamma_2^{(b)}$, we interchange $\boldsymbol{\varrho}\rightleftarrows\boldsymbol{\sigma}$, $\boldsymbol{\lambda}\to-\boldsymbol{\lambda}$:

$$\Gamma_2^{1(b)2}=-\tfrac{1}{2}\beta(JN^{-1})^2\left[\frac{V}{(2\pi)^3}\right]^3\int d\boldsymbol{\lambda}\int d\boldsymbol{\varrho}\int d\boldsymbol{\sigma}\Gamma_{\rho\sigma}{}^{\lambda-\sigma}\Gamma_{\lambda,\rho+\sigma-\lambda}{}^{\lambda-\sigma}(\epsilon_\rho+\epsilon_\sigma-\epsilon_{\sigma+\lambda}-\epsilon_{\rho-\lambda})^{-1}\times n_\lambda n_\rho n_\sigma+\tfrac{1}{4}\beta(JN^{-1})^2\left[\frac{V}{(2\pi)^3}\right]^2\sum_\lambda\int d\boldsymbol{\varrho}\int d\boldsymbol{\sigma}\Gamma_{\rho\sigma}{}^{\lambda}\Gamma_{\sigma+\lambda,\rho-\lambda}{}^{\lambda}(\epsilon_{\sigma+\lambda}+\epsilon_{\rho-\lambda}-\epsilon_\rho-\epsilon_\sigma)^{-1}n_\rho n_\sigma. \quad (69)$$

The first part of Eq. (69) is by dimensional estimate $\propto\theta^{9/2}$; therefore, we compute only the second term. By Eqs. (5) and (6),

$$\frac{\Gamma_{\rho\sigma}{}^{\lambda}\Gamma_{\sigma+\lambda,\rho-\lambda}{}^{\lambda}}{\epsilon_{\sigma+\lambda}+\epsilon_{\rho-\lambda}-\epsilon_\rho-\epsilon_\sigma}=(JS)^{-1}\left[\frac{(\Gamma_{\rho\sigma}{}^{\lambda})^2}{\gamma_\rho+\gamma_\sigma-\gamma_{\sigma+\lambda}-\gamma_{\rho-\lambda}}-\Gamma_{\rho\sigma}{}^{\lambda}\right]. \quad (70)$$

By means of Eq. (66), we now pass with the $\boldsymbol{\varrho}$ and $\boldsymbol{\sigma}$ to the new variables, and obtain

$$\Gamma_2^{(b)}=\frac{1}{8\kappa^5S^2}\frac{o^2}{(2\pi)^6(\beta JS)^4}\sum_{n,r=0}^{\infty}e^{-\beta L(n+r+2)}\sum_\lambda\int d\boldsymbol{\varrho}e^{-(n+1)\rho^2}\int d\boldsymbol{\sigma}e^{-(r+1)\sigma^2}\times(\gamma_0-\gamma_\lambda)^{-1}\delta^{-4}[\textstyle\sum_\delta\exp(i\boldsymbol{\delta}\cdot\boldsymbol{\lambda})(\boldsymbol{\delta}\cdot\boldsymbol{\varrho})(\boldsymbol{\delta}\cdot\boldsymbol{\sigma})]^2+O(\theta^{9/2}), \quad (71)$$

with

$$o=1,\ 2^{-1/2},\ 4\times3^{-3/2}, \quad (72)$$

for the simple, face-centered, and body-centered cubic lattices, respectively. Summation over nearest neighbors and integrations over the $\boldsymbol{\varrho}$ and $\boldsymbol{\sigma}$ yield

$$\Gamma_2^{(b)}=\tfrac{3}{4}\pi\nu NS^{-2}\phi\theta^4\sum_{n,r=0}^{\infty}(n+1)^{-5/2}(r+1)^{-5/2}e^{-\beta L(n+r+2)}+O(\theta^{9/2}) \quad (73)$$

or

$$M_2(T)=-\tfrac{3}{2}\pi\nu mS^{-3}\phi\zeta(3/2)\zeta(5/2)\theta^4+O(\theta^{9/2}), \quad (74)$$

with

$$\phi=\frac{1}{24\pi^3}\int\int_{-\pi}^{\pi}\int dxdydz\frac{\cos^2x}{1-\frac{1}{3}(\cos x+\cos y+\cos z)}=\tfrac{2}{3}\Gamma^s+\tfrac{1}{3}\alpha_s, \quad (75)$$

