Representations of the S Matrix in Terms of Its Angular Momentum Poles*

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The asymptotic distribution of poles of the nonrelativistic S matrix for potential scattering in the complex angular momentum plane is investigated, and so is the nature of the pole trajectories near $E=0$. As a consequence of the behavior of the distant poles and of their residues the S matrix is shown to be representable in the form of an infinite (Weierstrass-Hadamard) product as well as in the form of a (Mittag-Leffler) series of partial fractions.

1. INTRODUCTION

THE analytic continuation of the scattering matrix into the complex angular momentum plane has lately been explored by a large number of authors.[1–5] Its usefulness for several different purposes has been well established. The present paper adds to the discussion by a detailed investigation of the behavior of the Regge poles in the left-hand half of the angular momentum plane for nonrelativistic potential scattering of spin-zero particles, and by subsequent representations of the S matrix in terms of its poles.

Before proceeding, we briefly summarize the essential facts concerning Regge poles for potential scattering and their trajectories as they are known at present.

If the potential is such that there is a pole of the S matrix in the region $\mathrm{Re}l>-\frac{1}{2}$ (an attractive potential *always* leads at least to one such pole[3,4]), then it must remain on the real axis so long as $E<0$. For $E>0$ it must leave the real axis and turn into the upper half of the complex plane. If it leaves at $l<0$ it turns backwards[3]; if at $0<l<\frac{1}{2}$, it leaves forward at a finite angle; if at $l>\frac{1}{2}$, forward at zero angle, osculating the axis more and more closely the larger the l value at which it leaves.[3,6] A trajectory can never cross or touch the real axis when $E>0$. If the potential can be analytically continued into the complex r plane up to the imaginary axis, and if it obeys a certain bound there, the trajectory must turn back to the left eventually and either cross the axis $\mathrm{Re}l=-\frac{1}{2}$ or else approach it asymptotically.[1,7] If the potential has no such analytic continuation, the trajectory need not turn back. In the case of the square well it is known not to do so.[8] The number of trajectories that enter the upper right-hand quadrant of the $\lambda=l+\frac{1}{2}$ plane at $E=0$ and $\lambda>0$ is always finite.[3]

If r times the potential has a certain number of finite derivatives at $r=0$, then the S matrix can be analytically continued to $\mathrm{Re}l<-\frac{1}{2}$, the distance depending linearly on the number of finite derivatives of rV.[3] Let us assume for simplicity that all the derivatives of rV are finite at $r=0$. Then S is a meromorphic function of l in the whole complex l plane.[3,9,10] In the region $\mathrm{Re}l<-\frac{1}{2}$ there are infinitely many poles for each fixed energy, with no finite accumulation point. When $E<0$ the poles there need not necessarily lie on the real axis; if they do not, however, they must occur in complex conjugate pairs.[3] When $E>0$ the poles there are generally not on the real axis, but they need not lie in the upper half-plane. It follows from the symmetry relations (2.7) that they can cross the real axis at negative integral or half-integral l values, but only there. In the case of a simple Yukawa potential, trajectories are known from numerical computation to cross at such points.[11] As the energy tends to $+\infty$ or $-\infty$ each trajectory must either lead to infinity or else it must terminate at a negative integral value of l.[3,9] Depending on the number of derivatives of rV which *vanish* at $r=0$, some of the negative integers are ruled out as possible trajectory terminals.[3] For example, if the potential is *constant* for $0\leqslant r\leqslant r_0$, as in the case of a square well, no trajectories can end in the finite l plane; they must all terminate at infinity.[3,8]

We are going to prove a number of new results concerning the Regge poles near and to the left of $l=-\frac{1}{2}$. After setting up the preliminaries in Sec. 2, we discuss in Sec. 3 the question whether for negative energies the poles must lie on the real axis. In Sec. 4 the behavior of the Jost function for large complex $|l|$ is investigated and from this we determine the asymptotic distribution of Regge poles under various assumptions on the potential. Section 5 deals with the distribution of the poles near zero energy and as a function of the potential strength. There are generally infinitely many trajectories that arrive at $l=-\frac{1}{2}$ when $E=0$.[11a] Section 6 exploits our results on the asymptotic pole distribution to write the S matrix as an infinite product. In Sec. 7 the asymptotic behavior of the residues is examined

