Shell-Model Analysis of the Level Structure of Ar^{41}†

Y. SHADMI*
Institute for Advanced Study, Princeton, New Jersey

AND

I. TALMI‡
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey

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Levels of Ar^{41} experimentally observed up to excitation energy of 4.5 MeV are considered. Using experimental data from neighboring nuclei the positions of 40 odd-parity states are calculated in the framework of the shell model. Good agreement is obtained for the low-lying levels. Approximate treatment of the $d_{3/2}^2 f_{7/2}^2 p_{3/2}$ and $d_{3/2}^2 f_{7/2}^2 p_{1/2}$ configurations in addition to the $d_{3/2}^2 f_{7/2}^3$ ground configuration gives a semiquantitative account of all odd parity 3/2 and 1/2 levels up to 4.5 MeV. Many odd-parity levels with $J>3/2$ which were not yet found experimentally are predicted to be in this region.

I. INTRODUCTION

In a recent paper by Kashy *et al.*[1] the level structure of Ar^{41} is determined from the $Ar^{40}(d,p)Ar^{41}$ reaction. Twenty-four excited levels are found up to 4 MeV excitation energy. Of these, 17 show a stripping pattern and their l_n values are given. The most striking result is that 12 of these levels have $l_n=1$ and are therefore odd parity 3/2 or 1/2 levels.

Nuclei in which the protons are in the $1d_{3/2}$ orbit and the neutrons start filling the $1f_{7/2}$ shell have been treated with considerable success in the framework of the jj-coupling shell model.[2–4] It is therefore interesting to see whether the shell model can account for so many low lying odd parity levels with spins 3/2 and 1/2. Another question is whether there should be in the region below 4 MeV excitation in Ar^{41} other levels which were not yet observed. The assignment of J values (either 3/2 or 1/2) to levels with $l_n=1$ stripping was done in reference 1 rather arbitrarily. In particular, all levels below a certain energy were assigned $J=3/2$ and all higher levels $J=1/2$. It was mentioned there that most of these assignments have only a small effect on the location of the single particle levels. Nevertheless, it is interesting to see whether more accurate J assignments can be made on the basis of the shell model. In the following we shall consider these problems.

In the ground state of Ar^{41}, the protons are expected to be in the $d_{3/2}^2$ configuration and the neutrons in the $f_{7/2}^3$ configuration outside closed shells. All the levels of this configuration have odd parity. Even parity levels can be obtained by excitation of one of the nucleons into a higher orbit. Such even parity levels were actually observed.[1] We shall confine our treatment only to the odd parity states. There is much information on $1f_{7/2}^n$ configurations in this region[5,6] as well as on the $d_{3/2}$–$f_{7/2}$ interaction. We shall try to extend previous calculations to the more complicated case of Ar^{41}.

We shall use throughout the jj-coupling shell model assuming two body effective interactions between nucleons. Rather than use an arbitrary phenomenological potential we shall use matrix elements for the effective interaction as determined from neighboring nuclei.[7] In particular, the matrix elements of the $d_{3/2}$–$f_{7/2}$ interaction will be taken from the level schemes of K^{40} and Cl^{38} where excellent agreement has been obtained between theory and experiment.

II. STRUCTURE OF THE ENERGY MATRIX IN THE SHELL MODEL

A possible complete set of wave functions in the $d_{3/2}^2\, f_{7/2}^3$ configuration is the following. We can take an antisymmetric state of the protons with a definite spin J_p, an antisymmetric state of the neutrons with definite spin J_n and couple J_p and J_n to a total J. In antisymmetric states, J_p can be either 0 or 2 and J_n can be 3/2, 5/2, 7/2, 9/2, 11/2, 15/2. In these states, $d_{3/2}^2(J_p) f_{7/2}^3(J_n) J$, the mutual interaction energy of the protons, $V(d_{3/2}^2 J_p)$ is diagonal as well as that of the neutrons, $V(f_{7/2}^3 J_n)$. The actual values of these interaction energies will be taken from neighboring nuclei in which these configurations appear. The kinetic energy as well as the interaction of the $d_{3/2}$ protons and $f_{7/2}$ neutrons with the closed shells have the same value in all states of a given configuration. Since we are considering only level spacings, these energies will thus be omitted. However, the proton–neutron interaction has, in addition to diagonal elements, nonvanishing nondiagonal elements between states with

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* On leave of absence from the Hebrew University, Jerusalem, Israel.

‡ On leave from the Weizmann Institute of Science, Rehovoth, Israel.

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the same J but different values of J_p and J_n. For every value of J we have to write down the energy matrix whose rows and columns are characterized by J_p and J_n. The actual eigenstates and eigenvalues will be found only after diagonalization of these matrices. Before considering the results of the exact solution, let us first see what is the general picture expected to emerge from the exact calculations.

The energies of the $d_{3/2}^2$ configuration can be taken from Ar^{38} which has closed neutron shells. The first excited $J=2$ level in Ar^{38} lies 2.16 MeV above $J=0$ ground state. We shall thus take the energy difference between the interaction energy of the protons in the state $J_p=0$ and the states with $J_p=2$ to be 2.16 MeV. The energy spacing between two states with the same value of J_n will be dominated by this rather large difference. The proton–neutron interaction will, in general, have different expectation values in such states but these differences do not amount to more than a few tenths of MeV. We therefore expect that the low lying J_n states of the $f_{7/2}^3$ configuration when coupled to $J_p=0$ will give rise to the low lying levels of Ar^{41} and that the actual eigenstates will have rather small $J_p=2$ admixtures. We consider next the structure of the $f_{7/2}^3$ configuration.

