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Delta-Function Potential in a Box*

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The problem of a particle interacting with a potential of delta-function shape and having a tensor component is considered for box boundary conditions. The limits of small potential and large box dimensions are considered, especially in their effect on the mixing parameter.

INTRODUCTION

THE study of infinite nuclear matter has received a good deal of attention in the past few years for use both as a proving ground for methods of attack on problems involving the finite nucleus and as a means for testing some ideas about the nucleon-nucleon force.[1] In order to connect the properties of this nuclear matter with the real world, one must necessarily extrapolate information obtained from real nuclei. Usually, the volume energy term in an empirical mass formula together with the equilibrium density at the center of heavy nuclei (corrected for Coulomb effects) are taken to be the correct extrapolations. The goal of nuclear matter calculations, then, is to reproduce these numbers by some calculation scheme usually closely related to the Brueckner method. In order to define the calculation, boundary conditions must be introduced. For the infinite system it is not at all clear what the appropriate boundary conditions should be. Indeed, the boundary conditions pertaining to free-particle scattering (outgoing waves) are quite plausible. However, if infinite nuclear matter is considered to be a limit of finite nuclear matter, it is clear that the boundary conditions should be such that the wave function vanishes for large distances (say $r>R$) and then allow R to approach infinity.

One method of insuring this is to require the wave function to vanish at some finite distance R and allow the boundary to become very large only after calculations have been performed. This problem has been previously considered,[2] and various appropriate limits evaluated. The purpose of this paper is to gain an understanding of how these limits are approached. To do this, the simple case of a particle interacting with a delta-function potential in a spherical box is considered and particular attention is paid to the behavior of the mixing parameters, both as the potential is lessened and as the boundary is allowed to recede.

STATEMENT OF THE PROBLEM

Let us consider a particle of mass m interacting with a potential $V(\bar{r})$ in a spherical box of radius R. The boundary conditions then are

$$\Psi(\bar{r})=0, \quad r\geqslant R. \tag{1}$$

The potential chosen has a tensor component and parameters fixed so as to fit low-energy nucleon-nucleon data when $\epsilon=1$.

$$(2m/\hbar^2)V(\bar{r})=-\epsilon[C\delta(r-a)+T\delta(r-a)S_{12}], \tag{2}$$

ϵ is a parameter which is allowed to vary in order to change the strength of the potential. S_{12} is the usual tensor operator. C, T, and a have previously been given by Bolsterli,[3] and are $C=-1.013725$ F^{-1}, $T=1.197775$ F^{-1}, and $a=1.78865$ F. The delta-function shape is chosen for simplicity in performing the calculations.

Writing, as is usual,

$$\Psi(\bar{r})=[u(r)/r]\,{}^3S_1+[w(r)/r]\,{}^3D_1 \tag{3}$$

for the exact wave function and defining $k^2=2mE/\hbar^2$, the Schrödinger equation becomes the two coupled equations

$$\begin{aligned} &d^2u/dr^2+k^2u=-\epsilon[C\delta(r-a)u(r)+8^{\frac{1}{2}}T\delta(r-a)w(r)],\\ &d^2w/dr^2+(k^2-6/r^2)w\\ &\quad=-\epsilon[(C-2T)\delta(r-a)w(r)+8^{\frac{1}{2}}T\delta(r-a)u(r)], \end{aligned} \tag{4}$$

where the notation ${}^3S_1$ and ${}^3D_1$ refers to the properly normalized coupled angular wave functions with the eigenvalues $S=1$, $J=1$, $L=0$, 2, respectively. (Only the coupled S and D waves will be considered in this paper.)

The solution of these equations is most easily effected by means of the Green's function technique; the proper Green's functions being

$$G_l(r_<,r_>)=kr_<r_>j_l(kr_<)\times[n_l(kr_>)-j_l(kr_>)n_l(kR)/j_l(kR)]. \tag{5}$$

The (unnormalized) solutions are then given by

$$\begin{aligned} u(r)&=-\epsilon[CG_0(r,a)u(a)+8^{\frac{1}{2}}TG_0(r,a)w(a)],\\ w(r)&=-\epsilon[8^{\frac{1}{2}}TG_2(r,a)u(a)+(C-2T)G_2(r,a)w(a)]. \end{aligned} \tag{6}$$

Thus, the eigenvalue condition becomes the condition that these two equations are consistent at $r=a$. This requires

$$[1+\epsilon CG_0(a,a)][1+\epsilon(C-2T)G_2(a,a)]-8\epsilon^2T^2G_0(a,a)G_2(a,a)=0, \tag{7}$$

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[1] See especially K. A. Brueckner and J. L. Gammel, Phys. Rev. **109**, 1023 (1958) and K. A. Brueckner, in *The Many-Body Problem*, edited by C. DeWitt and P. Nozières (John Wiley & Sons, Inc., New York, 1959), pp. 47–164.

