Structure of Low-Lying 2^+ States in Even-Even Nuclei and Beta-Gamma Angular Correlations*

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(Received 22 May 1962; revised manuscript received 8 November 1962)

The ground states with spin 2^- of the odd-odd nuclei decay into those of the even-even nuclei through several channels, $2^- \xrightarrow{\beta_1} 0^+$, $2^- \xrightarrow{\beta_2} 2^+ \xrightarrow{\gamma} 0^+$, etc. In the $2^- \xrightarrow{\beta_2} 2^+$ transition, all first forbidden matrix elements other than B_{ij} are reduced in magnitude from their normal values, if the j selection rule of the j-j coupling shell model or the K selection rule of the rotational excitation in the collective model is realized in the main configurations of states 2^- and 2^+. In such a case, the present theory distinguishes whether the β_2 decay obeys the j selection rule or the K selection rule. Furthermore, if the j selection rule holds, the theory gives the configuration of the low-lying 2^+ state. This is done by combining the data on $\beta_2-\gamma$ directional correlations and branching ratio of β_1 and β_2, with the calculated value for $(\int B_{ij})_1/(\int B_{ij})_2$ which is nuclear-model dependent. The first excited 2^+ states in Se^{76}, Sr^{86}, Te^{122}, and Xe^{126} are studied in comparison with available data on beta decays of As^{76}, Rb^{86}, Sb^{122}, and I^{126}.

I. INTRODUCTION

BETA-GAMMA angular correlations have two important areas of application. One is to test the fundamental properties of the law of nature, that is, to test parity nonconservation, charge-conjugation noninvariance,[1] and time-reversal invariance.[2] These properties have been studied since 1957 and are well established now. It has been also confirmed that the beta decay is due to the VA interaction[3] with a two-component neutrino.

The other area is the study of nuclear structure. That is, the formulas for beta-gamma angular correlations are dependent on the decay scheme and various nuclear matrix elements. The formulas allow us to use the experimental data for determining spins and parities of nuclear states and the magnitudes of the nuclear matrix elements. These quantities, especially the last ones, give us a firm basis for study of nuclear structure. Experiments on beta-gamma angular correlations have been performed since 1949. Though the accuracy in these experiments was low in the early 1950's, it has been greatly improved in the last few years and many data have been accumulated. On the other hand, the theoretical works performed are mostly concerned with the formulation of the theory of beta-gamma angular correlations.[4,5]

It is our purpose in this paper to study theoretically the structure of the low-lying 2^+ states in even-even nuclei by using the data on beta-gamma directional correlations and branching ratio of beta decays. As is well known, there are several ways to explain these 2^+ states. One could assume rotational excitation in the collective model[6] or some kind of particle excitation in the j-j coupling shell model.[7] Among these models one

* Work partially supported by the U. S. Atomic Energy Commission.

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[1] T. D. Lee and C. N. Yang, Phys. Rev. **104**, 254 (1956). For experiments on beta-circularly polarized gamma correlation, see, e.g., C. S. Wu, in *Proceeding of the Rehovoth Conference on Nuclear Structure*, edited by M. J. Lipkin (Interscience Publishers, Inc., New York, 1958), p. 346.

[2] M. Morita and R. S. Morita, Phys. Rev. **107**, 1316 (1957); **110**, 461 (1958); E. Ambler, R. W. Hayward, D. D. Hoppes, and R. P. Hudson, *ibid.* **110**, 787 (1958).

[3] M. Goldhaber, L. Grodzins, and A. W. Sunyar, Phys. Rev. **109**, 1015 (1958). M. Morita, R. S. Morita, and M. Yamada, *ibid.* **111**, 237 (1958); M. Morita and R. S. Morita, *ibid.* **111**, 1130 (1958). N. E. Booth, G. W. Hutchinson, A. M. Seger, G. G. Shute, and D. H. White, Nucl. Phys. **11**, 341 (1959).

[4] M. Morita, Progr. Theoret. Phys. (Kyoto) **14**, 27 (1955). M. Morita and R. S. Morita, Phys. Rev. **107**, 1316 (1957); **109**, 2048 (1958).

[5] It is noted here that some of the later publications which review the first forbidden beta decays have misprints and miscalculations. In the paper by T. Kotani [Phys. Rev. **114**, 795 (1959)], $(u$-$z)$ in (A5) should be read as $(u$-$x)$. This changes the spectrum shape factor and all beta-gamma angular correlations drastically. In the paper by H. A. Weidenmüller [Rev. Mod. Phys. **33**, 574 (1961)], many corrections are necessary. For example, the spectrum shape factors $C(W)$ in Figs. 2 and 3 should be identical. The reason is that the $C(W)$ is bilinear with respect to the nuclear parameters and there are no interferences of different rank matrices. With his choice of parameters, there should be no difference for $C(W)$. The anisotropy $\epsilon(W)$ in Fig. 10 should change its sign, since Y and V in Figs. 9 and 10 are the same in magnitude but different in sign. Our numerical calculations indicate that several other figures are wrong. The present authors do not know whether these errors are purely computational. Also, in his notation, $x=C_V\int i\mathbf{r}$, etc., can never be unity.

[6] A. Bohr, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **26**, No. 14 (1952); A. Bohr and B. R. Mottelson, *ibid.* **27**, No. 16 (1953).