The spectrum of the $f_{7/2}^3$ configuration can be calculated if we take the matrix elements of the interaction from nuclei with $f_{7/2}^2$ configurations. Such calculations were carried out and give a consistent description of the spectra of the Ca isotopes as well as of nuclei with 28 neutrons. Starting with Ca^{42} or Ti^{50} (with the proton $f_{7/2}^2$ configuration) the spectra of Ca^{43}, Ca^{45}, V^{51}, and Mn^{53} can be calculated. Very good agreement is obtained for the energies of the $J=7/2$ ground states (binding energies) as well as for the energies of the first excited states with $J=5/2$.[5] These first excited 5/2 states lie about 0.3 MeV above the 7/2 ground states. Thus, we expect the ground state of Ar^{41} to be a 7/2 state which, due to the large $J_p=0$ component, can be easily excited by a stripping reaction with $l_n=3$. The first excited state is expected to be a 5/2 state. This state is not expected to be strongly excited by stripping. The only possible way for such a reaction to take place is through the $1f_{5/2}$ admixture in this 5/2 state. Since $f_{7/2}$–$f_{5/2}$ single nucleon separation is very large (higher than 6 MeV), very small $f_{5/2}$ admixture is expected in the 5/2 first excited state. The situation is very similar to that in Ca^{43} where no stripping is observed into the 5/2 state at 0.37 MeV. As we shall see later, this situation is not changed by the more detailed calculation, which will also predict the spacing between the 5/2 and 7/2 states. Turning to the experimental material we see (Table I of reference 1) that the ground state is strongly excited by $l_n=3$ stripping and that the first excited state (at 0.17 MeV) has indeed a very small width.

The situation with regard to the higher excited states of the $f_{7/2}^3$ configuration is not so clear cut. Taking the $f_{7/2}^2$ levels from Ca^{42} to be at 1.52 MeV ($J=2$), 2.75 MeV ($J=4$) and 3.18 MeV ($J=6$) and using tables of coefficients of fractional parentage (c.f.p.)[8] we calculate the following positions of the $f_{7/2}^3$ levels. $J=7/2$ (ground state), $J=5/2$ 0.30 MeV, $J=3/2$ 1.20 MeV, $J=11/2$ 1.75 MeV, $J=9/2$ 2.00 MeV, and $J=15/2$ 3.10 MeV. Whereas the calculated position of the 5/2 state agrees well with the experimental one, the calculated 3/2 level is much higher than the position of the observed 3/2 level in Ca^{43} (the other levels are not known experimentally). This is a definite case of configuration interaction due to the low lying $2p_{3/2}$ orbit (1.95 MeV above $1f_{7/2}$ level in Ca^{41}). The 3/2 level considered is strongly admixed with the lowest level of the $f_{7/2}^2\, p_{3/2}$ configuration. This accounts (for the time being only qualitatively) for its lowered energy as well as for its rather large width for $l_n=1$ stripping.

It turns out that the same situation occurs also in Ar^{41}. The position of the second excited level is found to be much lower than the calculated one. It also shows considerable $l_n=1$ stripping most probably due to admixtures of the 3/2 level of the $d_{3/2}^2\, f_{7/2}^2\, p_{3/2}$ configuration. This perturbing state, which has mostly $J_p=0$, lies according to the experimental data at 1.35 MeV, and is distinguished by its large width for $l_n=1$ stripping. The spacing between these two states (at 0.52 and 1.35 MeV) is considerably smaller than the corresponding spacing in Ca^{43} (0.59 and 2.05 MeV). As a result, the configuration admixtures are larger and the relative stripping width of the low 3/2 state in Ar^{41} is considerably bigger than that in Ca^{43}.

We see already how the fact that the $d_{3/2}^2\, f_{7/2}^2\, p_{3/2}$ configuration lies so low affects strongly level positions. We shall have to take this effect into our considerations.

The combination of $J_p=0$ with the various values of J_n gives states with $J=7/2$, 5/2, 3/2, 11/2, 9/2 and 15/2. Similarly, the coupling of $J_p=2$ with the various values of J_n leads to 29 different states. The states for which $J_p=2$ and $J_n=15/2$ are expected to lie roughly $2.16+3.10=5.36$ MeV above the ground state. This is rather high and therefore these states were omitted from the calculation.

In order to carry out the exact calculation we have to adopt some set of energies of the $f_{7/2}^3$ configuration. If we take the values calculated from Ca^{42} we obtain the spacings given above. We carried out once the diagonalization of the energy matrices using these values. These spacings are based on the level spacings of only one nucleus (Ca^{42}) and thus may not be very reliable. In order to see whether the final results are very sensitive to the $f_{7/2}^3$ energy levels we carried out another diagonalization using the observed level spacings in V^{51} which has the $f_{7/2}^3$ proton configurations.

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The levels taken from the V^{51} spectrum are[6]: $J=7/2$ (ground state), $J=5/2$ 0.32 MeV, $J=3/2$ 0.93 MeV, $J=11/2$ 1.61 MeV, $J=9/2$ 1.81 MeV, and $J=15/2$ 2.70 MeV. Although the agreement with the calculated values from Ca^{42} is rather good, there are some differences in the energies. It is interesting to see that the order of $J=9/2$ and $J=11/2$ levels is the same as calculated from Ca^{42}. In both cases the 9/2 level is *above* the 11/2 level.

