Interaction Picture in Gravitational Theory*†

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The problem of defining an interaction picture for a quantized version of the Einstein theory of gravitation is considered. The quantization method involves the use of the De Donder condition, as formulated by Fock, as an auxiliary condition on the state vectors. The theory is formulated in a Lorentz-covariant way by assuming the validity of Fock's conjecture that the De Donder-Fock coordinate condition determines the coordinate system up to a Lorentz transformation. It is shown that the interaction operator which appears in the Tomonaga-Schwinger equation obeyed by the interaction-picture state vectors satisfies a necessary integrability condition. Some problems involved in imposing the auxiliary condition on the interaction-picture state vectors are also considered.

I. INTRODUCTION

MOST discussions of the quantization of the Einstein theory of gravitation have been carried out within the framework of the Heisenberg picture.[1] This is certainly the most natural procedure, and it parallels the approach used in the early development of quantum electrodynamics. On the other hand, one of the most important advances in that theory was the introduction of the interaction picture by Tomonaga[2] and Schwinger.[3] Since the interaction picture has been of great practical importance in quantum electrodynamics, and since there are many analogies between the electromagnetic and gravitational fields, it is desirable to investigate the possibility of obtaining such a formulation for gravitational theory.

In a relativistic field theory the interaction picture is obtained from the Heisenberg picture by a unitary transformation U which is a functional of a spacelike surface Σ. That is,

$$\Psi_I = U[\Sigma]\Psi_H, \tag{1.1}$$

where Ψ_I and Ψ_H are, respectively, the state vectors of the interaction picture and the Heisenberg picture. The Σ dependence of U was first suggested by Tomonaga[2] as a covariant generalization of the time dependence of the analogous transformations employed in nonrelativistic quantum mechanics. The corresponding generalization of the Schrödinger equation takes the form[4]

$$i\hbar c\{\delta\Psi_I[\Sigma]/\delta\Sigma(x)\} = {}^\circ W(x,\Sigma)\Psi_I[\Sigma], \tag{1.2}$$

where $W(x,\Sigma)$ is the covariant interaction operator.[5] The prescript (°) on any operator means that it is to be taken in the interaction picture. The functional derivative can be conveniently defined as follows: Let δF be the variation induced in F by an infinitesimal variation of the surface Σ. The variation of the surface can be generated by displacing each point x^μ on Σ by $x^\mu \to x^\mu + \xi^\mu$, where ξ^μ is an infinitesimal four-vector. Finally, the functional derivative is defined by

$$\delta F = \int_\Sigma d\Sigma_\mu\, \xi^\mu(x) \frac{\delta F[\Sigma]}{\delta\Sigma(x)}, \tag{1.3}$$

where $d\Sigma_\mu$ is the covariant surface element. For the special case that Σ has the form $x^0 =$ constant, we obtain

$$\delta F = \int d^3x\, \xi^0(x) \frac{\delta F[\Sigma]}{\delta\Sigma(x)}. \tag{1.4}$$

Since Ψ_H is independent[6] of Σ, we can obtain from (1.1) and (1.2) an equation which involves only Heisenberg-picture operators:

$$i\hbar c\,\delta U[\Sigma]/\delta\Sigma(x) = U[\Sigma]W(x,\Sigma), \tag{1.5}$$

where $W(x,\Sigma) = U^{-1}[\Sigma]\, {}^\circ W(x,\Sigma) U[\Sigma]$.

For an arbitrarily chosen operator $W(x,\Sigma)$ the functional differential equation (1.5) will not necessarily possess a unique solution, even if U is prescribed on some initial surface Σ_0. Therefore, the operator W, besides being a covariant generalization of the interaction Hamiltonian, must also satisfy a condition which guarantees the integrability of (1.5). In the following, we define the appropriate interaction operator for the Einstein theory and show that it satisfies the required integrability condition, at least for the classical limit in which commutators are replaced by Poisson brackets. This last restriction is made in order to avoid the problem of factor ordering, which is particularly serious in gravitational theory because of the complicated form of the interaction operator. Strictly speaking, what we have established is a necessary but not sufficient condition for the complete quantum mechanical proof of integrability. We must therefore assume that an

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[1] For exceptions, see S. N. Gupta, Proc. Phys. Soc. (London) **A 65**, 608, 161 (1952); and the unpublished work of R. P. Feynman.
[2] S. Tomonaga, Progr. Theoret. Phys. (Kyoto) **1**, 27 (1946).
[3] J. Schwinger, Phys. Rev. **74**, 1439 (1948); **75**, 651 (1949).
[4] See, for example: S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Row, Peterson and Company, Evanston, Illinois, 1961), p. 420.
[5] In quantum electrodynamics, $W(x,\Sigma)$ is actually independent of Σ; but this is not always the case. See for example, references 7 and 8.
[6] Except for reductions of the wave function which occur when new experimental information is taken into account.

ordering of factors can be found which allows the quantum mechanical analog of our proof to be carried out.

In addition to the consideration of the integrability of (1.5), we very briefly consider some of the properties of the interaction picture. In particular, we point out some of the problems involved in imposing the coordinate condition which we have used in the quantization of the theory.

It is convenient at this point to review, very briefly, some earlier work which is pertinent to the present paper. As we have already mentioned,[1] the interaction picture is used in the unpublished work of Feynman and in several papers by Gupta. In Gupta's papers it is assumed that the interaction picture can be introduced in the usual way, and this formulation of the theory is used in the consideration of several self-energy problems. The technique employed in these calculations involves an expansion in powers of the gravitational coupling constant, and only linear terms are considered. In contrast to this procedure, we will work entirely in the Heisenberg picture (except in Sec. VII), and no use is made of the expansion technique.

In connection with the question of the integrability of the Tomonaga-Schwinger equation, mention should be made of the work of Belinfante[7,8] and Matthews.[9] The latter author proved, for several meson theories, that the S matrix is independent of the surface normals which appear explicitly in the definition of the interaction operator. It should be stressed that Matthews' result does not constitute a general proof that any field theory will lead to an S matrix which is independent of surface normals. In fact, it is a simple matter to invent interaction Hamiltonians which generate surface-dependent S matrices. This means that each case must be tried on its own merits. Furthermore, the method used by Matthews is not immediately applicable to the problem of proving the uniqueness of the operator $U(t)$ for finite times. Such a proof has been given by Belinfante for the special cases of vector meson theory and the gauge-independent version of quantum electrodynamics. The present paper is an extension, using slightly different methods, of Belinfante's results to the Einstein theory.

A more detailed account of the material presented here is available in the form of a research report.[10]

II. QUANTIZATION

The general covariance of the Einstein theory leads to constraints[11] which make it impossible to carry out the usual canonical quantization procedure. In order to quantize the theory, we use a method invented by Fermi[12] for the solution of the corresponding problem in quantum electrodynamics. This method consists of adding to the Einstein Lagrangian a term which is not invariant under arbitrary coordinate transformations. The constraints are then replaced by the imposition on the state vector of suitable auxiliary conditions which prevent the noninvariant term from having any effect on the values of physical quantities.

The modified gravitational Lagrangian is given by[13]

$$\mathfrak{L}_g=\frac{1}{\epsilon^2}\left[\tfrac{1}{8}\mathfrak{g}^{\lambda\nu}k_{\alpha\beta}k_{\rho\sigma}\mathfrak{g}^{\alpha\beta}{}_{,\lambda}\mathfrak{g}^{\rho\sigma}{}_{,\nu}+\tfrac{1}{2}k_{\alpha\lambda}\mathfrak{g}^{\lambda\nu}{}_{,\beta}\mathfrak{g}^{\alpha\beta}{}_{,\nu}-\tfrac{1}{4}\mathfrak{g}^{\rho\nu}k_{\alpha\lambda}k_{\beta\mu}\mathfrak{g}^{\lambda\mu}{}_{,\rho}\mathfrak{g}^{\alpha\beta}{}_{,\nu}\right]-\frac{1}{2\epsilon^2}k_{\mu\nu}\mathfrak{g}^{\mu\alpha}{}_{,\alpha}\mathfrak{g}^{\nu\beta}{}_{,\beta}, \quad (2.1)$$

where $\mathfrak{g}^{\mu\nu}\equiv(-g)^{1/2}g^{\mu\nu}$, and $k_{\mu\nu}\equiv(-g)^{-1/2}g_{\mu\nu}$, so that $k_{\mu\nu}\mathfrak{g}^{\nu\lambda}=\delta_\mu{}^\lambda$. The last term on the right-hand side of (2.1) is the noninvariant term which removes the constraints. We have expressed $\mathfrak{L}_g$ in terms of the $\mathfrak{g}^{\mu\nu}$ instead of the usual metric tensor $g_{\mu\nu}$, because of the appearance of the former in the classical De Donder condition:

$$\mathfrak{g}^{\mu\nu}{}_{,\nu}=0. \quad (2.2)$$

The noninvariant term in (2.1) has been chosen so that it vanishes when (2.2) is satisfied.