[2] N. Fukuda and R. G. Newton, Phys. Rev. **103**, 1558 (1956).

[3] M. Bolsterli, Phys. Rev. **114**, 1605 (1959).

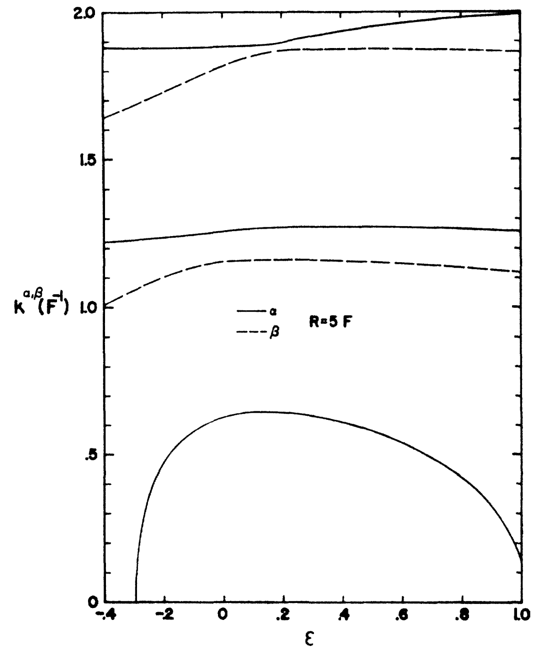

FIG. 1. The wave number $k^{\alpha,\beta}(\mathrm{F}^{-1})$ corresponding to the lowest few eigenvalues of the potential of Eq. (2) as a function of the parameter ϵ. The radius of the box is 5 F.

which will, henceforth, be called the eigenvalue equation since it determines the value of k. Once k has been determined from (7) it may be used in (6) to determine the wave function with arbitrary normalization.

In order to gain an understanding of the problem and to develop a terminology, let us consider the solutions of (7). Those solutions are shown in Fig. 1, where k is plotted as a function of ϵ for a boundary at $R=5$ F. It is seen that, except for the lowest eigenvalue, the eigenvalues are grouped into pairs. For ϵ not equal to zero, these levels have wave functions which are some mixture of the angular S and D functions. (Of course, this is also true in the more familiar case of scattering with no boundary, and is a direct consequence of the presence of the tensor operator in the potential.) If ϵ is allowed to approach zero, each of these levels *continuously* approaches a level whose wave function contains only one value of the orbital angular momentum. The eigenvalue at $\epsilon=0$ is called the "unperturbed" value and will be designated by $k_0{}^{S,D}$, the superscript indicating to which value of angular momentum the level corresponds. For $\epsilon\neq 0$, the eigenvalue which becomes $k_0{}^S(k_0{}^D)$ as $\epsilon\rightarrow 0$ is called $k^\alpha(k^\beta)$ and its wave function the $\alpha-(\beta-)$ wave. The solutions for $\epsilon=0$ are easily obtained and are called the unperturbed solutions. $k_0{}^{S,D}$ are given by

$$j_0(k_0{}^S R)=0,\quad j_2(k_0{}^D R)=0. \tag{8}$$

The unperturbed level can be used to completely specify a certain perturbed level in the presence of an interaction.