The matrix elements of the $d_{3/2}$ proton–$f_{7/2}$ neutron interaction were taken from K^{40}. Only the *differences* of the $d_{3/2}^{-1} f_{7/2}$ matrix elements enter the calculation of *spacings* of energy levels. These differences are given by the K^{40} energy levels.[2,3] Using the definition $V_J = \langle d_{3/2}\, f_{7/2} J | V | d_{3/2}\, f_{7/2} J \rangle$ we obtain (in MeV)

$$V_3 - V_2 = 0.75, \quad V_4 - V_2 = 1.32, \quad V_5 - V_2 = 0.70.$$

The actual construction of the energy matrices is described in detail in reference 4. We adopt here a somewhat different procedure which is explained in the Appendix. The numerical matrices were diagonalized on the IBM 650 Computer at Princeton University.

III. RESULTS AND DISCUSSION

The results of the diagonalization of the energy matrices are presented in Tables I and II. Table I contains the results for levels with either $J=1/2$ or $J=3/2$ which can be excited by $l_n=1$ stripping. Table II contains the information on levels with $J>3/2$. Since we would like to see whether the numbers of the various J levels can be understood we include also levels which should belong to the $d_{3/2}^2 f_{7/2}^2 p_{3/2}$ configuration. The positions of these levels can be only roughly estimated. We characterize these states by the values of J_p and J_n' of the $f_{7/2}^2$ neutrons along with the value of J. We estimate the relative positions of such states by considering the energies of the $d_{3/2}^2$ and the $f_{7/2}^2$ configurations in the states with J_p and J_n', respectively. We ignore in this zero-order approximation the difference between the interaction energy of the $d_{3/2}$ protons and one $f_{7/2}$ neutron and that of the $d_{3/2}$ protons and one $p_{3/2}$ neutron. Also, states with the same values of J_p and J_n' should have different energies due to the interaction between the $f_{7/2}$ and $p_{3/2}$ neutrons. We have no information on the strength of this interaction. We only know from other cases[7] that the interaction between protons in different orbits is rather *small* and *repulsive* on the average. The spread of such states is not expected to be more than a few tenths of MeV.

Far more serious is the interaction between the states of the $d_{3/2}^2 f_{7/2}^2 p_{3/2}$ configuration and the $d_{3/2}^2 f_{7/2}^3$ configuration. We do not treat this interaction in the present paper. Our results will therefore be mostly semiquantitative. Only if certain states are sufficiently removed from states with the same J of the perturbing configuration will our results be quantitatively significant. This we believe is the case for the lowest states with $J=7/2$, 5/2, 9/2, and 11/2. Another configuration that must also be taken into consideration is the $d_{3/2}^2 f_{7/2}^2 p_{1/2}$ configuration. The splitting between the $p_{3/2}$ and $p_{1/2}$ states is believed to be about 2 MeV.[1] We adopt this value and again give rough estimates of the positions of levels of this configuration.

In the first columns of Tables I and II the configurations of the various states are listed. In the next column a more precise characterization of the states is given. For the $d_{3/2}^2 f_{7/2}^2 p_{3/2}$ and $d_{3/2}^2 f_{7/2}^2 p_{1/2}$ configurations the values of J_p, J_n' as well as the spin of the p neutron are listed. In the case of the $d_{3/2}^2 f_{7/2}^3$ configuration the J_p and J_n of the dominant term of the eigenfunction are given. If two or more such states are important they are all listed in the order of their importance (if a state has more than 50% in probability

TABLE I. Observed and calculated levels with $J=\frac{1}{2}, \frac{3}{2}$.

Configuration	Dominant states	J	Level positions in MeV: Calculated I	Calculated II	Observed	Level number
$d_{3/2}^2 f_{7/2}^3$	(0, 3/2)	3/2	1.30	1.05	0.517	2
$d_{3/2}^2 f_{7/2}^2 p_{3/2}$	(0, 0, 3/2)	3/2	~1.1		1.354	4
$d_{3/2}^2 f_{7/2}^2 p_{3/2}$	(0, 2, 3/2)	3/2	~2.6		2.402	8
$d_{3/2}^2 f_{7/2}^2 p_{3/2}$	(0, 2, 3/2)	1/2	~2.6		2.701	9
$d_{3/2}^2 f_{7/2}^2 p_{3/2}$	(2, 0, 3/2)	3/2	~3.2		2.740	10
$d_{3/2}^2 f_{7/2}^2 p_{3/2}$	(2, 0, 3/2)	1/2	~3.2		2.955	12
$d_{3/2}^2 f_{7/2}^3$	(2, 7/2)+(2, 5/2)	3/2	3.27	3.31	3.017	13
$d_{3/2}^2 f_{7/2}^2 p_{1/2}$	(0, 0, 1/2)	1/2	~3.2		3.335	15
$d_{3/2}^2 f_{7/2}^3$	(2, 5/2)	1/2	3.49	3.39	3.293	14
$d_{3/2}^2 f_{7/2}^3$	(2, 5/2)+(2, 7/2)+(2, 3/2)	3/2	3.50	3.46	3.438	17
$d_{3/2}^2 f_{7/2}^3$	(2, 3/2)	1/2	4.29	4.01	3.979	24
$d_{3/2}^2 f_{7/2}^3$	(2, 3/2)	3/2	4.32	4.08	3.808	21
$d_{3/2}^2 f_{7/2}^2 p_{1/2}$	(0, 2, 1/2)	3/2	~4.7		4.280	28

TABLE II. Observed and calculated levels with $J=5/2$–$15/2$.