In order to have a tractable model of the interaction of gravitation with matter, we introduce a neutral scalar field with the generally covariant Lagrangian

$$\mathfrak{L}_m=-\tfrac{1}{2}\mathfrak{g}^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}-\tfrac{1}{2}\mu^2(-g)^{1/2}\varphi^2,$$

where μ is the mass of the particle associated with the field φ. The total Lagrangian $\mathfrak{L}$ is then the sum of $\mathfrak{L}_g$ and $\mathfrak{L}_m$.

We can now go over to the canonical formalism by introducing the momenta conjugate to $\mathfrak{g}^{\mu\nu}$ and φ:

$$\begin{aligned}\pi_{\mu\nu}&\equiv\partial\mathfrak{L}/\partial\mathfrak{g}^{\mu\nu}{}_{,0},\\ p&\equiv\partial\mathfrak{L}/\partial\varphi_{,0}.\end{aligned} \quad (2.3)$$

The canonical commutation relations are

$$\begin{aligned}[\mathfrak{g}^{\mu\nu}(\mathbf{x},x^0),\,\pi_{\alpha\beta}(\mathbf{x}',x^0)]&=i\hbar c\Delta_{\alpha\beta}{}^{\mu\nu}\delta_3(\mathbf{x}-\mathbf{x}'),\\ [\varphi(\mathbf{x},x^0),\,p(\mathbf{x}',x^0)]&=i\hbar c\delta_3(\mathbf{x}-\mathbf{x}'),\end{aligned} \quad (2.4)$$

where $\Delta_{\alpha\beta}{}^{\mu\nu}\equiv\frac{1}{2}(\delta_\alpha{}^\mu\delta_\beta{}^\nu+\delta_\alpha{}^\nu\delta_\beta{}^\mu)$, and $\delta_3(x)$ is the three-dimensional delta function.

The Hamiltonian is defined in the usual way by

$$\mathfrak{H}=\pi_{\mu\nu}\mathfrak{g}^{\mu\nu}{}_{,0}+p\varphi_{,0}-\mathfrak{L}. \quad (2.5)$$

In writing out the Hamiltonian it is convenient to introduce the abbreviation

$$Z_{\mu\nu\alpha\beta}{}^{\rho\sigma}=\epsilon^{-2}\left[\mathfrak{g}^{\rho\sigma}k_{\mu\nu}k_{\alpha\beta}-\tfrac{1}{4}\mathfrak{g}^{\rho\sigma}k_{\alpha(\mu}k_{\nu)\beta}+\tfrac{1}{4}\delta_{(\alpha}{}^\rho k_{\beta)(\mu}\delta_{\nu)}{}^\sigma-\tfrac{1}{4}\delta_{(\alpha}{}^\sigma k_{\beta)(\mu}\delta_{\nu)}{}^\rho\right],$$

[7] F. J. Belinfante, Phys. Rev. **76**, 66 (1949).
[8] F. J. Belinfante, Phys. Rev. **84**, 644 (1951).
[9] P. T. Matthews, Phys. Rev. **76**, 684 (1949).
[10] F. J. Belinfante and J. C. Garrison, "The Interaction Picture in Gravitational Theory and Some Related Topics," National Science Foundation Research Report (unpublished).
[11] P. G. Bergmann, Phys. Rev. **75**, 680 (1949).
[12] E. Fermi, Rend. Accad. nazl. Lincei **9**, 881 (1929).
[13] The signature of the metric is taken to be $(-1, 1, 1, 1)$; ordinary partial derivatives are denoted by commas; $\epsilon^2\equiv16\pi Gc^{-4}$, where G is the Newtonian gravitational constant; greek indices run over (0,1,2,3); and latin indices run over (1,2,3).

where

$$k_{\alpha(\mu}k_{\nu)\beta}\equiv k_{\alpha\mu}k_{\nu\beta}+k_{\alpha\nu}k_{\mu\beta}.$$

We also express the time derivatives $\varphi_{,0}$ and $\mathfrak{g}^{\mu\nu}{}_{,0}$ in terms of the canonical momenta by

$$\varphi_{,0}=-(\mathfrak{g}^{00})^{-1}[p+\mathfrak{g}^{0j}\varphi_{,j}], \quad (2.7)$$

$$\mathfrak{g}^{\mu\nu}{}_{,0}=V_{(1)}{}^{\mu\nu}+V_{(0)}{}^{\mu\nu}, \quad (2.8)$$

with

$$\begin{aligned} V_{(1)}{}^{\mu\nu}&=-2\epsilon^2(\mathfrak{g}^{00})^{-1}[\mathfrak{g}^{\mu\sigma}\mathfrak{g}^{\nu\rho}-\tfrac{1}{2}\mathfrak{g}^{\mu\nu}\mathfrak{g}^{\rho\sigma}]\pi_{\rho\sigma},\\ V_{(0)}{}^{\mu\nu}&=-(\mathfrak{g}^{00})^{-1}[\mathfrak{g}^{0k}\mathfrak{g}^{\mu\nu}{}_{,k}+\mathfrak{g}^{0(\mu}\mathfrak{g}^{\nu)k}-\mathfrak{g}^{k(\mu}\mathfrak{g}^{\nu)0}{}_{,k}]. \end{aligned} \quad (2.9)$$

In terms of these quantities the Hamiltonian is given by

$$\mathcal{H}=\mathcal{H}_g+\mathcal{H}_m=\sum_{s=0}^{2}\mathcal{H}_g{}^{(s)}+\mathcal{H}_m, \quad (2.10)$$

with

$$\mathcal{H}_g{}^{(2)}=\tfrac{1}{2}V_{(1)}{}^{\alpha\beta}\pi_{\alpha\beta}, \quad (2.11a)$$

$$\mathcal{H}_g{}^{(1)}=V_{(0)}{}^{\alpha\beta}\pi_{\alpha\beta}, \quad (2.11b)$$

$$\mathcal{H}_g{}^{(0)}=-\tfrac{1}{2}Z_{\alpha\beta\rho\sigma}{}^{0j}V_{(0)}{}^{\alpha\beta}\mathfrak{g}^{\rho\sigma}{}_{,j}-\tfrac{1}{2}Z_{\alpha\beta\rho\sigma}{}^{ij}\mathfrak{g}^{\alpha\beta}{}_{,i}\mathfrak{g}^{\rho\sigma}{}_{,j}, \quad (2.11c)$$

$$\begin{aligned}\mathcal{H}_m=(-2\mathfrak{g}^{00})^{-1}p^2-(\mathfrak{g}^{00})^{-1}\mathfrak{g}^{0j}p\,\varphi_{,j}-\tfrac{1}{2}(\mathfrak{g}^{00})^{-1}(\mathfrak{g}^{0j}\varphi_{,j})^2\\ +\tfrac{1}{2}\mathfrak{g}^{ij}\varphi_{,i}\varphi_{,j}+\tfrac{1}{2}(-g)^{1/2}\mu^2\varphi^2. \end{aligned} \quad (2.11d)$$