If the size of the box is allowed to increase, two things happen to the unperturbed levels: Each of the levels drops in energy and the density of the levels increases (the level spacing decreases). This is easily seen for S waves by considering Eq. (8). One then has

$$k_{0n}{}^S=n\pi/R,$$
and
$$k_{0n+1}{}^S-k_{0n}{}^S=\pi/R, \tag{9}$$

where the subscript n has been added to designate the nth S level. A very similar thing happens for the D waves; we merely replace $n\pi$ by the zeros of $j_2(x)$, a_n. Of more interest is how the D levels approach the S levels for large R. If one considers a given pair of levels (say the nth D level and $n+1$th S level), the spacing between them decreases as $1/R$ in the same way as indicated by (9),

$$k_{0n+1}{}^S-k_{0n}{}^D=[(n+1)\pi-a_n]/R=\mathrm{const}/R. \tag{10}$$

On the other hand, one is generally more interested in a level near a fixed value of k (rather than a particular level as was considered above). In order to be precise, one must fix a value of $k_0{}^S$ and then select a sequence of R's such that there is always an S level at $k_0{}^S$. One can then show[4]

$$k_0{}^S-k_0{}^D \underset{R\rightarrow\infty}{\longrightarrow} 3/k_0{}^D R^2, \tag{11}$$

and the levels approach one another as $1/R^2$. The fact

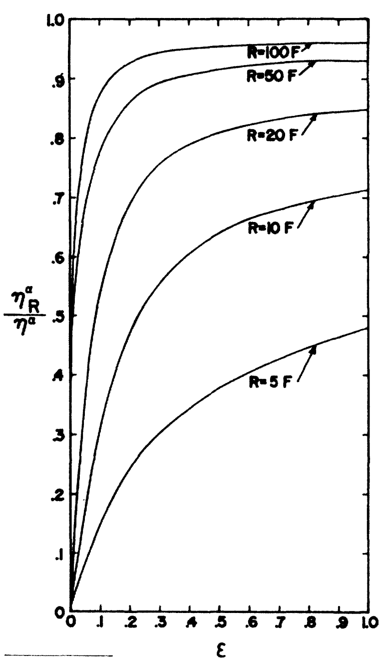

FIG. 2. The ratio of the mixing parameter for the box to that for free scattering as a function of the parameter ϵ. Several values of the box radius are shown. Only α waves are given as the curves for β waves are nearly identical.

[4] This can be understood when one notes that for a fixed R

$$k_{0n+1}{}^S-k_{0n}{}^D \underset{n\rightarrow\infty}{\longrightarrow} 3/k_0{}^D R^2.$$

Thus, the expression in (11) is valid when R is large enough so that a sufficient number of levels have been brought below $k_0{}^S$ so that the above approximation is valid.

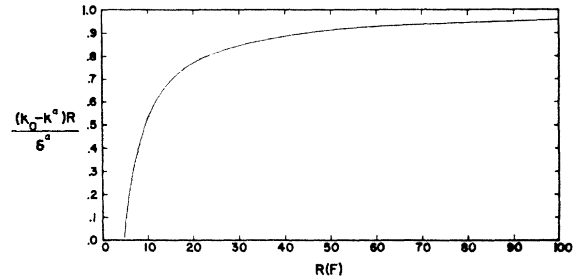

FIG. 3. The quantity $(k_0-k^\alpha)R/\delta^\alpha$ as a function of R for $\epsilon=1$. The β-wave curve is not given as it is quite similar.

that these levels are nondegenerate (even though very close together) enables one to use ordinary perturbation theory type expressions. That is, the unperturbed wave functions *cannot* be combinations of S and D waves but must be either pure S or pure D waves.

The specific problem treated here can now be defined: The characteristics of a given level ($k_0{}^S=2\pi/5=1.256637$ F^{-1}) are investigated for values of R. In particular, the reaction matrix and mixing parameter will be considered and their behavior studied as functions of R and ϵ.

SOLUTION AND RESULTS

Making use of Eqs. (5) and (7) the solution (6) can be written in the form (using a new normalization)

$$\frac{u_\alpha(r)}{r}=j_0(k^\alpha r)-\frac{j_0(k^\alpha R)}{n_0(k^\alpha R)}n_0(k^\alpha r),$$

$$\frac{w_\alpha(r)}{r}=\eta_R{}^\alpha\left[j_2(k^\alpha r)-\frac{j_2(k^\alpha R)}{n_2(k^\alpha R)}n_2(k^\alpha r)\right],$$

and (12)

$$\frac{u_\beta(r)}{r}=\eta_R{}^\beta\left[j_0(k^\beta r)-\frac{j_0(k^\beta R)}{n_0(k^\beta R)}n_0(k^\beta r)\right],$$

$$\frac{u_\beta(r)}{r}=j_2(k^\beta r)-\frac{j_2(k^\beta R)}{n_2(k^\beta R)}n_2(k^\beta r),$$