Configuration	Dominant states	J	Level positions in MeV Calculated I	Calculated II	Observed	Level number
$d_{3/2}^2 f_{7/2}^3$	(0, 7/2)	7/2	0.00	0.00	0.0	0
$d_{3/2}^2 f_{7/2}^3$	(0, 5/2)	5/2	0.17	0.23	0.171	1
$d_{3/2}^2 f_{7/2}^3$	(0, 9/2)+(2, 7/2)	9/2	1.76	1.64	1.636	5
$d_{3/2}^2 f_{7/2}^3$	(0, 11/2)	11/2	1.86	1.74	1.988	7
$d_{3/2}^2 f_{7/2}^2 p_{3/2}$	(0, 2, 3/2)	7/2	~2.6			
$d_{3/2}^2 f_{7/2}^2 p_{3/2}$	(0, 2, 3/2)	5/2	~2.6			
$d_{3/2}^2 f_{7/2}^3$	(2, 7/2)	11/2	2.72	2.70		
$d_{3/2}^2 f_{7/2}^3$	(2, 7/2)	7/2	2.86	2.87		
$d_{3/2}^2 f_{7/2}^3$	(2, 5/2)+(2, 7/2)	9/2	2.89	2.94		
$d_{3/2}^2 f_{7/2}^3$	(2, 7/2)+(2, 5/2)	5/2	3.11	3.14		
$d_{3/2}^2 f_{7/2}^2 p_{3/2}$	(2, 0, 3/2)	7/2	~3.2			
$d_{3/2}^2 f_{7/2}^2 p_{3/2}$	(2, 0, 3/2)	5/2	~3.2			
$d_{3/2}^2 f_{7/2}^3$	(2, 5/2)	7/2	3.28	3.32	3.393	16
$d_{3/2}^2 f_{7/2}^3$	(2, 7/2)+(0, 9/2)+(2, 5/2)	9/2	3.41	3.37		
$d_{3/2}^2 f_{7/2}^3$	(2, 5/2)+(2, 7/2)+(2, 3/2)	5/2	3.65	3.64		
$d_{3/2}^2 f_{7/2}^3$	(0, 15/2)	15/2	3.65	3.26		
$d_{3/2}^2 f_{7/2}^2 p_{3/2}$	(0, 4, 3/2)	11/2	~3.85			
$d_{3/2}^2 f_{7/2}^2 p_{3/2}$	(0, 4, 3/2)	9/2	~3.85			
$d_{3/2}^2 f_{7/2}^2 p_{3/2}$	(0, 4, 3/2)	7/2	~3.85			
$d_{3/2}^2 f_{7/2}^2 p_{3/2}$	(0, 4, 3/2)	5/2	~3.85			
$d_{3/2}^2 f_{7/2}^3$	(2, 3/2)	7/2	3.95	3.67		
$d_{3/2}^2 f_{7/2}^3$	(2, 3/2)	5/2	4.15	3.90		
$d_{3/2}^2 f_{7/2}^3$	(2, 11/2)	15/2	4.21	4.07		
$d_{3/2}^2 f_{7/2}^3$	(2, 9/2)	5/2	4.42	4.27		
$d_{3/2}^2 f_{7/2}^3$	(2, 9/2)+(2, 11/2)	13/2	4.47	4.28		
$d_{3/2}^2 f_{7/2}^3$	(2, 11/2)+(2, 9/2)	13/2	4.50	4.31		
$d_{3/2}^2 f_{7/2}^3$	(2, 11/2)	9/2	4.53	4.39		

it is underlined). The third column gives the J values of the various states. The calculated energies are given in the fourth and fifth columns. The energies calculated by using the Ca^{42} levels are given in the fourth column, whereas those calculated by using the V^{51} spectrum are listed in the fifth. The observed energies are given in column 6. In several cases we can establish the correspondence between calculated and experimental energies of single levels. At higher energies, however, where there are many levels and our calculations are less reliable the correspondence is less clear. We can only say that a certain given number of calculated levels in a certain range corresponds to the same number of experimental levels. The last column lists the level numbers as given in Table I of reference 1.

Let us first consider the consistency of the two theoretical calculations I and II. A glance at columns 4 and 5 shows that the calculated levels are rather close. In particular the order of levels is almost the same in both calculations. It also turns out that the admixtures of the states with various values of J_p and J_n are very close in both cases. Thus, the list of dominant states, in column 2, can serve for both calculations.

Comparing the theoretical predictions with the experimental values we see good agreement for the low lying levels, with the notable exception of the 3/2 case discussed above. There is very good agreement between the calculated position of the 5/2 state (Table II) and that of the first excited levels of Ar^{41}. As explained above, this state should be very weakly excited by $l_n=3$ stripping and indeed, no stripping has been observed experimentally leading to this state. The calculated positions of the $J=9/2$ and $J=11/2$ levels agree quite well with the energies of levels 5 and 7 of Table I in reference 1. The order of these levels is inverted as compared with V^{51} due to the $d_{3/2}$–$f_{7/2}$ interaction. None of these levels shows stripping which is consistent with our spin assignments. Since this order was obtained in both calculations we feel sure that it is the correct one in our model. It would be rather interesting to have the spins of these levels measured experimentally.

Let us consider now the 3/2 and 1/2 levels listed in Table I. The experimental position of the first 3/2 state (No. 2) is much lower than the calculated one. Most probably this is due to its strong interaction with

the state at 1.35 MeV (No. 4). This latter state has a large width for $l_n=1$ stripping and, therefore, identified as the ground state of the $d_{3/2}^2 f_{7/2}^2 p_{3/2}$ configuration ("single-particle $p_{3/2}$ state"). Starting from this state we can estimate the position of other levels of this configuration as explained above. Both 3/2 states (Nos. 2 and 4) were probably pushed down by the (0, 2, 3/2) and (2, 0, 3/2) states. These latter states are tentatively identified as levels No. 8 and 10 which also show rather large width for $l_n=1$ stripping.