The corresponding free-field Hamiltonian is given by

$$H=H_g+H_m=\sum_{s=0}^{2}H_g{}^{(s)}+H_m, \quad (2.12)$$

with

$$H_g{}^{(2)}=\epsilon^2(\gamma^{\mu\alpha}\gamma^{\nu\beta}-\tfrac{1}{2}\gamma^{\mu\nu}\gamma^{\alpha\beta})\pi_{\mu\nu}\pi_{\alpha\beta}, \quad (2.13a)$$

$$H_g{}^{(1)}=2[\gamma^{i\alpha}\mathfrak{g}^{\beta j}{}_{,j}-\gamma^{j\alpha}\mathfrak{g}^{\beta i}{}_{,j}]\pi_{\alpha\beta}, \quad (2.13b)$$

$$\begin{aligned}H_g{}^{(0)}=\epsilon^{-2}[-\tfrac{1}{8}\gamma^{ij}\gamma_{\alpha\beta}\gamma_{\rho\sigma}\mathfrak{g}^{\alpha\beta}{}_{,i}\mathfrak{g}^{\rho\sigma}{}_{,j}+\tfrac{1}{2}\mathfrak{g}^{0s}{}_{,k}\mathfrak{g}^{0k}{}_{,s}\\ -\mathfrak{g}^{ks}{}_{,s}\mathfrak{g}^{00}{}_{,k}+\tfrac{1}{2}\mathfrak{g}^{0k}{}_{,k}\mathfrak{g}^{0s}{}_{,s}+\tfrac{1}{4}\gamma^{ij}\gamma_{\alpha\mu}\gamma_{\beta\nu}\mathfrak{g}^{\alpha\beta}{}_{,i}\mathfrak{g}^{\mu\nu}{}_{,j}\\ +\tfrac{1}{2}\gamma_{\alpha\beta}\gamma^{ij}\mathfrak{g}^{\alpha 0}{}_{,i}\mathfrak{g}^{\beta 0}{}_{,j}-\tfrac{1}{2}\gamma_{\alpha\beta}\mathfrak{g}^{\beta j}{}_{,s}\mathfrak{g}^{\alpha s}{}_{,j}], \end{aligned} \quad (2.13c)$$

$$H_m=\tfrac{1}{2}p^2+\tfrac{1}{2}\gamma^{ij}\varphi_{,i}\varphi_{,j}+\tfrac{1}{2}\mu^2\varphi^2, \quad (2.13d)$$

where $\gamma^{\mu\nu}\equiv\mathrm{diag}\,(-1,1,1,1)$.

The appearance of the coupling constant ϵ^2 in (2.13) is actually spurious. If we were to introduce the quantities $h^{\mu\nu}$ defined by

$$h^{\mu\nu}=\epsilon^{-1}\mathfrak{g}^{\mu\nu}-\gamma^{\mu\nu}, \quad (2.14)$$

and express H_g in terms of them and their conjugate momenta, then the apparent dependence on ϵ^2 would be eliminated.

III. AUXILIARY CONDITION

The De Donder condition has been used by a number of authors[1,14,15] to restrict the choice of coordinate systems otherwise allowed by the general covariance of the Einstein theory. The use of this condition has recently been advocated by Fock,[16] who goes beyond the previous work by also employing boundary conditions adapted to the physical nature of the problem. Following Fock, we consider only those physical situations for which there exist coordinate systems having the following properties:

$$\mathfrak{g}^{\mu\nu}-\gamma^{\mu\nu}\to O(1/r)\quad\text{as}\quad r\to\infty, \quad (3.1a)$$

$$\mathfrak{g}^{\mu\nu}{}_{,\lambda}\to O(1/r)\quad\text{as}\quad r\to\infty, \quad (3.1b)$$

$$\lim_{r\to\infty}\left\{\frac{\partial}{\partial r}[r(\mathfrak{g}^{\mu\nu}-\gamma^{\mu\nu})]+\frac{\partial}{\partial x^0}[r(\mathfrak{g}^{\mu\nu}-\gamma^{\mu\nu})]\right\}=0. \quad (3.1c)$$

In (3.1c), (x^0+r) may lie in any arbitrary interval. We denote the flat-space metric by $\gamma^{\mu\nu}\equiv\mathrm{diag}(-1,1,1,1)$, and define r by $r\equiv[(x^1)^2+(x^2)^2+(x^3)^2]^{1/2}$. The first two conditions require the asymptotic flatness of space-time, and the third is interpreted by Fock as guaranteeing the absence of incoming gravitational waves at infinity. Coordinates satisfying these boundary conditions as well as the De Donder condition (2.2) are called harmonic.

The importance for our work of Fock's version of the De Donder condition lies in his conjecture that the only coordinate transformations which preserve (2.2) as well as the boundary conditions (3.1) are the inhomogeneous Lorentz transformations. If Fock is right, the invariance group of the theory will effectively be reduced to the Lorentz group; and we will then be free to use the general methods developed to deal with Lorentz-covariant field theories.

Fock's conjecture has been verified for static, spherically symmetric, singularity-free metrics, as well as for several other special cases.[17] In the following work, we assume the validity of his conjecture for a class of metrics large enough to be of general interest. It should be understood that Fock's conjecture in no way contradicts the principle of general covariance, since it merely gives a prescription for choosing a particular class of coordinate systems in which to work out the generally covariant theory. It should also be emphasized that the boundary conditions (3.1) are essentially physical in nature; they exclude, for example, various non-Euclidean topologies such as the closed universes considered in cosmology.

IV. INTEGRABILITY CONDITION

In Lorentz-covariant field theories it has always been found that the interaction operator could be written in the form

$$W(x,\Sigma)=W^{\mu\nu}(x)N_\mu(x)N_\nu(x), \quad (4.1)$$

[14] V. Fock, J. Phys. (U.S.S.R.) **1**, 81 (1939).

[15] A. Papapetrou, Proc. Roy. Irish Acad. **A 52**, 11 (1948).

[16] V. Fock, *The Thoery of Space Time and Gravitation* (Pergamon Press, New York, 1959), p. 346.

[17] F. J. Belinfante and J. C. Garrison, Phys. Rev. **125**, 1124 (1962).

where the symmetric tensor $W^{\mu\nu}(x)$ describes the interaction and $N_\mu(x)$ is the unit normal to Σ at the point x. We assume that (4.1) is also valid in our case, but the definition of $N_\mu(x)$ requires some further discussion.

If the spacelike surface Σ is implicitly represented by the scalar equation

$$\sigma(x)=0, \tag{4.2}$$

then the unit normal is usually defined by

$$N_\mu=-(-g^{\alpha\beta}\sigma_{,\alpha}\sigma_{,\beta})^{-1/2}\sigma_{,\mu}, \tag{4.3}$$

so that

$$g^{\mu\nu}N_\mu N_\nu=-1. \tag{4.4}$$

Since in our theory the gravitational field is quantized, it follows from (4.3) that the vector N_μ is a q number; this fact makes any correspondence with flat-space field theories rather difficult to attain. This difficulty can be avoided by making use of the fact that we have already restricted ourselves to harmonic coordinate systems, in which there is a naturally defined flat-space metric $\gamma^{\mu\nu}$.[18] Thus, we drop (4.3) and instead define N_μ by

$$N_\mu=-(-\gamma^{\alpha\beta}\sigma_{,\alpha}\sigma_{,\beta})^{-1/2}\sigma_{,\mu}. \tag{4.5}$$

Then N_μ is normalized by

$$\gamma^{\mu\nu}N_\mu N_\nu=-1. \tag{4.6}$$

The definition (4.5) does not make sense unless $\sigma(x)$ satisfies

$$\gamma^{\alpha\beta}\sigma_{,\alpha}\sigma_{,\beta}<0. \tag{4.7}$$

A surface which satisfies (4.7) is called quasi-spacelike. There are examples[10] of spacelike surfaces which are not quasi-spacelike; therefore, (4.7) imposes a nontrivial restriction on the set of surfaces which are admissible in our theory. Instead of searching for the most general spacelike surfaces which satisfy (4.7), we further restrict ourselves to the use of those surfaces which have the form $x^0=$ constant in some harmonic coordinate system. This automatically guarantees the satisfaction of (4.7), and it is analogous to the restriction to flat surfaces usually made in Lorentz-covariant theories. For this reason, we call such surfaces quasi-flat.