where $\eta_R{}^\alpha$ and $\eta_R{}^\beta$ are the mixing parameters given by

$$\eta_R{}^\alpha=\frac{8^{\frac{1}{2}}\epsilon Tk^\alpha a^2[j_0(k^\alpha a)-(j_0(k^\alpha R)/n_0(k^\alpha R))n_0(k^\alpha a)]j_2(k^\alpha a)n_2(k^\alpha R)/j_2(k^\alpha R)}{1+\epsilon(C-2T)G_2(a,a)},$$

$$\eta_R{}^\beta=\frac{8^{\frac{1}{2}}\epsilon Tk^\beta a^2[j_2(k^\beta a)-(j_2(k^\beta R)/n_2(k^\beta R))n_2(k^\beta a)]j_0(k^\beta a)n_0(k^\beta R)/j_0(k^\beta R)}{1+\epsilon CG_0(a,a)}. \quad (13)$$

If one solves the same problem but with scattering-type boundary conditions (instead of the box), the eigenwaves would be as given in (12) except one must replace

$$\begin{aligned}&k^{\alpha,\beta} \text{ by } k_0,\\ &j_l(k^{\alpha,\beta}R)/n_l(k^{\alpha,\beta}R) \text{ by } \tan\delta^{\alpha,\beta},\\ &\eta_R{}^{\alpha,\beta} \text{ by } \eta^{\alpha,\beta},\end{aligned} \quad (14)$$

where $\eta^{\alpha,\beta}$ and $\delta^{\alpha,\beta}$ are the usual eigenstate mixing parameter and phase shift.

Fukuda and Newton[2] have shown that as $R\to\infty$

$$k^{\alpha,\beta}\to k_0;\quad j_l(k^{\alpha,\beta}R)/n_l(k^{\alpha,\beta}R)\to\tan\delta^{\alpha,\beta};\quad \text{and}\quad \eta_R{}^{\alpha,\beta}\to\eta^{\alpha,\beta},$$

so that Eq. (14) then shows that the eigenwave scattering solution is the appropriate approximation to the solution of a particle in a box of large radius, and that thus for infinite nuclear matter one should use the eigenwave function.

This brings up a more subtle point, however. The unperturbed solution from which a given perturbed solution has evolved while an interaction is turned on is important for the application of the separation method in the nuclear matter problem.[5] For this consideration, it is quite important whether the problem is solved for finite R first and then the limit $R\to\infty$ taken or if one lets $R\to\infty$ and then solves the problem. In the former case the levels are all nondegenerate (except, of course, for the $2J+1$ fold m_J degeneracy which is always present in a system which has no external interactions), and thus the unperturbed level can be only a pure S or D state but not a linear combination.

In the latter case the unperturbed S and D states are degenerate so that any linear combination of the two is also degenerate. Indeed, one finds that in order to reach the eigenstate from these now degenerate levels as $\epsilon\to 1$, one must begin from a mixed state (e.g., the pure S state will not pass to the eigenstate but to some other mixture). Putting it in another way, $\lim_{\epsilon\to 0}\eta^{\alpha,\beta}\neq 0$. These facts can be expressed by saying that the two limits $\epsilon\to 0$ and $R\to\infty$ do not commute.

How the various limits are approached can be seen in Fig. 2, where $\eta_R{}^\alpha/\eta^\alpha$ is plotted as a function of ϵ for various values of R. ($\eta_R{}^\beta/\eta^\beta$ is not given as it has properties quite similar to $\eta_R{}^\alpha/\eta^\alpha$.) As observed above, except for small enough ϵ, this ratio approaches 1 as R gets large in accord with the more generally proved theorem.[2] These curves indicate that the correct unperturbed wave functions are indeed the pure angular momentum eigenfunctions as were used in previous separation method calculations,[5] if the box solutions are accepted as a reasonable extrapolation to nuclear matter.

[5] B. L. Scott and S. A. Moszkowski, Ann. Phys. (N. Y.) **14**, 107 (1961); S. A. Moszkowski, and B. L. Scott, *ibid.* **11**, 65 (1960).