Another level that has a rather large width is No. 15 at 3.34 MeV. This fact along with its position suggests that it is the ground state of the $d_{3/2}^2 f_{7/2}^2 p_{1/2}$ configuration ("single-particle $p_{1/2}$ state"). This state is probably strongly admixed with the (2, 3/2) state of the $d_{3/2}^2 f_{7/2}^3$ configuration. Levels 15 and 24, which both have rather large widths, are probably admixtures of these two states. From the position of the (0, 0, 1/2) state, the positions of other states of the $d_{3/2}^2 f_{7/2}^2 p_{1/2}$ configuration can be estimated.

Looking at the predicted levels of Table I we see that with our crude assumptions it is very difficult to compare the predicted positions with experimental energies. The identification made in Table I should be considered as tentative only. A more careful consideration of the $d_{3/2}$–$p_{3/2}$ and the $f_{7/2}$–$p_{3/2}$ interaction, as well as configuration interaction, can change the calculated positions of the levels by several hundreds of KeV. Nevertheless, even the approximation used gives some picture of the positions of the 1/2 and 3/2 levels in the energy range considered which agrees quite well with the experimental findings. There are altogether 13 odd parity levels with $J=3/2$ and $J=1/2$ up to 4.5 MeV excitation in Ar^{41} and the existence of all of them, and even their approximate positions, can be accounted for as states of the $d_{3/2}^2 f_{7/2}^3$ ground configuration as well as of the $d_{3/2}^2 f_{7/2}^2 p_{3/2}$ and $d_{3/2}^2 f_{7/2}^2 p_{1/2}$ configurations. All these are expected to be the lowest configurations in the shell model. These configurations have many more levels, several of which have spins $J=3/2$ and $J=1/2$. These levels are predicted, however, to be around 4.5 MeV above the ground state of Ar^{41} and higher. We also see that no clear cut separation into 3/2 and 1/2 levels is expected. There are 8 calculated levels with spins 3/2 and only 5 levels with 1/2. It is clear, however, that the lower levels that show large widths are expected to be 3/2 levels and the higher ones with large widths are expected to have spin 1/2. This principle was used in the identifications made in Table I (e.g., in choosing the spins for levels 21 and 24).

The same considerations apply also to the levels in Table II, with $J>3/2$. We mentioned already the good agreement in the positions of the 5/2, 9/2, and 11/2 levels. Above these we predict the approximate positions of 23 odd parity levels with spins ranging from 5/2 to 15/2. None of these levels can be obtained by $l_n=1$ stripping and thus they are expected to have small cross sections for the (d,p) reaction. As mentioned in the Introduction, not many odd parity levels were found in reference 1. In the range between 2 and 4.5 MeV there is only one level with $l_n=3$ stripping, at 3.39 MeV. This level agrees very well with a predicted 7/2 level with $J_p=2$, $J_n=5/2$ of the $d_{3/2}^2 f_{7/2}^3$ configuration. However, other predicted levels with $J=7/2$ are not observed by stripping. In the range considered (2–4.5 MeV) there are 4 levels with probable even parity and only 9 other levels with very small cross sections (or isotropic angular distributions). We, therefore conclude that probably as many as 14 more odd parity levels would be observed in Ar^{41} below 4.5 MeV. It is possible to identify the 9 observed levels as some of the predicted states. However, because of the missing levels there is no unique way to do it and we do not list these 9 levels in Table II.

Not much can be predicted quantitatively without considering all interactions of the three configurations mentioned above. To do this, more information about the $d_{3/2}$–$p_{3/2}$, $f_{7/2}$–$p_{3/2}$, $d_{3/2}$–$p_{1/2}$ and $f_{7/2}$–$p_{1/2}$ interactions is required. Such information can be hopefully obtained from the experimental data of simpler configurations (involving less nucleons outside closed shells) in neighboring nuclei. The analysis presented above, although it gives only a semiquantitative description of the situation, is still instructive. It shows that at higher excitation energies, states of different configurations are rather close so that the interaction between them must be taken into account. However, if we have enough information from low lying states we may still be able to obtain quantitative agreement even at higher energies. Although it is not yet possible to accomplish this more complete program, even the present results illustrate the capability of the shell model to give an explanation not only of the low lying levels but of all the levels of a rather complicated nuclear spectrum.

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APPENDIX. THE CALCULATION OF THE $d_{3/2}$–$f_{7/2}$ INTERACTION MATRICES

Any interaction between a $d_{3/2}$ proton and a $f_{7/2}$ neutron can be expanded in terms of scalar products of irreducible tensor operators as follows. We define[9]

[9] See, for example, A. de-Shalit and I. Talmi, *Nuclear Shell Theory* (Academic Press Inc., New York, 1962), p. 235. This book will be referred to as ST in the following.

$$V_J=\langle jj'J|V|jj'J\rangle=\langle jj'J|\sum_k(\mathbf{u}_1^{(k)}\cdot\mathbf{u}_2^{(k)})F^k|jj'J\rangle$$
$$=\sum_k(-1)^{j+j'+J}\begin{Bmatrix}j & j' & J\\ j' & j & k\end{Bmatrix}F^k. \quad (1)$$