$$\phi=\frac{1}{48\pi^3}\int\int_{-\pi}^{\pi}\int dxdydz\frac{1+2\cos^2x\cos^2y-2\cos^2x+\cos^2x\cos y\cos z}{1-\frac{1}{3}(\cos x\cos y+\cos x\cos z+\cos y\cos z)}=\tfrac{2}{3}\Gamma^f+\tfrac{1}{3}\alpha_f, \quad (76)$$

$$\phi=\frac{1}{24\pi^3}\int\int_{-\pi}^{\pi}\int dxdydz\frac{3\cos^2x\cos^2y\cos^2z+2\cos^2x-4\cos^2x\cos^2y}{1-\cos x\cos y\cos z}=\tfrac{2}{3}\Gamma^b+\tfrac{1}{3}\alpha_b, \quad (77)$$

for the three types of cubic lattice. The quantities Γ and α are taken from Dyson's[2,15] theory:

$$G_0(\boldsymbol{\delta})=N^{-1}\sum_{\rho}(\gamma_0-\gamma_\rho)^{-1}\exp(i\boldsymbol{\delta}\cdot\boldsymbol{\rho}),\quad \alpha=\sum_{\delta}G_0(\boldsymbol{\delta})=N^{-1}\sum_{\rho}(\gamma_0-\gamma_\rho)^{-1}\gamma_\rho. \tag{78}$$

Explicitly,

$$\alpha_s=-1+\frac{1}{8\pi^3}\int\int\int_{-\pi}^{\pi}\frac{dxdydz}{1-\frac{1}{3}(\cos x+\cos y+\cos z)}=0.52, \tag{79}$$

$$\alpha_f=-1+\frac{1}{8\pi^3}\int\int\int_{-\pi}^{\pi}\frac{dxdydz}{1-\frac{1}{3}(\cos x\cos y+\cos x\cos z+\cos y\cos z)}=0.34, \tag{80}$$

$$\alpha_b=-1+\frac{1}{8\pi^3}\int\int\int_{-\pi}^{\pi}\frac{dxdydz}{1-\cos x\cos y\cos z}=0.39, \tag{81}$$

$$\Gamma^s=\frac{1}{8\pi^3}\int\int\int dxdydz\frac{\cos x(1-\cos y)}{1-\frac{1}{3}(\cos x+\cos y+\cos z)}\approx\frac{1}{5}, \tag{82}$$

$$\Gamma^f=\frac{1}{24\pi^3}\int\int\int_{-\pi}^{\pi}dxdydz\frac{\cos x\cos y(2-\cos 2z)-\cos 2x}{1-\frac{1}{3}(\cos x\cos y+\cos x\cos y+\cos y\cos z)}\approx\frac{1}{12}, \tag{83}$$

$$\Gamma^b=\frac{1}{16\pi^3}\int\int\int_{-\pi}^{\pi}dxdydz\frac{2\cos x\cos y\cos z-\cos 2x(1+\cos 2y)}{1-\cos x\cos y\cos z}\approx\frac{1}{8}. \tag{84}$$

Similar methods can be used to compute the share in the magnetization of the diagrams $\Gamma_3^{(d)}$, $\Gamma_4^{(e)}$, etc.

By Eqs. (44), (51), (60), and (74), the spontaneous magnetization becomes

$$\begin{aligned}M(T)=(m/S)\{&S-\zeta(3/2)\theta^{3/2}-\tfrac{3}{4}\pi\nu\zeta(5/2)\theta^{5/2}\\&-\omega\pi^2\nu^2\zeta(7/2)\theta^{7/2}-\tfrac{3}{2}\pi\nu S^{-1}\\&\times[1+(3S)^{-1}(2\Gamma+\alpha)+\cdots]\\&\times\zeta(3/2)\zeta(5/2)\theta^4+O(\theta^{9/2})\}.\end{aligned} \tag{85}$$

A detailed account of the present investigation, giving the calculations in full, will be published in Acta Physica Polonica.

Note added in proof. In a new paper "Some Remarks on the Spin Wave Theory for Cubic Ferromagnetics" [Acta Physica Polonica, (to be published)], the present author carried out summation of the infinite series of graphs contributing to the spontaneous magnetization as the fourth power of the absolute temperature, obtaining full agreement with Dyson's magnetization formula.

ACKNOWLEDGMENTS

The preliminary application of the dimensional estimate was insufficiently strict; hence contributions to the magnetization from the $\Gamma_3^{(d)}$, $\Gamma_4^{(e)}$, etc., diagrams were too small. The author wishes to express his best thanks to Professor Freeman J. Dyson for detecting this error and for his other valuable remarks.

The author thanks Professor S. Szczeniowski for his kind interest throughout the present investigation, and is indebted to K. Flatau, B.Sc., for the English translation.