[7] M. G. Mayer and J. H. D. Jensen, *Elementary Theory of Nuclear Shell Structure* (John Wiley & Sons, Inc., New York, 1955).

could find what kind of excitation is realized in each nucleus by studying beta decays of the odd-odd nucleus with spin and parity 2^-. As is shown in Fig. 1, the ground state of the odd-odd nucleus decays by emitting β_1 to the ground state of the even-even nucleus or by emitting β_2 to the low-lying 2^+ state, which in most cases is the first excited state. It may also decay to the other excited states. We can measure the $\beta_2-\gamma$ directional correlation in the decay scheme $2^- \xrightarrow{\beta_2} 2^+ \xrightarrow{\gamma} 0^+$ and the branching ratios, a_1 and a_2, in β_1 and β_2 decays. In analysis of these data, we adopt the so-called modified B_{ij} approximation.[8] We introduce two nuclear parameters X and Y which are relative contributions from nuclear matrix elements with rank zero and one, respectively, compared with that of $(\int B_{ij})$. The beta-gamma directional correlation data give one relation between X and Y. A calculated value of $(\int B_{ij})_1/(\int B_{ij})_2$ combined with branching ratios, a_1 and a_2, gives another relation between X and Y. Here $(\int B_{ij})_1/(\int B_{ij})_2$ is nuclear-model dependent and the subscripts 1 and 2 refer to the β_1 and β_2 decays, respectively. If this set of equations have solutions for X and Y, the assumed nuclear model is possibly realized. In this way, we can examine each nuclear model and find the configuration of the 2^+ states.

In Sec. 2, the reason why we adopt the so-called modified B_{ij} approximation is described. In Sec. 3, we give the set of two equations in which the model-dependent quantity $(\int B_{ij})_1/(\int B_{ij})_2$ is related to X and Y. In Sec. 4, the quantity $(\int B_{ij})_1/(\int B_{ij})_2$ is given for several nuclear models. In Sec. 5, we discuss the nuclear structure of the first excited 2^+ states in Se^{76}, Sr^{86}, Te^{122}, and Xe^{126} by using experimental data on beta decays of As^{76},[9] Rb^{86},[10] Sb^{122},[11] and I^{126},[12] respectively. In Sec. 6, the conclusions are given.

2. MODIFIED B_{ij} APPROXIMATION

In the early 1950's, most of the experimental data on beta-gamma directional correlation were inaccurate. However, the first precision experiment was done on Sb^{124},[13] which was studied theoretically in detail by two of the present authors.[14] A calculation was made with assumptions of both first and second forbidden beta decays, since the ground state of Sb^{124} was believed to be 4^+ at that time. There were two important findings: (A) The ground state of Sb^{124} is 3^- and there is no possibility of 4^+,[15] and (B) the matrix element B_{ij} in this beta decay is large compared with the other first forbidden matrix elements. The case, $B_{ij}=0$, is excluded, since in this case we cannot explain the beta-gamma directional correlation data. The large B_{ij} compared with the other matrix elements can be understood qualitatively from the selection rule effect. In some configurations of the j-j coupling shell model, all matrix elements other than B_{ij} vanish due to the spin selection rule (j selection rule) for the single-particle operator. However, they cannot be identically zero, since the spectrum shape factor is almost energy independent in experiments, while that of B_{ij} is (p^2+q^2) and strongly energy dependent, p and q being the momenta of the electron and neutrino, respectively. Theoretically, small but nonzero values of the first forbidden matrix elements other than B_{ij} can be obtained by mixing configurations. Thus the so-called modified B_{ij} approximation was made.[8]

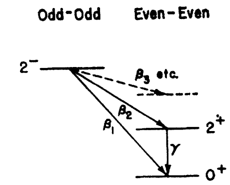

FIG. 1. Decay scheme.

In the modified B_{ij} approximation, we take into account the B_{ij} term and the leading terms of the other first forbidden matrix elements. These leading terms are the contribution of momentum-type matrices and Coulomb corrections for the coordinate-type matrices,[16] if $\alpha Z/2R \gg W_0$. Explicit formulas for beta-gamma directional correlations in this approximation are given in reference 8. Although we have discussed only the j selection rule, a similar consideration holds for the strong-coupling theory in the collective model, if the beta decay is, e.g., $\Delta j=0$ and $\Delta K=2$ (K selection rule).

We do not claim that the modified B_{ij} approximation is quite accurate for all purposes. However, we emphasize that the B_{ij} is sometimes large compared with the other matrix elements and in such cases, this approximation is convenient for *qualitative* discussion of the beta decay. It also gives the right order of magnitude for anisotropy for all electron energies. A slight modification of the present approximation may be necessary to explain the experimental data in detail. For example, the energy dependence of beta-gamma directional correlation is given in the form

$$W(\theta)=1+\epsilon P_2(\cos\theta),$$

with

$$\epsilon=(p^2/W)(l+mW),$$

[8] Z. Matumoto, M. Morita, and M. Yamada, Bull. Kobayasi Inst. Phys. Res. **5**, 210 (1955).

[9] H. Rose, Phil. Mag. **44**, 739 (1953).

[10] H. J. Fischbeck and R. G. Wilkinson, Phys. Rev. **120**, 1762 (1960). P. C. Simms, A. Namenson, and C. S. Wu, Bull. Am. Phys. Soc. **7**, 34 (1962), have obtained almost the same data.

[11] R. M. Steffen, Phys. Rev. **123**, 1787 (1961).

[12] D. T. Stevenson and M. Deutsch, Phys. Rev. **84**, 1071 (1951).

[13] E. K. Darby and W. Opechowski, Phys. Rev. **83**, 676 (1951); D. T. Stevenson and M. Deutsch, *ibid.* **83**, 120 (1951).

[14] M. Morita and M. Yamada, Progr. Theoret. Phys. (Kyoto) **8**, 449 (1952); **10**, 641 (1953).

[15] If there is no $\cos^4\theta$ term in the beta-gamma directional correlation, the beta decay is the first forbidden transition and the spin and parity are 3^-. However, there is a possibility that the $\cos^4\theta$ term is so small that it could not be detected experimentally. This was considered in reference 14, but it was ruled out.

[16] The energy independence of the shape correction factor is given either by large Coulomb terms or by some cancellation of the Coulomb terms with a precarious balance among the energy-dependent terms. Here we assume the cancellation to be not the case.

where l and m are energy independent. W is the energy of the electron. In this case, we cannot discuss the m term in the present approximation, since the l term is about 10 times larger than the m term. Such a discussion is obviously beyond the validity of the present approximation. To do this, we need the formulas with all six nuclear matrix elements in the first forbidden transitions.