In (1), j characterizes the $d_{3/2}$ orbit, j' the $f_{7/2}$ orbit and the $\mathbf{u}^{(k)}$ are unit tensors operators defined by their reduced matrix elements $(j\|\mathbf{u}^{(k)}\|j')=\delta_{jj'}$. The summation over k goes from $k=0$ to the smaller of the values $2j$ and $2j'$. The parameters F^k characterize uniquely the interaction considered. Using the orthogonality properties of the Racah coefficients in (1) it is possible to express the F^k in terms of the matrix elements V_J. Thus, we have [ST, Eq. (22.51)]

$$F^k=(2k+1)\sum_J(-1)^{j+j'+J}(2J+1)\begin{Bmatrix}j & j' & J\\ j' & j & k\end{Bmatrix}V_J. \quad (2)$$

As seen from (1), F^0 contributes the same amount to all levels of a configuration. Since we are interested only in level *spacings* we shall omit F^0 from the following calculations. The other F^k are determined only by energy differences within the $d_{3/2}\,f_{7/2}$ configuration. These are taken from the observed level spectrum of K^{40}. The proton configuration of Ar^{41} is that of two $d_{3/2}$ protons and therefore equivalent to that of two $d_{3/2}$ proton holes. We can thus use in the calculations either $d_{3/2}$ protons of $d_{3/2}^{-1}$ proton holes. We choose arbitrarily the last possibility, so that $j=3/2$ characterizes now a hole state in the $d_{3/2}$ proton shell. We can thus take the differences of V_J directly from K^{40}. Using (2) we then obtain (in MeV)

$$F^1=1.62,\quad F^2=4.89,\quad F^3=-0.71.$$

Had we used instead the interaction between the $f_{7/2}$ neutron and a $d_{3/2}$ proton (either taken from Cl^{38} or calculated from K^{40}) the absolute values of F^k would have been the same. Only the coefficients of the odd tensors F^1 and F^3 would have changed their signs. The results of the calculation are the same in both cases. The numerical coefficients in the interaction matrices assume a much simpler form if instead of F^k, certain multiples of them are used. We, therefore, introduce the values of F_k defined by

$$\begin{aligned}F_1&=[(210)^{1/2}/1260]F^1=0.0186\text{ MeV},\\ F_2&=[(42)^{1/2}/2100]F^2=0.0151\text{ MeV},\\ F_3&=[(330)^{1/2}/11550]F^3=-0.00113\text{ MeV}.\end{aligned} \quad (3)$$

The matrix elements of the interaction in the $d_{3/2}^{-2}\,f_{7/2}^3$ configuration are now given by

$$\langle j^2(J_p)j'^3(J_n)J|\sum_{i=1}^{2}\sum_{h=1}^{3}V_{ih}|j^2(J_p')j^3(J_n')J\rangle$$
$$=\langle j^2(J_p)j'^3(J_n)J|\sum_{k=0}^{3}(\mathbf{U}_p^{(k)}\cdot\mathbf{U}_n^{(k)})F^k|j^2(J_p')j'^3(J_n')J\rangle, \quad (4)$$

where $\mathbf{U}_p^{(k)}=\mathbf{u}_1^{(k)}+\mathbf{u}_2^{(k)}$ operates on the wave functions of the two $d_{3/2}^{-1}$ proton holes and $\mathbf{U}_n^{(k)}=\mathbf{u}_1^{(k)}+\mathbf{u}_2^{(k)}+\mathbf{u}_3^{(k)}$ operates on those of the three $f_{7/2}$ neutrons. Using a well known formula [ST (15.5)], we obtain for the matrix elements (4) the form

$$\sum_k(-1)^{J_p'+J_n+J}(j^2J_p\|\mathbf{U}_p^{(k)}\|j^2J_p')$$
$$\times(j'^3J_n\|\mathbf{U}_n^{(k)}\|j'^3J_n')\begin{Bmatrix}J_p & J_n & J\\ J_n' & J_p' & k\end{Bmatrix}F^k. \quad (5)$$

Thus, the matrix elements (4) are given in terms of the reduced matrix elements of the single particle operators $\mathbf{U}^{(k)}$. The reason for using the tensor expansion in (1) becomes now apparent. We can make use of the selection rules on the reduced matrix elements of $\mathbf{U}^{(k)}$. Since the $d_{3/2}^{-2}$ configuration is in the middle of the $d_{3/2}$ shell, matrix elements of tensor operators with $k>0$, even, vanish between states with the same seniority [ST (28.41)]. The matrix elements of $\mathbf{U}_p^{(2)}$ vanish if $J_p=J_p'=2$ (for $J_p=J_p'=0$ the matrix elements of any $\mathbf{U}^{(k)}$ with $k>0$ vanish). Tensor operators with k odd are diagonal in the seniority scheme and their nonvanishing matrix elements are independent of n [ST (28.22)]. Thus, the matrix elements of $\mathbf{U}_n^{(1)}$ and $\mathbf{U}_n^{(3)}$ vanish between $J_n=7/2$ ($v=1$) and $J_n'=3/2$, 5/2, 9/2, 11/2, 15/2 ($v=3$). Moreover, $\mathbf{U}_n^{(1)}$ is proportional to J_n and is therefore diagonal in our scheme. These facts facilitate the actual computations.