In the theory of nuclear matter, the diagonal element of the reaction matrix gives the energy of interaction.

$$\Delta E = \langle\Phi_0|V|\Psi\rangle/\langle\Phi_0|\Phi_0\rangle. \tag{15}$$

This, of course, can also be shown by direct calculation in this case. As $R\rightarrow\infty$ the energy shift, ΔE, can be related to the phase shift[2]

$$(k^2-k_0^2)/k \xrightarrow[R\rightarrow\infty]{} -2\delta/R,$$

which can be written in the form

$$(k_0-k^{\alpha,\beta})/\delta^{\alpha,\beta} \xrightarrow[R\rightarrow\infty]{} 1. \tag{16}$$

The quantity $(k_0-k^\alpha)R/\delta^\alpha$ is plotted in Fig. 3 as a function of R for $\epsilon=1$. The behavior is similar for other values of ϵ.

CONCLUSION AND SUMMARY

Explicit results have been obtained for a particle interacting with a potential in a box. The mixing parameter was considered in the limits $R\rightarrow\infty$ and $\epsilon\rightarrow 0$. It was found that if $\epsilon\rightarrow 0$ for R finite then $\eta_R^{\alpha,\beta}\rightarrow 0$. This is a direct result of the nondegeneracy of the levels in the presence of the box boundary condition. On the other hand, one finds that $\eta_R^{\alpha,\beta}\rightarrow$ const if $\epsilon\rightarrow 0$ for $R=\infty$ (i.e., scattering boundary conditions). This results from the degeneracy of the S and D levels for an infinite region. Thus, it is important to keep in mind when performing calculations involving infinite systems precisely which limit is the correct extrapolation for the problem in hand.

These calculations were carried out on the Minneapolis-Honeywell 800 digital computer at the Computer Sciences Laboratory of the University of Southern California.

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Elementary Particle Classification Based upon a Massless Dirac Field

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Elementary particles are viewed as linearized excitations of a nonlinear massless Dirac field, whose ground state is assumed to contain a finite density of field quanta. Nucleons and antinucleons are viewed as single-particle and single-hole excitations, and other elementary particles as space-symmetric correlations of 2, 3, or 4 particles and holes. Hyperons are viewed as composite. The strangeness quantum number is idenitfied with half the excess of particles over holes in the excitation. The classification has room for 29 particles: the known elementary particles plus doubly charged K and μ mesons and antiparticles, a strange charged spin 1 boson and antiparticle, and a boson quintet of strangeness and charge 0, ± 1, ± 2.

INTRODUCTION

ELEMENTARY particles can be viewed as excitations of a simple substrate. Heisenberg[1] has suggested that the substrate is a nonlinear spinor field, and Nambu[2] and the author[3] have suggested in addition that it resembles the superconducting fluid described by Bardeen, Cooper, and Schrieffer.[4] The collective excitations of an interacting massless Dirac fluid (a quantized nonlinear massless Dirac field) can be classified under the assumption that the interaction conserves the number of quanta of each type in the fluid. The resulting spectrum of elementary excitations bears a strong resemblance to the spectrum of elementary particles.

[1] H.-P. Dürr, W. Heisenberg, H. Mitter, S. Schlieder, and K. Yamazaki, Z. Naturforsch. **14a**, 441 (1959). Earlier papers are referred to there.
[2] Y. Nambu, Phys. Rev. Letters **4**, 380 (1960). See also Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961); **124**, 246 (1961).
[3] J. C. Fisher, Bull. Am. Phys. Soc. **3**, 68 (1958).
[4] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **106**, 162 (1957).

MATHEMATICAL APPARATUS

The particle representation of a quantized massless Dirac field leads to a description of the possible states of the field in terms of two types of two-component particles. Each particle state can be characterized by a quantum number $t=\pm 1$ that denotes the type of particle, a quantum number $s=\pm 1$ that denotes whether its spin is parallel or antiparallel to its momentum, and a quantum number $q=(q_x,q_y,q_z)$ that denotes its momentum. The operator that creates a particle in state (t,s,q) is $c^\dagger(t,s,q)$, and its Hermitian adjoint is $c(t,s,q)$. The operators anticommute according to the relationships

$$[c^\dagger(t,s,q),\, c^\dagger(t',s',q')]_+ = [c(t,s,q),\, c(t',s',q')]_+ = 0,$$
$$[c^\dagger(t,s,q),\, c(t',s',q')]_+ = \delta_{tt'}\delta_{ss'}\delta(q-q').$$

The state with no particles is $|0\rangle$, and it satisfies the relationship

$$c(t,s,q)|0\rangle = 0.$$