For a similar reason, the modified B_{ij} approximation is invalid if the measured anisotropy is very small [for example, in the case of the beta decay of Au^{198}; see the third paper of reference 4].

We also mention that the so-called ξ approximation [the large Coulomb energy approximation] is a limiting case of the modified B_{ij} approximation. That is, the formulas for the former are reduced to those for the latter by setting $B_{ij}=0$, or equivalently by taking only the square terms of X and Y.

3. RELATIONS BETWEEN NUCLEAR PARAMETERS X AND Y

Let us consider the decay scheme given in Fig. 1. In the β_1 decay, $2^- \to 0^+$, there is only one matrix element,

$$C_A\left(\int B_{ij}\right)_1.$$

In the β_2 decay, $2^- \to 2^+$, there are six matrix elements,

$$C_A\int\gamma_5,\quad C_A\int\boldsymbol{\sigma}\cdot\mathbf{r},$$

$$C_V\int\mathbf{r},\quad C_V\int\boldsymbol{\alpha},\quad C_A\int\boldsymbol{\sigma}\times\mathbf{r},$$

and

$$C_A\left(\int B_{ij}\right)_2.$$

Therefore, the formulas for beta-gamma angular correlations are expressed with five nuclear parameters which are ratios among the above six matrix elements. In the modified B_{ij} approximation, however, we need only two parameters,

$$X=\left[\pm(\alpha Z/2R)i\int\boldsymbol{\sigma}\cdot\mathbf{r}-\int\gamma_5\right]\Big/ i\left(\int B_{ij}\right)_2, \quad (1)$$

and

$$Y=\left[\pm(\alpha Z/2R)C_A\int\boldsymbol{\sigma}\times\mathbf{r} + (\alpha Z/2R)iC_V\int\mathbf{r}\mp C_V\int\boldsymbol{\alpha}\right]\Big/ iC_A\left(\int B_{ij}\right)_2, \quad \text{for } \beta^{\mp}. \quad (2)$$

These parameters, X and Y, are the relative magnitudes of the contributions from matrices of rank zero and one with respect to that of $(\int B_{ij})_2$. In principle, one could obtain the values of these two parameters from two different experiments, and these could be compared with values calculated theoretically. However, the theoretical calculation requires a more precise knowledge of the radial part of the nuclear wave function than is possible, so that we do not use this method.

Instead, we introduce the term $(\int B_{ij})_1/(\int B_{ij})_2$ which can be calculated without knowing the nuclear radial wave function, and get a relation between X and Y which depends on the branching ratio and $(\int B_{ij})_1/(\int B_{ij})_2$. Another relationship between X and Y can be obtained from the beta-gamma directional correlation measurements. These two relations allow us to pick a value for $(\int B_{ij})_1/(\int B_{ij})_2$ which corresponds to a particular model.

Now let us find this set of equations. In the successive decays, $2^- \xrightarrow{\beta_2} 2^+ \xrightarrow{\gamma} 0^+$, the beta-gamma directional correlation is given by

$$W(\theta)=1+a(W)\cos^2\theta, \quad (3)$$

with anisotropy

$$a(W)=\frac{p^2}{W}\frac{-(1/2)(3/7)^{1/2}X+(3/4)(1/14)^{1/2}Y-(3/224)W}{X^2+Y^2+(1/2)(1/21)^{1/2}(p^2/W)X-(1/4)(1/14)^{1/2}(p^2/W)Y+(1/12)q^2+(59/672)p^2}. \quad (4)$$

This is derived from Eq. (57) of the third paper in reference 4. Here, the natural units $\hbar=m=c=1$ are adopted. Inserting the experimental value of $a(W)$ into Eq. (4) at a certain energy W, we have a relation between X and Y, which makes a circle in the parameter plane $X-Y$:

$$(X-X_0)^2+(Y-Y_0)^2=R^2, \quad (5)$$

with

$$X_0=-(1/4)(1/21)^{1/2}(p^2/W)[3+a(W)]/a(W), \quad (6)$$

$$Y_0=-(1/2)(3/2)^{1/2}X_0, \quad (7)$$

$$R^2=X_0^2+Y_0^2+(W/8)(3/7)^{1/2}X_0-(1/12)(q^2+p^2). \quad (8)$$

If we use $\epsilon(W)$ defined in $W(\theta)=1+\epsilon(W)P_2(\cos\theta)$, we replace $[3+a(W)]/2a(W)$ by $1/\epsilon(W)$ in Eq. (6).

The other relation between X and Y, which is again a circle in the $X-Y$ plane, is derived from the branching ratios of the β_1 and β_2 decays and the calculated value $(\int B_{ij})_1/(\int B_{ij})_2$. The sum of the branching ratios is unity:

$$a_1+a_2+\cdots=1. \quad (9)$$

Here $a_1, a_2, \cdots$, are the branching ratios of $\beta_1, \beta_2, \cdots$, respectively. The transition probability is inversely proportional to the half-life t of the initial state:

$$1/t_1+1/t_2+\cdots=1/t, \quad (10)$$

where $t_1, t_2, \cdots,$ are the half-lives for branches 1, 2, $\cdots$, respectively. Thus, we have

$$a_i = t/t_i. \tag{11}$$

For the β_1 decay, $2^- \to 0^+$, we have

$$f_{c1}t_1 = D/\sum_{ij} |B_{ij}|^2, \tag{12}$$

with the universal constant

$$D = (2\pi^3/C_A{}^2)\ln 2, \tag{13}$$

and f_{c1} is the corrected integrated Fermi function given in Eq. (15) below. We introduce the Fermi functions by