The reduced matrix elements of $\mathbf{U}_p^{(k)}$ are given by [ST (15.26) and (15.27)]

$$((3/2)^2J_p\|\mathbf{U}_p^{(k)}\|(3/2)^2J_p')$$
$$=(-1)^{k+1}2[(2J_p+1)(2J_p'+1)]^{\frac{1}{2}}\times\begin{Bmatrix}3/2 & 3/2 & k\\ J_p & J_p' & 3/2\end{Bmatrix}. \quad (6)$$

In the three-particle case we first expand the j^3 wave functions in coefficients of fractional parentage

$$\psi(j^3J_n)=\sum_{J_n''}[j^2(J_n'')jJ_n]j^3J_n]\psi(j^2(J_n'')j_3J_n) \quad (7)$$

Because of the antisymmetry, J_n'' must be even. We then obtain [ST (26.33)]

$$((3/2)^3J_n\|\mathbf{U}_n^{(k)}\|(3/2)^3J_n')$$
$$=(-1)^{3/2+J_n+k}[(2J_n+1)(2J_n'+1)]^{\frac{1}{2}}\times\sum_{J_n''}[(3/2)^2(J_n'')3/2J_n][(3/2)^3J_n]\times[(3/2)^2(J_n'')3/2J_n'][(3/2)^3J_n']\times\begin{Bmatrix}3/2 & 3/2 & k\\ J_n & J_n' & J_n''\end{Bmatrix}. \quad (8)$$

The values of the reduced matrix elements (7) and (8)

were calculated in all cases. These were used in (5) in order to construct the interaction matrices for the various values of J given below.

The numerical values of the matrix elements were obtained by using the values (3) for the F^k. To these matrices were added the diagonal elements of the mutual proton interaction in the J_p states of the $d_{3/2}{}^{-2}$ configuration and the mutual neutron interaction in the various J_n states of the $f_{7/2}{}^3$ configuration. The resulting matrices were then diagonalized and thus the various eigenvalues in the various J levels were obtained.

Matrices of the $d_{3/2}-f_{7/2}$ Interaction Energy

$J=1/2$

	(2, 3/2)	(2, 5/2)
(2, 3/2)	$9F_1-99F_3$	$-7(231)^{1/2}F_3$
(2, 5/2)	$-7(231)^{1/2}F_3$	$14F_1+63/2F_3$

$J=3/2$

	(0, 3/2)	(2, 3/2)	(2, 5/2)	(2, 7/2)
(0, 3/2)	0	$-21(5)^{1/2}F_2$	$-2(330)^{1/2}F_2$	$-30(2)^{1/2}F_2$
(2, 3/2)	$-21(5)^{1/2}F_2$	$6F_1+99F_3$	$-7(66)^{1/2}F_3$	0
(2, 5/2)	$-2(330)^{1/2}F_2$	$-7(66)^{1/2}F_3$	$11F_1-9F_3$	0
(2, 7/2)	$-30(2)^{1/2}F_2$	0	0	$18F_1-165/2F_3$

$J=5/2$

	(0, 5/2)	(2, 3/2)	(2, 5/2)	(2, 7/2)	(2, 9/2)
(0, 5/2)	0	$4(55)^{1/2}F_2$	$13/2(70)^{1/2}F_2$	$-10/3(462)^{1/2}F_2$	$-25/6(78)^{1/2}F_2$
(2, 3/2)	$4(55)^{1/2}F_2$	$F_1-297/7F_3$	$13/2(154)^{1/2}F_3$	0	$-15/14(4290)^{1/2}F_3$
(2, 5/2)	$13/2(70)^{1/2}F_2$	$13/2(154)^{1/2}F_3$	$6F_1-81/4F_3$	0	$-25/22(429)^{1/2}F_3$
(2, 7/2)	$-10/3(462)^{1/2}F_2$	0	0	$13F_1+55F_3$	0
(2, 9/2)	$-25/6(78)^{1/2}F_2$	$-15/14(4290)^{1/2}F_3$	$-25/22(429)^{1/2}F_3$	0	$-22F_1+1405/14F_3$

$J=7/2$

	(0, 7/2)	(2, 3/2)	(2, 5/2)	(2, 7/2)	(2, 9/2)	(2, 11/2)
(0, 7/2)	0	$-30F_2$	$5(154)^{1/2}F_2$	$5/3(105)^{1/2}F_2$	$25/9(390)^{1/2}F_2$	$16/3(35)^{1/2}F_2$
(2, 3/2)	$-30F_2$	$-6F_1+99/14F_3$	$-9/4(154)^{1/2}F_3$	0	$-165/28(390)^{1/2}F_3$	0
(2, 5/2)	$5(154)^{1/2}F_2$	$-9/4(154)^{1/2}F_3$	$-F_1+33/2F_3$	0	$-5/22(15015)^{1/2}F_3$	$-1025/132(110)^{1/2}F_3$
(2, 7/2)	$5/3(105)^{1/2}F_2$	0	0	$6F_1+55F_3$	0	0
(2, 9/2)	$25/9(390)^{1/2}F_2$	$-165/28(390)^{1/2}F_3$	$-5/22(15015)^{1/2}F_3$	0	$15F_1-7025/154F_3$	$175/44(546)^{1/2}F_3$
(2, 11/2)	$16/3(35)^{1/2}F_2$	0	$-1025/132(110)^{1/2}F_3$	0	$175/44(546)^{1/2}F_3$	$2600/121F_1-65/22F_3$