With the definition (4.5) for N_μ in mind, together with the consequent restriction to quasi-flat surfaces, we can go on to determine the integrability condition for (1.5). It is well known[2] that a sufficient condition for this integrability is given by[19]

$$\frac{\delta^2 U[\Sigma]}{\delta\Sigma(x)\delta\Sigma(x')}-\frac{\delta^2 U[\Sigma]}{\delta\Sigma(x')\delta\Sigma(x)}=0. \tag{4.8}$$

By making use of (1.5), we obtain from (4.8) the following condition on $W(x,\Sigma)$;

$$\frac{1}{i\hbar c}[W(x',\Sigma),\, W(x,\Sigma)]+\left\{\frac{\delta W(x,\Sigma)}{\delta\Sigma(x')}-\frac{\delta W(x',\Sigma)}{\delta\Sigma(x)}\right\}=0. \tag{4.9}$$

The surface dependence of $W(x,\Sigma)$ is given by (4.1), so that

$$\frac{\delta W(x,\Sigma)}{\delta\Sigma(x')}=2W^{\mu\nu}(x)N_\mu\frac{\delta N_\nu(x)}{\delta\Sigma(x')}. \tag{4.10}$$

We can calculate the functional derivative appearing on the right by making use of the defining equation (4.5). Thus,

$$\delta N_\mu=(\gamma^{\nu\lambda}\sigma_{,\nu}\sigma_{,\lambda})^{-3/2}\gamma^{\alpha\beta}\sigma_{,\alpha}(\delta\sigma)_{,\beta}+(\gamma^{\nu\lambda}\sigma_{,\nu}\sigma_{,\lambda})^{-1/2}(\delta\sigma)_{,\mu}, \tag{4.11}$$

where $\delta\sigma$ is the change in form of σ brought about by the infinitesimal variation $x^\mu\rightarrow x^\mu+\xi^\mu$ of the surface Σ. For $\delta\sigma$ we have

$$\delta\sigma=-\xi^\mu\sigma_{,\mu}. \tag{4.12}$$

We can now make use of the fact that Σ is required to be quasi-flat to carry out the rest of the calculation in the special harmonic coordinate system in which $\sigma(x)=x^0$. We then find

$$\delta N_\mu=-\delta_\mu{}^k\xi_{,k}{}^0. \tag{4.13}$$

A comparison of (4.13) with (1.4) leads to

$$\frac{\delta N_\mu(x)}{\delta\Sigma(x')}=-\delta_\mu{}^k\frac{\partial}{\partial x'^k}\delta_3(x-x'). \tag{4.14}$$

Substituting these results in (4.9), we find in our special coordinate system:

$$\frac{1}{i\hbar c}[W^{00}(x),\, W^{00}(x')]=2\left\{W^{0j}(x)\frac{\partial}{\partial x'^j}-W^{0j}(x')\frac{\partial}{\partial x^j}\right\}\delta_3(\mathbf{x}-\mathbf{x}'). \tag{4.15}$$

Any interaction tensor which satisfies (4.15) leads to unique solutions of the Tomonaga-Schwinger equation (1.5). In using (4.15) we will replace the left-hand side with the corresponding Poisson bracket; and we will also treat the right-hand side as a classical expression; that is, we ignore the problem of factor ordering.

V. INTERACTION TENSOR

In this section we construct the interaction tensor which appears in (4.15). It can be shown that a knowledge of W^{00}, together with the requirement of Lorentz covariance, serves to uniquely determine the remaining components of $W^{\mu\nu}$. Therefore, the first step in constructing $W^{\mu\nu}$ is to define W^{00}. The most obvious

[18] That is, if Fock's conjecture is right, the transformations which connect different harmonic coordinate systems are just exactly those which leave $\gamma_{\mu\nu}$ invariant.

[19] Since we are dealing with a very restricted class of surfaces, (4.8) is actually more stringent than is necessary. For a detailed discussion, see Appendix C of reference 10.

way of doing this is to set W^{00} equal to the interaction Hamiltonian,

$$W^{00}=\mathcal{H}_I\equiv\mathcal{H}-H\,; \tag{5.1}$$

and this is exactly what we do. However, it should be noted that such a procedure is not always satisfactory. There are several cases[7,8] in which it is necessary to add a three divergence to $\mathcal{H}_I$ in order to ensure integrability.

We express $W^{\mu\nu}$ as the difference between two symmetric tensors[20] $\mathcal{H}^{\mu\nu}$ and $H^{\mu\nu}$ which have $\mathcal{H}$ and H, respectively, for their 00 components. The total Hamiltonian $\mathcal{H}$ is related to the canonical energy-momentum tensor $t_\mu{}^\nu$ by

$$\mathcal{H}=-t_0{}^0, \tag{5.2}$$

where

$$t_\mu{}^\nu=\delta_\mu{}^\nu\mathcal{L}-\frac{\partial\mathcal{L}}{\partial\mathrm{g}^{\alpha\beta}{}_{,\nu}}\mathrm{g}^{\alpha\beta}{}_{,\mu}-\frac{\partial\mathcal{L}}{\partial\varphi_{,\nu}}\varphi_{,\mu}. \tag{5.3}$$

If we introduce the convention of raising and lowering tensor indices by means of $\gamma_{\mu\nu}$, then we can write

$$\mathcal{H}=-t_0{}^0=\gamma^{0\lambda}t_\lambda{}^0=t^{00}. \tag{5.4}$$

From this it is obvious that the desired symmetric tensor $\mathcal{H}^{\mu\nu}$ is given by

$$\mathcal{H}^{\mu\nu}=\tfrac{1}{2}(t^{\mu\nu}+t^{\nu\mu})=\tfrac{1}{2}(\gamma^{\mu\lambda}t_\lambda{}^\nu+\gamma^{\nu\lambda}t_\lambda{}^\mu). \tag{5.5}$$

We obtain the other tensor $H^{\mu\nu}$ by introducing the quantities

$$\pi_{\alpha\beta}{}^\lambda=\partial\mathcal{L}/\partial\mathrm{g}^{\alpha\beta}{}_{,\lambda},\qquad p^\lambda=\partial\mathcal{L}/\partial\varphi_{,\lambda}. \tag{5.6}$$

The components p^0 and $\pi_{\alpha\beta}{}^0$ are identical to the canonical momenta p and $\pi_{\alpha\beta}$. Upon replacing p and $\pi_{\alpha\beta}$ by p^0 and $\pi_{\alpha\beta}{}^0$ in (2.13), we can easily determine by inspection[10] the unique symmetric tensor $H^{\mu\nu}$ for which $H^{00}=H$.

Instead of giving the explicit expressions for $\mathcal{H}^{\mu\nu}$ and $H^{\mu\nu}$ separately, we write down directly the difference $W^{\mu\nu}$. In fact, we only give the W^{0k} since they are the only components that appear on the right-hand side of (4.15). In writing out the W^{0k}, it is convenient to use the decomposition (2.8) for the $\mathrm{g}^{\alpha\beta}{}_{,0}$ to write the $\pi_{\alpha\beta}{}^k$ as

$$\pi_{\alpha\beta}{}^k=\pi_{\alpha\beta}{}^{k(1)}+\pi_{\alpha\beta}{}^{k(0)}, \tag{5.7}$$

with

$$\pi_{\alpha\beta}{}^{k(1)}=Z_{\alpha\beta\mu\nu}{}^{0k}V_{(1)}{}^{\mu\nu},\qquad \pi_{\alpha\beta}{}^{k(0)}=Z_{\alpha\beta\mu\nu}{}^{0k}V_{(0)}{}^{\mu\nu}+Z_{\alpha\beta\mu\nu}{}^{kj}\mathrm{g}^{\mu\nu}{}_{,j}. \tag{5.8}$$

Furthermore, we collect terms in W^{0k} according to their degree in the canonical momenta $\pi_{\alpha\beta}$ and p.