$$f = \int_1^{W_0} F(Z,W)pWq^2dW, \tag{14}$$

and

$$f_c = \int_1^{W_0} F(Z,W)pWq^2(p^2+q^2)dW/12. \tag{15}$$

Here $F(Z,W)$, f, and f_c are the Fermi function, integrated Fermi function, and the corrected integrated Fermi function, respectively. The f and f_c take different values [f_1 and f_{c1}, f_2 and f_{c2}] for β_1 and β_2, since the maximum electron energies W_0 are different. Our definition of the matrix $|\int B_{ij}|^2$ differs from $\sum_{ij}|B_{ij}|^2$ by a constant factor,[17]

$$\sum_{ij} |B_{ij}|^2 = [(2j_1+1)/(2j+1)]\left|\int B_{ij}\right|^2, \tag{16}$$

in the $j \to j_1$ transition. For the β_1 decay

$$f_{c1}t_1 = 5D\bigg/\left|\left(\int B_{ij}\right)_1\right|^2, \tag{17}$$

since $j=2$ and $j_1=0$. For the β_2 decay, the spectrum shape factor in the present approximation is

$$C(W) = X^2+Y^2+(1/12)(p^2+q^2), \tag{18}$$

so that

$$\int_1^{W_0} F(Z,W)pWq^2[X^2+Y^2+(1/12)(p^2+q^2)]dW = D\left[t_2\left|\left(\int B_{ij}\right)_2\right|^2\right]^{-1} = a_2 D\left[t\left|\left(\int B_{ij}\right)_2\right|^2\right]^{-1}. \tag{19}$$

Inserting Eqs. (14) and (15) into Eq. (19), we have

$$(X^2+Y^2)f_2 + f_{c2} = a_2 D\left[t\left|\left(\int B_{ij}\right)_2\right|^2\right]^{-1}. \tag{20}$$

From Eqs. (11) and (17), we have

$$t = 5a_1 D\left[f_{c1}\left|\left(\int B_{ij}\right)_1\right|^2\right]^{-1}. \tag{21}$$

Inserting t into Eq. (20), we finally obtain

$$X^2+Y^2 = R'^2, \tag{22}$$

with

$$R'^2 = (f_{c2}/f_2)\left[(a_2 f_{c1}/5a_1 f_{c2}) \times \left|\left(\int B_{ij}\right)_1 \bigg/ \left(\int B_{ij}\right)_2\right|^2 - 1\right]. \tag{23}$$

In the expression for R', a_1 and a_2 are obtained from the experimental data; f_2, f_{c1}, and f_{c2} are calculated numerically; and $(\int B_{ij})_1/(\int B_{ij})_2$ is the calculated quantity that is nuclear-model dependent. The set of equations, (5) and (22), determine X and Y.

4. CALCULATION OF $(\int B_{ij})_1/(\int B_{ij})_2$

In this section, we give $(\int B_{ij})_1/(\int B_{ij})_2$ for different nuclear models. We define the reduced nuclear matrix elements $\int B_{ij}$ by

$$(j_1 m_1|\textstyle\sum_k T_{2M}{}^{(k)}|jm) = (j2mM|j_1 m_1)\left(\int B_{ij}\right)_1 \tag{24}$$

in the transition, $jm \to j_1 m_1$.[17] Here $T_{2M}{}^{(k)}$ is the tensor operator of rank two. The subscript M is the magnetic quantum number, while the superscript (k) refers to the kth nucleon.

A. Rotational Excitation of the Collective Model

We assign quantum numbers j and K for each nuclear state. The j, j_1, j_2, and L are nuclear spins of the initial state, the ground and excited states of the daughter nucleus, and the rank of the beta matrix element, respectively. The K's are also given with the same subscripts. Then the ratio is[8]

$$\left(\int B_{ij}\right)_1 \bigg/ \left(\int B_{ij}\right)_2 = [(2j_2+1)/(2j_1+1)]^{1/2} \times (jLKK_L|j_1K_1)/(jLKK_L|j_2K_2). \tag{25}$$

In our case, we have

$$L=j=j_2=2,\quad j_1=K_1=K_2=0,\quad K=-K_L=\pm 2. \tag{26}$$

[17] By definition, $\int B_{ij}$, $\int \mathbf{r}$, etc., are identical with the notation $\mathfrak{M}(B_{ij})$, $\mathfrak{M}(\mathbf{r})$, etc., in our previous publications, (e.g., reference 4). Since some people are reluctant to use the symbol $\mathfrak{M}$, we adopt the present one which looks more familiar. As a result, the notation, $\int \mathbf{r}$, in E. J. Konopinski and G. E. Uhlenbeck, Phys. Rev. **60**, 308 (1941), is different from ours by a constant factor. For example, in the beta decay $j \to j_1$, we have the relations

$$\left|\int B_{ij}\right|^2 \equiv |\mathfrak{M}(B_{ij})|^2 = [(2j+1)/(2j_1+1)](\textstyle\sum_{ij}|B_{ij}|^2)_{\text{K-U}},$$

$$\left|\int \mathbf{r}\right|^2 \equiv |\mathfrak{M}(\mathbf{r})|^2 = [(2j+1)/(2j_1+1)]\left(\left|\int \mathbf{r}\right|^2\right)_{\text{K-U}},$$

etc. This constant factor is important in the calculation of the ft values.

Therefore,

$$\left(\int B_{ij}\right)_1 \Big/ \left(\int B_{ij}\right)_2 = \left(\frac{7}{2}\right)^{1/2}. \quad (27)$$

This value is common for all nuclei which have the decay scheme given in Eq. (26).

B. *j-j* Coupling Shell Model

The calculation is more complicated in the j-j coupling shell model. We assume that the ground state of the daughter nucleus has seniority zero for both proton and neutron shells, and the first excited state has seniority two for the one of proton and neutron shells and zero for the other. The configurations of the parent and daughter nuclei are $(j_p)^{\pi-1}(j_n)^{\nu+1}$ and $(j_p)^{\pi}(j_n)^{\nu}$, respectively, π and ν being even integers. The j_p and j_n are the angular momenta of proton and neutron which are relevant to the beta decay. In these configurations, we can have two kinds of particle excitation. Either, protons are excited to have seniority two and neutrons have seniority zero. Or, protons have seniority zero and neutrons have seniority two.