$J=9/2$

	(0, 9/2)	(2, 5/2)	(2, 7/2)	(2, 9/2)	(2, 11/2)
(0, 3/2)	0	$-5/2(130)^{1/2}F_2$	$-50/9(78)^{1/2}F_2$	$-175/66(66)^{1/2}F_2$	$140/11(11)^{1/2}F_2$
(2, 5/2)	$-5/2(130)^{1/2}F_2$	$-10F_1-15/4F_3$	0	$-35/44(2145)^{1/2}F_3$	$-205/198(1430)^{1/2}F_3$
(2, 7/2)	$-50/9(78)^{1/2}F_2$	0	$-3F_1-65F_3$	0	0
(2, 9/2)	$-175/66(66)^{1/2}F_2$	$-35/44(2145)^{1/2}F_3$	0	$6F_1-1405/44F_3$	$-175/22(26)^{1/2}F_3$
(2, 11/2)	$140/11(11)^{1/2}F_2$	$-205/198(1430)^{1/2}F_3$	0	$-175/22(26)^{1/2}F_3$	$1700/121F_1+35/11F_3$

$J=11/2$

	(0, 11/2)	(2, 7/2)	(2, 9/2)	(2, 11/2)	(2, 15/2)
(0, 11/2)	0	$16/9(210)^{1/2}F_2$	$-70/33(330)^{1/2}F_2$	$-5/77(5005)^{1/2}F_2$	$-40/7(70)^{1/2}F_2$
(2, 7/2)	$16/9(210)^{1/2}F_2$	$-14F_1+35/2F_3$	0	0	0
(2, 9/2)	$-70/33(330)^{1/2}F_2$	0	$-5F_1+7025/143F_3$	$-25/143(546)^{1/2}F_3$	$-60/143(330)^{1/2}F_3$
(2, 11/2)	$-5/77(5005)^{1/2}F_2$	0	$-25/143(546)^{1/2}F_3$	$600/121F_1+135/77F_3$	$-3400/1001(286)^{1/2}F_3$
(2, 15/2)	$-40/7(70)^{1/2}F_2$	0	$-60/143(330)^{1/2}F_3$	$-3400/1001(286)^{1/2}F_3$	$170/11F_1+4845/13F_3$

$J=13/2$

	(2, 9/2)	(2, 11/2)	(2, 15/2)
(2, 9/2)	$-18F_1-4215/286F_3$	$375/143(39)^{1/2}F_3$	$-120/143(935)^{1/2}F_3$
(2, 11/2)	$375/143(39)^{1/2}F_3$	$-700/121F_1-255/77F_3$	$160/1001(14586)^{1/2}F_3$
(2, 15/2)	$-120/143(935)^{1/}F_3$	$160/1001(14586)^{1/2}F_3$	$105/11F_1-3135/91F_3$

$J=15/2$

	(0, 15/2)	(2, 11/2)	(2, 15/2)
(0, 15/2)	0	$-20/7(210)^{1/2}F_2$	$5/7(1785)^{1/2}F_2$
(2, 11/2)	$-20/7(210)^{1/2}F_2$	$-200/11F_1-5/14F_3$	$-60/7(34)^{1/2}F_3$
(2, 15/2)	$5/7(1785)^{1/2}F_2$	$-60/7(34)^{1/2}F_3$	$30/11F_1-95/7F_3$

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Quasi-Deuteron Model and the Internal Momentum Distribution of Nucleons in C^{12}

YOUNG S. KIM

Department of Physics and Astronomy, The Ohio State University, Columbus, Ohio

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The quasi-deuteron model was assumed in computing the internal momentum distribution of nucleons in C^{12} using the monoergic photodata of Cence and Moyer and the bremsstrahlung photodata of the Purdue group. The computed quasi-deuteron momentum distribution has the form $A\exp(-p^2/4ME_1)/(4\pi ME_1)^{1.5}+Bp^2\exp(-p^2/4ME_2)/1.5\pi^{1.5}(4ME_2)^{2.5}$, where E_1 is 1.5 MeV and E_2 is 5 MeV. The ratio B/A is approximately 2. These two components resemble the $(1s)$ and $(1p)$ wave functions of the shell model. The $(1s)$ component is associated with a binding energy 40 MeV while the $(1p)$ component is associated with a binding energy 10 MeV. The internal momentum distribution of nucleons in C^{12} was calculated from this quasi-deuteron momentum distribution assuming that the nucleons exist in proton-neutron pairs in the nucleus.

INTRODUCTION

SEVERAL independent sources of information are available on the internal momentum distribution of nucleons in light nuclei.[1–8] Two of these are the quasi-elastic proton-proton scattering data[6–8] and the nucleon photoproduction data.[1–5] In a quasi-elastic p-p scattering experiment (see, for example, reference 6) a target is bombarded by a fairly monochromatic proton beam and some suitable angular and energy correlations of the scattered and scattering protons are measured and analyzed to yield the proton binding energy and momentum distribution in the nucleus. These experiments indicate that there are several groups of protons in the nucleus with some characteristic binding energy and momentum distribution. The transparent nucleus model (impulse approximation) is usually employed in these analyses. The binding energy and momentum distribution thus computed are reasonably consistant with the predictions of the shell theory.[6]

In a nucleon photoproduction experiment, a target is bombarded by a photon beam (either bremsstrahlung[1–3] or monochromatized bremsstrahlung[4,5]) and a proton with a given momentum is detected with[2,3] or without[1,4,5] a neutron in coincidence. The n-p coincidence rate as a function of the detection geometry has been seen to peak where the free deuteron photodissociation n-p coincidence rate peaks.[2,3] The width of the n-p coincidence rate as a function of the neutron emission angle is considerably wider than the resolution of the detector system and can be related to the intranucleus momentum distribution of nucleons in nuclei. In addition to the momentum distribution, the energy spectrum of the singly detected protons yields information on the binding energy of nucleons in nuclei. Of course, the width of the angular correlation of n-p

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