$$W^{0k}=\sum_{s=0}^{2}W_{g(s)}{}^{0k}+\sum_{s=0}^{2}W_{m(s)}{}^{0k}, \tag{5.9}$$

with[21]

$$W_{g(2)}{}^{0k}=\pi_{\alpha\beta}{}^{k(1)}V_{(1)}{}^{\alpha\beta}-\epsilon^2 j^{\alpha\beta}\pi_{\alpha\beta}{}^{k(1)}+\gamma^{\alpha[0}V_{(1)}{}^{k]\beta}\pi_{\alpha\beta}, \tag{5.10a}$$

$$\begin{aligned}W_{g(1)}{}^{0k}={}&\tfrac{1}{2}\pi_{\alpha\beta}{}^{k(1)}V_{(0)}{}^{\alpha\beta}+\tfrac{1}{2}\pi_{\alpha\beta}{}^{k(0)}V_{(1)}{}^{\alpha\beta}-\tfrac{1}{2}\pi_{\alpha\beta}\mathrm{g}^{\alpha\beta}{}_{,k}\\&+\gamma^{\alpha[0}V_{(0)}{}^{k]\beta}\pi_{\alpha\beta}+\gamma^{\alpha[j}\mathrm{g}^{0]\beta}{}_{,j}\pi_{\alpha\beta}{}^{k(1)}+\gamma^{\alpha[j}\mathrm{g}^{k]\beta}{}_{,j}\pi_{\alpha\beta}\\&-\epsilon^2 j^{\alpha\beta}\pi_{\alpha\beta}{}^{k(0)}+\epsilon^{-2}\{-\tfrac{1}{8}\gamma_{\mu\nu}\mathrm{g}^{\mu\nu}{}_{,k}\gamma_{\alpha\beta}V_{(1)}{}^{\alpha\beta}\\&+\tfrac{1}{4}\gamma_{\alpha\rho}\gamma_{\beta\sigma}\mathrm{g}^{\alpha\beta}{}_{,k}V_{(1)}{}^{\rho\sigma}+\tfrac{1}{2}\gamma_{\alpha\beta}\mathrm{g}^{0\beta}{}_{,k}V_{(1)}{}^{\alpha0}\\&+\tfrac{1}{2}\mathrm{g}^{0k}{}_{,j}V_{(1)}{}^{0j}-\tfrac{1}{2}\mathrm{g}^{00}{}_{,j}V_{(1)}{}^{jk}+\tfrac{1}{2}\mathrm{g}^{0j}{}_{,j}V_{(1)}{}^{0k}\\&-\tfrac{1}{2}\gamma_{\alpha\beta}\mathrm{g}^{\alpha k}{}_{,j}V_{(1)}{}^{\beta j}-\tfrac{1}{2}V_{(1)}{}^{00}\mathrm{g}^{kj}{}_{,j}\},\end{aligned} \tag{5.10b}$$

$$\begin{aligned}W_{g(0)}{}^{0k}={}&\tfrac{1}{2}\pi_{\alpha\beta}{}^{k(0)}V_{(0)}{}^{\alpha\beta}+\gamma^{\alpha[j}\mathrm{g}^{0]\beta}{}_{,j}\pi_{\alpha\beta}{}^{k(0)}+\epsilon^{-2}\{\tfrac{1}{2}\gamma_{\alpha\beta}\mathrm{g}^{0\beta}{}_{,k}V_{(0)}{}^{0\alpha}\\&-\tfrac{1}{8}\gamma_{\alpha\beta}\mathrm{g}^{\alpha\beta}{}_{,k}\gamma_{\mu\nu}V_{(0)}{}^{\mu\nu}+\tfrac{1}{4}\gamma_{\alpha\rho}\gamma_{\beta\sigma}\mathrm{g}^{\alpha\beta}{}_{,k}V_{(0)}{}^{\rho\sigma}\\&+\tfrac{1}{2}\gamma_{\alpha\beta}\mathrm{g}^{\beta j}{}_{,k}\mathrm{g}^{\alpha0}{}_{,j}-\tfrac{1}{2}\gamma_{\alpha\beta}\mathrm{g}^{\alpha k}{}_{,j}V_{(0)}{}^{\beta j}-\tfrac{1}{2}\mathrm{g}^{00}{}_{,j}V_{(0)}{}^{jk}\\&+\tfrac{1}{2}\mathrm{g}^{0k}{}_{,j}V_{(0)}{}^{0j}-\tfrac{1}{2}\mathrm{g}^{kj}{}_{,i}\mathrm{g}^{0i}{}_{,j}+\tfrac{1}{2}V_{(0)}{}^{0k}\mathrm{g}^{0j}{}_{,j}+\mathrm{g}^{ij}{}_{,j}\mathrm{g}^{0k}{}_{,i}\\&-\tfrac{1}{2}\mathrm{g}^{kj}{}_{,j}V_{(0)}{}^{00}-\tfrac{1}{2}\mathrm{g}^{kj}{}_{,j}\mathrm{g}^{0i}{}_{,i}-\tfrac{1}{2}\gamma_{\alpha\beta}\gamma^{ij}\mathrm{g}^{\alpha k}{}_{,i}\mathrm{g}^{\beta0}{}_{,j}\},\end{aligned} \tag{5.10c}$$

$$W_{m(2)}{}^{0k}=-\tfrac{1}{2}(\mathrm{g}^{00})^{-2}\mathrm{g}^{0k}p^2-\tfrac{1}{2}(\mathrm{g}^{00})^{-1}\mathrm{g}^{0k}p^2, \tag{5.10d}$$

$$\begin{aligned}W_{m(1)}{}^{0k}={}&-\tfrac{1}{2}[1+(\mathrm{g}^{00})^{-1}]p\,\varphi_{,k}+\tfrac{1}{2}[1+(\mathrm{g}^{00})^{-1}]\mathrm{g}^{jk}\varphi_{,j}\\&-\tfrac{1}{2}(\mathrm{g}^{00})^{-1}\mathrm{g}^{0k}\mathrm{g}^{0j}\varphi_{,j}p-(\mathrm{g}^{00})^{-2}\mathrm{g}^{0k}\mathrm{g}^{0j}\varphi_{,j}p,\end{aligned} \tag{5.10e}$$

$$\begin{aligned}W_{m(0)}{}^{0k}={}&\tfrac{1}{2}(\mathrm{g}^{00})^{-1}\mathrm{g}^{0i}\mathrm{g}^{kj}\varphi_{,i}\varphi_{,j}-\tfrac{1}{2}(\mathrm{g}^{00})^{-1}\mathrm{g}^{0j}\varphi_{,j}\varphi_{,k}\\&-\tfrac{1}{2}[(\mathrm{g}^{00})^{-1}\mathrm{g}^{0j}\varphi_{,j}]^2\mathrm{g}^{0k}.\end{aligned} \tag{5.10f}$$

These expressions are to be compared with the results of the calculation of the commutator (Poisson bracket) on the left-hand side of (4.15).

VI. PROOF OF INTEGRABILITY

The remaining step in the proof that the interaction operator $W^{\mu\nu}$ leads to an integrable Tomonaga-Schwinger equation is the calculation of the commutator on the left-hand side of (4.15). This is a straightforward but tedious procedure in which there are several fortunate simplifications.

The interaction Hamiltonian depends only on the field operators, their first spatial gradients, and the canonical momenta; therefore, the commutator will have the general form:

$$(i\hbar c)^{-1}[W^{00}(x),\,W^{00}(x')]=\left\{F^k(x,x)\frac{\partial}{\partial x^k}-F^k(x',x')\frac{\partial}{\partial x'^k}\right\}\delta_3(\mathbf{x}-\mathbf{x}'). \tag{6.1}$$

No terms proportional to the delta function can appear in (6.1), since the left-hand side is antisymmetric in $\mathbf{x}$ and $\mathbf{x}'$. Upon comparison with (4.15), we see that it

[20] The objects considered here are only required to transform as tensors under the Lorentz group. More general transformations will not be considered.

[21] We introduce the abbreviations: $j^{\alpha\beta}\equiv(\gamma^{\alpha\mu}\gamma^{\beta\nu}-\tfrac{1}{2}\gamma^{\alpha\beta}\gamma^{\mu\nu})\pi_{\mu\nu}$, $A^{[\mu}B^{\nu]}\equiv A^\mu B^\nu-A^\nu B^\mu$.

is sufficient to establish

$$F^k(x,x)=2W^{0k}(x). \quad (6.2)$$

Thus we may ignore any term proportional to a delta function, and we may put $\mathbf{x}=\mathbf{x}'$ in the coefficients of the gradients of the delta function.