For the first case, we evaluate $(\int B_{ij})_1$ by using the method given by Racah.[18]

$$(2200|00)\left(\int B_{ij}\right)_1 = ((j_p)^{\pi}(v=0,0)(j_n)^{\nu}(v=0,0),00|\textstyle\sum_k T_{20}^{(k)}| \times(j_p)^{\pi-1}(v=1,j_p)(j_n)^{\nu+1}(v=1,f_n),20). \quad (28)$$

The symbols to the right of the tensor operator stand for the wave function of the initial state, i.e., $(\pi-1)$ protons are in the j_p orbit with seniority $v=1$ and angular momentum j_p, $(\nu+1)$ neutrons are in j_n orbit with seniority $v=1$ and angular momentum j_n, and these two shells make the resultant angular momentum two with its magnetic quantum number zero. On the left, both the proton and neutron shells have the seniority $v=0$ and the angular momentum zero, and the total system has zero angular momentum, which represents the final state. If we decompose the initial- and final-state wave functions in those of proton and neutron shells, Eq. (28) becomes

$$(2200|00)\left(\int B_{ij}\right)_1 = \sum_{\mu} ((j_p)^{\pi}(v=0,0),00|((j_n)^{\nu}(v=0,0),00|\textstyle\sum_k T_{20}^{(k)}| \times(j_p)^{\pi-1}(v=1,j_p),j_p-\mu) \times|(j_n)^{\nu+1}(v=1,j_n),j_n\mu)(j_pj_n-\mu\mu|20). \quad (29)$$

[18] G. Racah, Phys. Rev. **61**, 186 (1942); **62**, 438 (1942).

Furthermore, the neutron shell of the initial state is decomposed into ν neutrons and one neutron with the fractional parentage coefficient, and the proton shell for the final state is decomposed similarly.

$$\begin{aligned}(2200|00)\left(\int B_{ij}\right)_1 =& \sum_{\mu\mu'} \pi^{1/2}((j_p)^{\pi}(v=0,0),0\{|(j_p)^{\pi-1}(v=1,j_p)j_p,0) \\ &\times((j_p)^{\pi-1}(v=1,j_p),j_p-\mu'|(j_pj_p-\mu'\mu'|00) \\ &\times((j_n)^{\nu}(v=0,0),00|(j_p\mu'|T_{20}^{(k)}|j_n\mu) \\ &\times|(j_n)^{\nu}(v=0,0),00)(j_pj_n-\mu\mu|20) \\ &\times|(j_p)^{\pi-1}(v=1,j_p),j_p-\mu)(\nu+1)^{1/2} \\ &\times((j_n)^{\nu}(v=0,0)j_n,j_n|\}(j_n)^{\nu+1}(v=1,j_n),j_n). \quad (30)\end{aligned}$$

Here a factor $(\nu+1)^{1/2}$ comes from the fact that $(\nu+1)$ neutrons in the initial state are equivalent. $\pi^{1/2}$ comes from a similar consideration for protons. The fractional parentage coefficients are given by[19]

$$((j_p)^{\pi}(v=0,0),0\{|(j_p)^{\pi-1}(v=1,j_p)j_p,0)=1, \quad (31)$$

and

$$((j_n)^{\nu}(v=0,0)j_n,j_n|\}(j_n)^{\nu+1}(v=1,j_n),j_n) = [(2j_n+1-\nu)/(\nu+1)(2j_n+1)]^{1/2}. \quad (32)$$

We use the Wigner-Eckert theorem for the matrix element of the kth nucleon, the Racah coefficient, and the orthonormality of wave functions. Then Eq. (30) becomes

$$\begin{aligned}(2200|00)\left(\int B_{ij}\right)_1 =& \sum_{\mu} [\pi(2j_n+1-\nu)/(2j_n+1)(2j_p+1)]^{1/2}(j_p\|T_2^{(k)}\|j_n) \\ &\times(j_n2\mu0|j_p\mu)(j_pj_p-\mu\mu|00)(j_pj_n-\mu\mu|20) \\ =& (2200|00)(j_p\|T_2^{(k)}\|j_n) \\ &\times[5\pi(2j_n+1-\nu)/(2j_n+1)]^{1/2}W(2j_n0j_p,j_p2). \quad (33)\end{aligned}$$

Therefore,

$$\left(\int B_{ij}\right)_1 = (j_p\|T_2^{(k)}\|j_n)[5\pi(2j_n+1-\nu)/(2j_n+1)]^{1/2} \times W(2j_n0j_p,j_p2). \quad (34)$$

A similar calculation for the β_2 decay, $2^-\to 2^+$, gives us

$$\left(\int B_{ij}\right)_2 = (j_p\|T_2^{(k)}\|j_n) \times[10(2j_p+1-\pi)(2j_n+1-\nu)/(2j_p-1)(2j_n+1)]^{1/2} \times W(2j_n2j_p,j_p2). \quad (35)$$

Here we assume that the protons are excited (i.e., $v=2$ for proton shell).

[19] C. S. Schwartz and A. de Shalit, Phys. Rev. **94**, 1257 (1954).

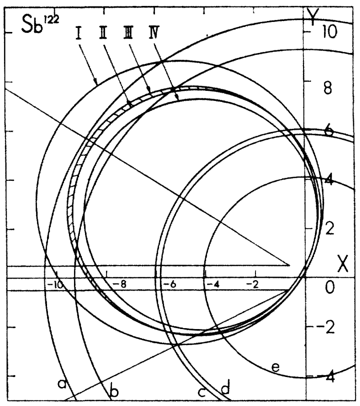

FIG. 2. Parameter plane in the case of $Sb^{122} \to Te^{122}$. The radii, R, and the center position, X_0 and Y_0, of the experimental circles I–IV are given in Table I. The radii, R', of the theoretical circles, *a*, *b*, *c*, *d*, *e*, are given in Table III. Here *a*, *b*, *c*, *d*, *e* refer to $(g_{7/2})^6(h_{11/2})^\nu$, $(g_{7/2})^\pi(h_{11/2})^2$, $(g_{7/2})^4(h_{11/2})^\nu$, $(g_{7/2})^\pi(h_{11/2})^4$, $(g_{7/2})^\pi(h_{11/2})^6$, respectively; see Sec. 5.