Using the definition (5.1) of W^{00}, we have[22]

$$[W^{00}(x),W^{00}(x')]=[\mathcal{H}(x),\mathcal{H}(x')]+[H(x),H(x')]-\{[\mathcal{H}(x),H(x')]-(x\leftrightarrow x')\}. \quad (6.3)$$

The simplest of these commutators is $[H,H']$.

$$(i\hbar c)^{-1}[H,H']=\{2\epsilon^{-2}\mathfrak{g}^{lj}{}_{,l}\mathfrak{g}^{0k}{}_{,j}-\tfrac{1}{2}\mathfrak{g}^{0l}{}_{,l}\mathfrak{g}^{jk}{}_{,j}-\tfrac{1}{2}\mathfrak{g}^{00}{}_{,j}\mathfrak{g}^{0j}{}_{,k}-\tfrac{1}{2}\mathfrak{g}^{jk}{}_{,l}\mathfrak{g}^{0l}{}_{,j}+\tfrac{1}{2}\mathfrak{g}^{0i}{}_{,l}\mathfrak{g}^{lj}{}_{,k}\gamma_{ij}+\tfrac{1}{2}\gamma^{ij}\mathfrak{g}^{00}{}_{,i}\mathfrak{g}^{0k}{}_{,j}-\tfrac{1}{2}\gamma_{ij}\gamma^{lm}\mathfrak{g}^{0i}{}_{,l}\mathfrak{g}^{jk}{}_{,m}-2(\gamma^{kl}\pi_{l\rho}\mathfrak{g}^{j\rho}{}_{,j}+\tfrac{1}{2}\pi_{\mu\nu}\mathfrak{g}^{\mu\nu}{}_{,k}-\pi_{\rho j}\gamma^{jl}\mathfrak{g}^{\rho k}{}_{,l})-p\,\varphi_{,k}\}\times\frac{\partial}{\partial x'^k}\delta_3(\mathbf{x}-\mathbf{x}')-(\mathbf{x}\leftrightarrow\mathbf{x}'). \quad (6.4)$$

We next consider the cross terms $[\mathcal{H},H']$.

$$[\mathcal{H},H']=[\mathcal{H}_g,H_g']+[\mathcal{H}_m,H_g']+[\mathcal{H}_g,H_m']+[\mathcal{H}_m,H_m']. \quad (6.5)$$

Since $\mathcal{H}_m$ contains no gradients of the $\mathfrak{g}^{\mu\nu}$, the commutator $[\mathcal{H}_m,\mathcal{H}_g']$ can only lead to terms proportional to δ_3; and since H_m' contains no gravitational field operators at all, $[\mathcal{H}_g,H_m']$ vanishes identically. This leaves only the first and fourth terms in (6.5). For the latter, we find

$$(i\hbar c)^{-1}[\mathcal{H}_m,H_m']=\{(\mathfrak{g}^{00})^{-1}p\,\varphi_{,k}+(\mathfrak{g}^{00})^{-1}\mathfrak{g}^{0k}p^2+(\mathfrak{g}^{00})^{-1}\mathfrak{g}^{0j}\varphi_{,j}\varphi_{,k}+(\mathfrak{g}^{00})^{-1}\mathfrak{g}^{0k}\mathfrak{g}^{0j}\varphi_{,j}p-\mathfrak{g}^{kj}\varphi_{,j}p\}(\partial/\partial x'^k)\delta_3. \quad (6.6)$$

The evaluation of the remaining term is more difficult. According to (2.11) and (2.13), we can write

$$[\mathcal{H}_g,H_g']=\sum_{s=0}^{2}\sum_{r=0}^{2}[\mathcal{H}_g{}^{(s)},H_g{}^{(r)\prime}]. \quad (6.7)$$

We list below the commutators which appear on the right-hand side of (6.7), with the exception of those which either vanish or are proportional to δ_3:

$$(i\hbar c)^{-1}[\mathcal{H}_g{}^{(2)},H_g{}^{(1)\prime}]=-2\gamma^{\alpha[0}V_{(1)}{}^{k]\beta}\pi_{\alpha\beta}(\partial/\partial x'^k)\delta_3, \quad (6.8a)$$

$$(i\hbar c)^{-1}[\mathcal{H}_g{}^{(2)},H_g{}^{(0)\prime}]=\epsilon^{-2}\{\tfrac{1}{4}\gamma_{\alpha\beta}V_{(1)}{}^{\alpha\beta}\gamma_{\mu\nu}\mathfrak{g}^{\mu\nu}{}_{,k}-\mathfrak{g}^{0k}{}_{,j}V_{(1)}{}^{0j}+\mathfrak{g}^{00}{}_{,j}V_{(1)}{}^{kj}+\mathfrak{g}^{kj}{}_{,j}V_{(1)}{}^{00}-\mathfrak{g}^{0j}{}_{,j}V_{(1)}{}^{0k}-\tfrac{1}{2}\gamma_{\alpha\mu}\gamma_{\beta\nu}\mathfrak{g}^{\mu\nu}{}_{,k}V_{(1)}{}^{\alpha\beta}-\gamma_{\rho\sigma}\mathfrak{g}^{0\rho}{}_{,k}V_{(1)}{}^{0\sigma}+\gamma_{\rho\sigma}\mathfrak{g}^{\rho k}{}_{,j}V_{(1)}{}^{\sigma j}\}(\partial/\partial x'^k)\delta_3, \quad (6.8b)$$

$$(i\hbar c)^{-1}[\mathcal{H}_g{}^{(1)},H_g{}^{(2)\prime}]=2(\mathfrak{g}^{00})^{-1}\epsilon^2\{\mathfrak{g}^{0k}j^{\alpha\beta}\pi_{\alpha\beta}+2\mathfrak{g}^{\alpha[0}j^{k]\beta}\pi_{\alpha\beta}\}(\partial/\partial x'^k)\delta_3, \quad (6.8c)$$

$$(i\hbar c)^{-1}[\mathcal{H}_g{}^{(1)},H_g{}^{(1)\prime}]=2\{V_{(0)}{}^{k\rho}\pi_{\rho0}+V_{(0)}{}^{0\rho}\gamma^{kl}\pi_{l\rho}-\pi_{0\rho}{}^{k(1)}\mathfrak{g}^{l\rho}{}_{,l}-\pi_{\rho l}{}^{k(1)}\gamma^{lj}\mathfrak{g}^{\rho0}{}_{,j}\}(\partial/\partial x'^k)\delta_3, \quad (6.8d)$$

$$(i\hbar c)^{-1}[\mathcal{H}_g{}^{(1)},H_g{}^{(0)\prime}]=\epsilon^{-2}\{\tfrac{1}{4}\gamma_{\alpha\beta}V_{(0)}{}^{\alpha\beta}\gamma_{\mu\nu}\mathfrak{g}^{\mu\nu}{}_{,k}-V_{(0)}{}^{0j}\mathfrak{g}^{0k}{}_{,j}+\mathfrak{g}^{00}{}_{,j}V_{(0)}{}^{kj}+\mathfrak{g}^{kj}{}_{,j}V_{(0)}{}^{00}-\mathfrak{g}^{0j}{}_{,j}V_{(0)}{}^{0k}+\gamma_{\rho\sigma}\mathfrak{g}^{0\rho}{}_{,k}V_{(0)}{}^{\sigma0}-\tfrac{1}{2}\gamma_{\mu\alpha}\gamma_{\nu\beta}\mathfrak{g}^{\mu\nu}{}_{,k}V_{(0)}{}^{\alpha\beta}-\gamma_{\rho\sigma}\mathfrak{g}^{\rho k}{}_{,j}V_{(0)}{}^{j\sigma}\}\times(\partial/\partial x'^k)\delta_3, \quad (6.8e)$$

$$(i\hbar c)^{-1}[\mathcal{H}_g{}^{(0)},H_g{}^{(2)\prime}]=2\epsilon^2j^{\mu\nu}\pi_{\mu\nu}{}^{k(0)}(\partial/\partial x'^k)\delta_3, \quad (6.8f)$$

$$(i\hbar c)^{-1}[\mathcal{H}_g{}^{(0)},H_g{}^{(1)\prime}]=-2\{\pi_{0\beta}{}^{k(0)}\mathfrak{g}^{j\beta}{}_{,j}+\pi_{j\beta}{}^{k(0)}\gamma^{jl}\mathfrak{g}^{\beta0}{}_{,l}\}(\partial/\partial x'^k)\delta_3. \quad (6.8g)$$

The combination of Eqs. (6.8a) through (6.8g) determines that part of $[\mathcal{H}_g,H_g']$ which contributes to (6.3). This completes the calculation of $[\mathcal{H},H']$.

We must finally calculate $[\mathcal{H},\mathcal{H}']$.

$$[\mathcal{H},\mathcal{H}']=[\mathcal{H}_g,\mathcal{H}_g']+\{[\mathcal{H}_g,\mathcal{H}_m']-(x\leftrightarrow x')\}+[\mathcal{H}_m,\mathcal{H}_m']. \quad (6.9)$$

The commutator $[\mathcal{H}_g,\mathcal{H}_m']$ contains only δ_3 terms; therefore, the cross terms in (6.9) vanish identically. For $[\mathcal{H}_m,\mathcal{H}_m']$ we find

$$(i\hbar c)^{-1}[\mathcal{H}_m,\mathcal{H}_m']=\{(\mathfrak{g}^{00})^{-1}\mathfrak{g}^{0l}\mathfrak{g}^{jk}\varphi_{,l}\varphi_{,j}+(\mathfrak{g}^{00})^{-1}\mathfrak{g}^{kj}\varphi_{,j}p-(\mathfrak{g}^{00})^{-2}\mathfrak{g}^{0i}\mathfrak{g}^{0j}\mathfrak{g}^{0k}\varphi_{,i}\varphi_{,j}-(\mathfrak{g}^{00})^{-2}\mathfrak{g}^{0k}p^2-2(\mathfrak{g}^{00})^{-2}\mathfrak{g}^{0k}\mathfrak{g}^{0j}\varphi_{,j}p\}(\partial/\partial x'^k)\delta_3-(x\leftrightarrow x'). \quad (6.10)$$

In the calculation of $[\mathcal{H}_g,\mathcal{H}_g']$ we must again resort to using (2.11) for $\mathcal{H}_g$.

$$[\mathcal{H}_g,\mathcal{H}_g']=\sum_{r=0}^{2}\sum_{s=0}^{2}[\mathcal{H}_g{}^{(r)},\mathcal{H}_g{}^{(s)\prime}]. \quad (6.11)$$

We give below those nonvanishing commutators which are not simply proportional to δ_3:

$$(i\hbar c)^{-1}[\mathcal{H}_g{}^{(2)},\mathcal{H}_g{}^{(1)}]'=\{(\mathfrak{g}^{00})^{-1}\mathfrak{g}^{k0}V_{(1)}{}^{\alpha\beta}\pi_{\alpha\beta}\}\times(\partial/\partial x'^k)\delta_3, \quad (6.12a)$$

$$(i\hbar c)^{-1}[\mathcal{H}_g{}^{(2)},\mathcal{H}_g{}^{(0)\prime}]=V_{(1)}{}^{\alpha\beta}\pi_{\alpha\beta}{}^{k(0)}(\partial/\partial x'^k)\delta_3, \quad (6.12b)$$

$$(i\hbar c)^{-1}[\mathcal{H}_g{}^{(1)},\mathcal{H}_g{}^{(1)\prime}]=\pi_{\mu\nu}{}^{k(1)}V_{(0)}{}^{\mu\nu}(\partial/\partial x'^k)\delta_3-(x\leftrightarrow x'), \quad (6.12c)$$

$$(i\hbar c)^{-1}[\mathcal{H}_g{}^{(1)},\mathcal{H}_g{}^{(0)\prime}]=V_{(0)}{}^{\mu\nu}\pi_{\mu\nu}{}^{k(0)}(\partial/\partial x'^k)\delta_3. \quad (6.12d)$$

The quantities $F^k(x,x)$ can now be determined by simply inspecting the results given above. For example, the terms of second degree in p are given by

$$F_{m(2)}{}^k=-(\mathfrak{g}^{00})^{-1}\mathfrak{g}^{0k}p^2-(\mathfrak{g}^{00})^{-2}\mathfrak{g}^{0k}p^2. \quad (6.13)$$

We see from (5.10a) that this is just $2W_{m(2)}{}^{0k}$. The same result holds for the other powers of p, as well as for all the powers of $\pi_{\alpha\beta}$; therefore, we conclude that

$$F^k(x,x)=2W^{0k}(x). \quad (6.14)$$

This ensures the validity of (4.15) and completes the proof of integrability.

[22] In the interests of brevity from now on we write: $A(x')=A'$, $A(x)=A$, $\delta_3(x-x')=\delta_3$, and $A(x,x')-A(x',x)=A(x,x')-(x\leftrightarrow x')$.

VII. INTERACTION PICTURE

In this section, we briefly consider the definition and properties of the interaction picture. If $F(x)$ is any Heisenberg-picture operator, then the corresponding operator in the interaction picture is defined by

$$^\circ F(x,\Sigma)=U[\Sigma]F(x)U^{-1}[\Sigma]. \tag{7.1}$$

The explicit surface dependence of $^\circ F(x,\Sigma)$ implies that the definition of the interaction picture must include a convention for choosing Σ. Furthermore, this convention must lead to a free-field equation of motion for $^\circ F$; otherwise the interaction picture would not be useful. This can be accomplished as follows: In each coordinate system, we choose Σ to be the constant-time surface which passes through x. Thus[23]

$$^\circ F(x)=U[x^{0\prime}=x^0]F(x)U^{-1}[x^{0\prime}=x^0]. \tag{7.2}$$

The operator $^\circ F(x)$ defined in this way satisfies

$$i\hbar c\ {}^\circ F_{,0}=\left[{}^\circ F,\int_{x^{0\prime}=x^0} d^3x'\ {}^\circ H(x')\right]. \tag{7.3}$$

Applying this result to $^\circ\varphi$, $^\circ p$, $^\circ\mathfrak{g}^{\mu\nu}$, and $^\circ\pi_{\mu\nu}$, we obtain the canonical field equations for the free fields. After elimination of the $^\circ\pi_{\mu\nu}$ and $^\circ p$, we find the expected second-order field equations:

$$\Box\ {}^\circ h^{\mu\nu}=0,\quad (\Box-\mu^2)\,{}^\circ\varphi=0, \tag{7.4}$$

with $h^{\mu\nu}$ defined by (2.14), and

$$\Box\equiv\gamma^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}.$$

Since the surface Σ used in defining $^\circ F(x)$ is chosen anew in each coordinate system, it is necessary to exercise some caution with regard to the transformation properties of the interaction-picture operators. For example, the $^\circ\pi_{\mu\nu}$ do not transform in the same way as the $\pi_{\mu\nu}$. However, an explicit calculation shows that the quantities $^\circ h^{\mu\nu}$, $^\circ h^{\mu\nu}{}_{,\lambda}$, $^\circ\varphi$, and $^\circ\varphi_{,\lambda}$ do behave in the appropriate way under Lorentz transformations.[10]

From the field equations, together with the canonical commutation rules, we find in the standard way the following covariant commutation relations[24]:

$$\begin{aligned}[{}^\circ h^{\mu\nu}(x),{}^\circ h^{\rho\sigma}(x')]&=i\hbar c\{\gamma^{\mu(\rho}\gamma^{\sigma)\nu}-\gamma^{\mu\nu}\gamma^{\rho\sigma}\}D(x-x'),\\ [{}^\circ\varphi(x),{}^\circ\varphi(x')]&=i\hbar c\Delta(x-x').\end{aligned} \tag{7.5}$$

We can also introduce an invariant splitting into positive- and negative-frequency parts:

$$\begin{aligned}{}^\circ h^{\mu\nu}&={}^\circ h^{\mu\nu(+)}+{}^\circ h^{\mu\nu(-)},\\ {}^\circ\varphi&={}^\circ\varphi^{(+)}+{}^\circ\varphi^{(-)}.\end{aligned} \tag{7.6}$$

The commutation rules (7.5), together with the invariant decomposition (7.6), could now be used to set up the machinery of covariant perturbation theory. We do not exploit this possibility here; instead, we briefly consider some of the problems connected with finding the correct quantum mechanical form of the auxiliary condition. In doing so, we will be guided by the analogies which exist between the Einstein and Maxwell theories. In the first place, it is clear that we must require that (2.2) hold for expectation values. That is, we require that all physically admissible state vectors Ψ_H satisfy

$$\langle\Psi_H|\mathfrak{g}^{\mu\nu}{}_{,\nu}|\Psi_H\rangle=0. \tag{7.7}$$

It is known that the corresponding condition in quantum electrodynamics,

$$\langle\Psi_H|A^\lambda{}_{,\lambda}|\Psi_H\rangle=0, \tag{7.8}$$

does not suffice to eliminate all the effects of the non-gauge-invariant term in the Lagrangian. In that case, it is necessary to impose the Gupta-Bleuler condition:

$$A^{\lambda(-)}{}_{,\lambda}\Psi_H=0. \tag{7.9}$$

It is possible to define $A^{\lambda(\pm)}{}_{,\lambda}$ in the Heisenberg picture because of the fact that $A^\lambda{}_{,\lambda}$ satisfies

$$\Box A^\lambda{}_{,\lambda}=0. \tag{7.10}$$

Unfortunately, the corresponding gravitational quantity $h^{\mu\nu}{}_{,\nu}$ does not satisfy any such simple equation; therefore, the simple condition (7.9) has no analog in the gravitational case.[25] In order to find a condition which can be taken over to the gravitational case, we consider the form taken by (7.9) in the interaction picture:

$$\left[{}^\circ A^{\lambda(-)}{}_{,\lambda}(x)-\int d^3x'\ {}^\circ\rho(x')D^{(-)}(x-x')\right]\times\Psi_I[\Sigma]=0, \tag{7.11}$$

where $^\circ\rho$ is the charge density operator. We now show that such a condition can also be found in the present context.