Finally, the ratio of matrix elements, $\int B_{ij}$ for β_1 and β_2 decays is

$$\left(\int B_{ij}\right)_1 \Big/ \left(\int B_{ij}\right)_2 = (-)^{j_p-j_n}[\pi(2j_p-1)/10(2j_p+1)(2j_p+1-\pi)]^{1/2} \times[W(22j_pj_p,2j_n)]^{-1} \text{ for protons excited.} \quad (36)$$

This formula does not depend on how many neutrons are in the j_n orbit, since these neutrons have seniority zero in the 2^+ state. On the other hand, if the neutrons are excited to have seniority two, we have the following formula by a similar calculation:

$$\left(\int B_{ij}\right)_1 \Big/ \left(\int B_{ij}\right)_2 = (-)^{1+j_p-j_n}[(2j_n-1)(2j_n+1-\nu)/10\nu(2j_n+1)]^{1/2} \times[W(22j_nj_n,2j_p)]^{-1} \text{ for neutrons excited.} \quad (37)$$

This does not depend on how many protons are in the j_p orbit.

5. STRUCTURE OF THE FIRST EXCITED 2^+ STATES IN Se^{76}, Sr^{86}, Te^{122}, AND Xe^{126}

In this section, we discuss the structure of the first excited states of Se^{76}, Sr^{86}, Te^{122}, and Xe^{126} using the available data on beta decays of As^{76},[9] Rb^{86},[10] Sb^{122},[11] and I^{126},[12] which have the relevant decay scheme shown in Fig. 1. The analysis of the data is explicitly given in the case of Sb^{122}. The results are also given in the other three cases.

Sb^{122}

As has been shown in Sec. 3, the X and Y obey two relations, Eqs. (5) and (22), both of which are circles in the $X-Y$ plane. For convention, we call them the *experimental circle* and *theoretical circle*. The experimental circle, Eq. (5), is fully determined by the measured value of $a(W)$, while the theoretical circle, Eq. (22), depends on $(\int B_{ij})_1/(\int B_{ij})_2$ and the branching ratio of the beta decays. Since the experimental data for $a(W)$ always have a certain inaccuracy due to errors, the experimental circle has a corresponding thickness. For example, we have $a(W)=0.50-0.57$ at $W=2.00$ (in units of mc^2) given by Steffen.[11] Correspondingly, all points in the area between the experimental circles, I and II in Fig. 2, fit $a(W)$ at $W=2.00$. Similarly, the experimental circles, III and IV, correspond to $W=3.20$, where $a(W)=0.109-0.117$.[11] [For the experimental circles, X_0, Y_0, R are summarized in Table I.] Therefore, a point on the overlap of the above two areas fits $a(W)$ at $W=2.00$ and 3.20.

Next, we can draw the theoretical circles. Numerically integrated values of various Fermi functions are given in Table II with the measured branching ratios.[20] In the third column of Table III, $(\int B_{ij})_1/(\int B_{ij})_2$ is listed for several nuclear models, as given by Eqs. (27), (36), and (37). The radii R' of the theoretical circles are calculated using Eq. (23) and are given in the fourth

TABLE I. Experimental circles. The radius R and the center position, X_0 and Y_0, are calculated with Eqs. (6)–(8). Experimental data $a(W)_{exp}$, are given in references 10 and 11 for Rb^{86} and Sb^{122}, respectively. For Sb^{122}, the numbers are the same as those in Fig. 2. W is expressed in units of mc^2.

No.	W	$a(W)_{exp}$	X_0	Y_0	R
Rb^{86}					
I	1.31	0.077	−1.20	0.74	1.31
II	1.31	0.093	−1.00	0.61	1.06
III	2.03	0.268	−1.03	0.63	1.01
IV	2.03	0.298	−0.94	0.57	0.88
Sb^{122}					
I	2.00	0.050	−5.0	3.1	5.8
II	2.00	0.057	−4.4	2.7	5.0
III	3.20	0.109	−4.5	2.8	5.1
IV	3.20	0.117	−4.2	2.6	4.7

TABLE II. Calculated values of Fermi functions. The Fermi functions, with definitions in Eqs. (14) and (15), are calculated numerically. The maximum energies, W_0 (in units of mc^2) and the branching ratios, a_1 and a_2, of the β_1 and β_2 decays are taken from reference 20.

Beta decay	W_0 in β_1	W_0 in β_2	a_1 (%)	a_2 (%)	f_{c1}	f_2	f_{2c}
${}_{33}As^{76} \to {}_{34}Se^{76}$	6.81	5.72	56.4	30.6	2500	496	720
${}_{37}Rb^{86} \to {}_{38}Sr^{86}$	4.57	2.40	91	9	160	5.03	0.924
${}_{51}Sb^{122} \to {}_{52}Te^{122}$	4.86	3.74	30	63	431	110	62.1
${}_{53}I^{126} \to {}_{54}Xe^{126}$	3.45	2.69	9	29	36.4	19.9	4.95

[20] For branching ratios, a_1 and a_2, and the maximum energy W_0, we use K. Way *et al.*, *Nuclear Data Tables*, National Research Council (National Bureau of Standards, Washington, D. C., 1960). For the Fermi function, $F(Z,W)$, we use *Tables for the Analysis of Beta Spectra* (National Bureau of Standards, Washington, D. C., 1952).