First of all, we note that the condition (7.7), which holds for all times, can be replaced by two initial conditions which hold at some one time:

$$\begin{aligned}\langle\Psi_H|\mathfrak{g}^{\mu\nu}{}_{,\nu}|\Psi_H\rangle&=0,\\ \langle\Psi_H|\mathfrak{g}^{\mu\nu}{}_{,0\nu}|\Psi_H\rangle&=0.\end{aligned} \tag{7.12}$$

[23] We no longer indicate the explicit Σ dependence of $^\circ F(x,\Sigma)$.

[24] The D and Δ functions used here are those defined by J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Reading, Massachusetts, 1955), Appendix A. We also use their definition of positive- and negative-frequency parts for field operators. Note that with their conventions, the negative-frequency part represents an annihilation operator.

[25] The contrary assertion found in the first of the papers of Gupta mentioned in footnote 1 is incorrect. The apparent proof that $\mathfrak{g}^{\mu\nu}{}_{,\nu}$ satisfies a wave equation is based on the use of a special form of the gravitational field equations [Gupta's Eq. (14)] which is obtained by dropping terms proportional to $\mathfrak{g}^{\mu\nu}{}_{,\nu}$ in the preceding equation. This obviously means that the wave equation on p. 614 of Gupta's paper is a simple identity $0=0$, since the quantity $\mathfrak{g}^{\mu\nu}{}_{,\nu}$ itself has already been set equal to zero.

The conditions (7.12), together with the field equations, guarantee the validity of (7.7) for all times. If we express the operators appearing in (7.12) in terms of the canonical field operators and then transform to the interaction picture, we find

$$\begin{aligned}\langle\Psi_I[\Sigma]|\,{}^\circ h^{\mu\lambda}{}_{,\lambda}+\epsilon\,{}^\circ B^\mu|\Psi_I[\Sigma]\rangle&=0,\\ \langle\Psi_I[\Sigma]|\,{}^\circ h^{\mu\lambda}{}_{,\lambda 0}+\epsilon\,{}^\circ E^\mu|\Psi_I[\Sigma]\rangle&=0,\end{aligned}\tag{7.13}$$

with

$${}^\circ B^\mu\equiv[{}^\circ h^{0(\alpha}\gamma^{\beta)\mu}-\gamma^{0\mu}\,{}^\circ h^{\alpha\beta}]\,{}^\circ p_{\alpha\beta},\tag{7.14a}$$

$$\begin{aligned}{}^\circ E^\mu\equiv&-\delta_0{}^\mu\,{}^\circ\mathcal{H}+\gamma^{\mu k}({}^\circ p_{\alpha\beta}\,{}^\circ h^{\alpha\beta}{}_{,k}+{}^\circ\varphi_{,0}\,{}^\circ\varphi_{,k})\\ &+2(\partial/\partial x^k)\{{}^\circ h^{\lambda k}\gamma^{\mu\rho}\,{}^\circ p_{\rho\lambda}-\gamma^{k\mu}\,{}^\circ h^{\alpha\beta}\,{}^\circ p_{\alpha\beta}\},\end{aligned}\tag{7.14b}$$

$${}^\circ p_{\alpha\beta}\equiv\tfrac{1}{4}(\gamma_{\alpha\mu}\gamma_{\beta\nu}-\gamma_{\alpha\beta}\gamma_{\mu\nu})\,{}^\circ h^{\mu\nu}{}_{,0}+\tfrac{1}{2}\,{}^\circ h_{,j}{}^{\nu[0}\delta_{(\alpha}{}^{j]}\gamma_{\beta)\nu}.\tag{7.14c}$$

By making use of the field equations (7.4) for ${}^\circ h^{\mu\nu}$ and the properties of the D function, we can replace (7.13) by the following condition, which holds for all surfaces Σ and points x (i.e., x need not lie on Σ):

$$\langle\Psi_I[\Sigma]|\Omega^\mu(x,\Sigma)|\Psi_I[\Sigma]\rangle=0,\tag{7.15}$$

where the operator $\Omega^\mu(x,\Sigma)$ is defined by

$$\Omega^\mu={}^\circ h^{\mu\nu}{}_{,\nu}+\int_\Sigma d^3x'\,\frac{\partial}{\partial x'^0}D(x-x')\epsilon\,{}^\circ B^\mu(x')-\int_\Sigma d^3x'\,\epsilon\,{}^\circ E^\mu(x')D(x-x').\tag{7.16}$$

Since Ω^μ satisfies

$$\Box\Omega^\mu=0,\tag{7.17}$$

we can define positive- and negative-frequency parts $\Omega^{\mu(\pm)}$. The analog of (7.11) is then given by

$$\Omega^{\mu(-)}\Psi_I[\Sigma]=0,\tag{7.18}$$

which obviously guarantees the validity of (7.15).

It is one thing to obtain the condition (7.18); but it is quite another thing, as a glance at the expression for Ω^μ shows, to prove that (7.18) actually eliminates from the theory the unphysical consequences of the noninvariant term in the Lagrangian. It would presumably be possible to construct such a proof by using perturbation theory (expanding Ω^μ in powers of ϵ), but this has not yet been done. However, Gupta has been able to give such a proof in the free-field approximation in which (7.18) simplifies to[1]

$${}^\circ h^{\mu\nu(-)}{}_{,\nu}\Psi_I=0.\tag{7.19}$$

In quantum electrodynamics, the analogous simplification of (7.11) is usually justified by remarking that one is only interested in scattering states for which the second term in (7.11) is negligible. This argument seems rather dubious in the present case because of the complicated gravitational self-interactions which are present in Ω^μ. An alternative procedure in the electrodynamic case is to perform a unitary transformation which brings (7.11) into a form in which the second term is missing. This amounts to a covariant separation of the Coulomb interaction from the interaction between electrons and the transverse radiation field.[26] Such a clear separation does not seem very likely in gravitational theory, since the "longitudinal" part of the gravitational field will possess energy and will therefore interact with the "transverse" parts of the field.

VIII. SUMMARY AND DISCUSSION

In this paper, we have attempted to cast the Einstein theory into a form which resembles, as closely as possible, a Lorentz-covariant field theory. In order to accomplish this, we have assumed the validity of Fock's conjecture that the De Donder condition, together with the boundary conditions (3.1), reduces the invariance of the theory to the Lorentz group. We were thus able to introduce a naturally defined flat-space metric $\gamma_{\mu\nu}$ which was used in the definition of various quantities such as the unit normals to spacelike surfaces.

The physical assumptions underlying Fock's conjecture forced us to restrict our attention to universes for which the spatial geometry is asymptotically Euclidean. Furthermore, the spacelike surfaces on which we define the state vectors of the interaction picture were restricted to those satisfying the condition (4.7). Therefore, our work does not apply to universes with essentially non-Euclidean topologies. The possibilities of quantizing the theory and defining an interaction picture for these more general geometries must be separately investigated.

Within the framework of our assumptions, we have established that the interaction operator defined here satisfies a condition which is necessary for the integrability of the Tomonaga-Schwinger equation. The proof of sufficiency cannot be made until some systematic method of handling the problem of factor ordering is available. One way of approaching this problem would be to expand the interaction operator in a series of products of field operators; (i.e., in powers of the coupling constant) transform to the interaction picture; and then order the operators by using the Wick product for each term in the series. The integrability condition (4.15) would then have to be satisfied to each order in the coupling constant. A similar expansion might also be used to prove that the auxiliary condition (7.18), which is the quantum mechanical form of the classical De Donder condition, actually eliminates the unphysical effects of the noninvariant term in the Lagrangian. In particular, we note that the commutation relations (7.5) require the use of an indefinite metric. It must therefore be shown that states of negative norm are eliminated by the auxiliary condition.

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[26] See reference 24, Appendix A.