TABLE III. Beta decays of Sb^{122} and I^{126}. Experimental data for $a(W)$ are given in references 11 and 12 for Sb^{122} and I^{126}, respectively. The spectrum shape factors $C(W)$ are nearly isotropic in experiments for both cases. $(\int B_{ij})_1/(\int B_{ij})_2$ are given by Eqs. (27), (36), and (37). The radius R' of the theoretical circle is calculated by Eq. (23). X and Y for Sb^{122} are found in Fig. 2. When a set of X and Y give a good (fair) fit with data, it is "Good" ("Fair"), otherwise "No." The spectrum shape factor is checked only if $a(W)$ is "Good."

Nuclear model	Configuration of the 2^+ state p n	$\left\lvert\frac{(\int B_{ij})_1}{(\int B_{ij})_2}\right\rvert^2$	$Sb^{122} \rightarrow Te^{122}$ R'	X	Y	$a(W)$	$C(W)$	$I^{126} \rightarrow Xe^{126}$ R'	$a(W)$	$C(W)$
Rotational excitation		7/2	2.28			No		1.97	Good	Good
Protons are excited in j-j coupling shell model	$(g_{7/2})^6(h_{11/2})^\nu$	135/2	10.50			No		8.90	No	
	$(g_{7/2})^4(h_{11/2})^\nu$	45/2	6.03	-5.7	-2.2	Fair	Good	5.12	No	
	$(g_{7/2})^2(h_{11/2})^\nu$	15/2	3.43			No		2.93	Good	Good
Neutrons are excited in j-j coupling shell model	$(g_{7/2})^\pi(h_{11/2})^{10}$	55/26	1.71			No		1.50	Good	Good
	$(g_{7/2})^\pi(h_{11/2})^8$	275/52	2.85			No		2.45	Good	Good
	$(g_{7/2})^\pi(h_{11/2})^6$	275/26	4.10			No		3.49	Good	Good
	$(g_{7/2})^\pi(h_{11/2})^4$	275/13	5.84	-5.4	-2.3	Fair	Good	4.97	No	
	$(g_{7/2})^\pi(h_{11/2})^2$	1375/26	9.29	-9.2	0.8	Good	Good	7.87	No	

column. The R' may have an uncertainty [of the order of 10%] due to the assumptions involved in the theoretical calculation. Now if a theoretical circle crosses the overlapping area of experimental curves, the assumed nuclear model is possibly realized. With this cross point (X,Y) [given in the fifth and sixth columns of Table III], the beta-gamma directional correlation can be recalculated and checked with the experimental data. If the energy dependence is well reproduced, this model is "Good" for $a(W)$ as is shown in the seventh column of Table III. If there is no cross point of the theoretical and experimental circles, the model is improbable and is marked by "No." If $a(W)$ is "Good," the spectrum shape factor $C(W)$ is checked and the result is given in the eighth column of Table III. A model for which both $a(W)$ and $C(W)$ are "Good" is the most probable. As a result, the 2^+ state with 0.56-Mev level in Te^{122} is probably $(g_{7/2})^\pi(h_{11/2})^2$, but the possibility of $(g_{7/2})^\pi(h_{11/2})^4$ or $(g_{7/2})^4(h_{11/2})^\nu$ cannot be ruled out. It is interesting to note that the $(g_{7/2})^\pi(h_{11/2})^2$ is the simplest configuration among the above three possibilities, if the level sequence in the j-j coupling shell model is $2d_{3/2}$, $1h_{11/2}$, $3s_{1/2}$ [from the lower level to the higher one]. On the other hand, if the level sequence is $1h_{11/2}$, $3s_{1/2}$, $2d_{3/2}$, the simplest configuration becomes $(g_{7/2})^\pi(h_{11/2})^6$, which can poorly explain the data on the beta-gamma directional correlation.

As is seen in Fig. 2, there are, generally, two sets of X and Y for each nuclear model. In our theory, we do not need to discriminate between these two sets for examining nuclear models. Of course, the second experiments, such as the beta-circularly polarized gamma correlation or the gamma-ray angular distribution following beta-ray emission from the oriented nuclei, can determine a unique set of X and Y. For example, we adopt the data on the angular distributions of the 0.56-MeV gamma ray following the beta ray emission from oriented Sb^{122}.[21] This gives each interior region of the two triangles in Fig. 2. Finally we have $X=-9.2$ and $Y=0.8$ for $(g_{7/2})^\pi(h_{11/2})^2$. The other set (for which $|Y|>|X|$) is ruled out.

Steffen emphasized the proportionality of the anisotropy, $a(W)$, to p^2/W.[11] In the modified B_{ij} approximation, this holds practically, if $|X|$ or $|Y|$ are considerably larger than unity. In other words, formulas in the modified B_{ij} approximation reduce to those for the large Coulomb energy approximation in this condition. In fact, $|X|$ and $|Y|$ for Sb^{122} are relatively large compared with those for Rb^{86}; see Table IV. This means that the selection rule is less effective and the theoretical values $(\int B_{ij})_1/(\int B_{ij})_2$ are less accurate.

Rb^{86}

The experimental values of the anisotropy in the beta-gamma directional correlation for $Rb^{86} \rightarrow Sr^{86}$ are also given in a great precision by Fischbeck and Wilkinson,[10] and Simms, Namenson, and Wu.[10] The result of the analysis is given in Table IV. The configuration of the 1.08-MeV level of Sr^{86} is uniquely determined as $(f_{5/2})^\pi(g_{9/2})^6$. In this particular configuration, two neutrons have to be excited across the main shell of $N=50$. On the other hand, the simplest configuration is either $(f_{5/2})^\pi(g_{9/2})^8$ or $(f_{5/2})^4(g_{9/2})^\nu$. As is shown in Table IV, both configurations cannot explain either $a(W)$ or $C(W)$. One could, therefore, consider the following four possibilities: (A) for some reason, the level sequence in the j-j coupling shell model is so arranged that the two neutrons are really excited across the main shell of $N=50$, (B) the assumed models are too simple, (C) there is some cancellation among the six first forbidden nuclear matrix elements, (D) the selection rule effects are not dominant. In the case (C) or (D), the modified B_{ij} approximation is invalid. *Note added in proof.* After this work was completed, accurate data on the beta-circularly polarized gamma correlation have been reported by Rogers and Boehm.[22] An analysis taking these data into account will make the above situation more clear.

[21] G. E. Bradley, F. M. Pipkin, and R. E. Simpson, Phys. Rev. 123, 1824 (1958).

[22] J. D. Rogers and F. Boehm, Phys. Letters 1, 113 (1962); and (private communication) to M. Morita.

TABLE IV. Beta decays of As^{76} and Rb^{86}. Experimental data are given in references 9 and 10 for As^{76} and Rb^{86}, respectively. See also caption of Table III.

Nuclear model	Configuration of the 2^+ state p n	$\left\lvert \frac{(\int B_{ij})_1}{(\int B_{ij})_2} \right\rvert^2$	R'	$Rb^{86} \to Sr^{86}$ X	Y	$a(W)$	$C(W)$	$As^{76} \to Se^{76}$ R'	$a(W)$	$C(W)$
Rotational excitation		7/2	1.69	−1.68	−0.15	Fair	No	0.68	No	
Protons are excited in j-j coupling shell model	$(f_{5/2})^4(g_{9/2})^\nu$	196/5	5.81	None	None	No		4.47	Good	Good
	$(f_{5/2})^2(g_{9/2})^\nu$	49/5	2.88	None	None	No		1.98	Good	No
Neutrons are excited in j-j coupling shell model	$(f_{5/2})^\pi(g_{9/2})^8$	21/11	1.21	−1.14	−0.40	Fair	No	No real value	No	
				−0.60	1.56	Fair	No			
	$(f_{5/2})^\pi(g_{9/2})^6$	56/11	2.06	−2.01	0.44	Good	Good	1.16	Good	No
				−1.29	1.61	Good	Good			
	$(f_{5/2})^\pi(g_{9/2})^4$	126/11	3.12	None	None	No		2.20	Good	No
	$(f_{5/2})^\pi(g_{9/2})^2$	336/11	5.13	None	None	No		3.91	Good	Good

As^{76}

Compared with the data on Rb^{86} [10] and Sb^{122},[11] the data on As^{76} [9] are inaccurate. However, as is shown in Table IV, we can conclude that either $(f_{5/2})^\pi(g_{9/2})^2$ or $(f_{5/2})^4(g_{9/2})^\nu$ is probable for the 0.56-MeV level of Se^{76}. The former is again the most simple configuration which can be assumed on the basis of the j-j coupling shell model. *Note added in proof.* Refined data of the anisotropy $a(W)$ have been recently reported by Fischbeck and Newsome.[23] They adopted the analysis given here with their new data and the data on the angular distribution of the 0.56-MeV gamma ray following the beta ray from dynamically oriented As^{76}.[24] They have also the same configurations as ours with the values $-2.2 \gtrsim Y \gtrsim -2.8$ and $-3.2 \gtrsim X \gtrsim -4.2$. The data on the beta-circularly polarized gamma correlation[25] are also consistent with these values of X and Y.

I^{126}

The data for $a(W)$ which we have used are also old.[12] As is shown in Table III, the rotational excitation, $(g_{7/2})^2(h_{11/2})^\nu$, $(g_{7/2})^\pi(h_{11/2})^6$, $(g_{7/2})^\pi(h_{11/2})^8$, and $(g_{7/2})^\pi \times (h_{11/2})^{10}$ are not excluded on the basis of our analysis. More precise experiments on beta decay of I^{126} are thus recommended in order to study the 0.39-MeV level of Xe^{126}.

TABLE V. Comparison of $\alpha Z/2R$ with W_0.[a]

Parent nuclei	$\alpha Z/2R$	W_0 in β_2
$_{33}As^{76}$	9.38	5.72
$_{37}Rb^{86}$	10.1	2.40
$_{51}Sb^{122}$	12.3	3.74
$_{53}I^{126}$	12.6	2.69

[a] $R = 1.2 \times 10^{-13} A^{1/3}$ cm is used.

6. CONCLUSION

We have shown the possibility of examining nuclear models with the data on beta-gamma directional correlation. In fact, we have tested the rotational excitation in the collective model and several kinds of particle excitation in the j-j coupling shell model. In the study of the 2^+ states in even-even nuclei with available data, we find that the rotational excitation is ruled out in the cases of Se^{76}, Sr^{86}, and Te^{122}, while the simplest configurations which can be assumed on the basis of the j-j coupling shell model are not inconsistent with the data in the cases of Se^{76}, Te^{122}, and I^{126}. We can also examine the other nuclear models if the $(\int B_{ij})_1/(\int B_{ij})_2$ is calculable. For example, it will be worth studying the vibrational excitations in the collective model for the 2^+ states in even-even nuclei.

The modified B_{ij} approximation assumes $\alpha Z/2R \gg W_0$ which is checked in Table V. This approximation may be less effective in the case of As^{76} compared with the other three cases. An analysis similar to that of the present paper will be, in principle, possible with all six nuclear matrix elements in the first forbidden beta decays.

ACKNOWLEDGMENTS

The authors would like to express their sincere thanks to Miss Julia Thacher and Miss Harriet Bougen for computational work, and to Dr. Karl Eklund for reading the manuscript. One of the authors (M. M.) is deeply indebted to Professor H. J. Fischbeck and Professor F. Boehm for making available data on As^{76} and Rb^{86}, respectively, prior to publication.

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[24] E. P. Pipkin, G. E. Bradley, and R. E. Simpson, Nucl. Phys. **27**, 353 (1961).
[25] M. Delabaye, J. P. Deutsch, P. Lipnik, J. phys. radium **23**, 257 (1962).