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[1] T. Regge, Nuovo Cimento 14, 951 (1959).
[2] For a list of references, see references 3 and 5.
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[5] B. R. Desai and R. G. Newton, Phys. Rev. 129, 1437 (1963).
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[7] T. Regge, Nuovo Cimento 18, 947 (1960).
[8] A. O. Barut and F. Calogero, Phys. Rev. 128, 1383 (1962).
[9] S. Mandelstam (to be published).
[10] H. Cheng, Phys. Rev. 127, 647 (1962); E. J. Squires, Nuovo Cimento 25, 242 (1962).
[11] A. Ahmadzadeh, P. G. Burke, and C. Tate (to be published).
[11a] A detailed investigation of the motion of Regge poles as $E\rightarrow 0$ is given by the present authors in the forthcoming publication "Threshold Motion of Regge Poles," Indiana University preprint.

and S is written as a series of partial fractions. There an Appendix concerned with the branch point at $k=0$.

Both the product and the partial fraction representation of the S matrix may be useful in approximate evaluations of the scattering amplitude by means of the Watson transform and in carrying out (if possible) the Mandelstam[9] program of pushing the imaginary axis integral further to the left. About these applications we have noting to say at present.

2. PRELIMINARIES

We are using the same notation as in reference 3. At this point we shall merely list those functions defined and discussed there which are relevant for our present purpose.

A generalized Jost function which approaches unity as $k\to\infty$ is $\mathfrak{f}(\lambda,k)$ which has the integral representation

$$\mathfrak{f}(\lambda,k)=1-i(\pi/2k)^{1/2}e^{\frac{1}{2}\pi(\lambda+\frac{1}{2})i}\times\int_0^\infty dr\, r^{1/2}V(r)J_\lambda(kr)f(\lambda,k;r). \quad (2.1)$$

Here $\lambda=l+\frac{1}{2}$, k is the wave number, so that $E=k^2$ in units where $\hbar=2m=1$; f is the irregular solution of the Schrödinger equation whose zero-order version is

$$f_0(\lambda,k;r)=(\tfrac{1}{2}\pi kr)^{1/2}H_\lambda^{(2)}(kr)e^{-\frac{1}{2}\pi(\lambda+\frac{1}{2})i}; \quad (2.2)$$

J_λ and $H_\lambda^{(2)}$ are the Bessel function and the Hankel function of the second kind, respectively. f is the unique solution of the integral equation

$$f(\lambda,k;r)=f_0(\lambda,k;r)-\int_r^\infty dr'\, g_\lambda(k;r,r')V(r')f(\lambda,k;r'), \quad (2.3)$$

where

$$\begin{aligned}g_\lambda(k;r,r')&=\tfrac{1}{2}\pi(rr')^{1/2}[J_\lambda(kr)Y_\lambda(kr')-J_\lambda(kr')Y_\lambda(kr)]\\&=\tfrac{1}{2}\pi(rr')^{1/2}[J_\lambda(kr)J_{-\lambda}(kr')-J_\lambda(kr')J_{-\lambda}(kr)]/\sin\pi\lambda. \quad (2.4)\end{aligned}$$

In general, $\mathfrak{f}$ has simple poles at the negative half-integral values of λ. It is, therefore, useful to introduce

$$\mathfrak{f}_e(\lambda,k)\equiv\mathfrak{f}(\lambda,k)/\Gamma(\tfrac{1}{2}+\lambda), \quad (2.5)$$

which, if all the derivatives of rV exist at $r=0$, is an entire function of λ for fixed k. The S matrix is given by

$$S(\lambda,k)=\mathfrak{f}(\lambda,k)/\mathfrak{f}(\lambda,-k)=\mathfrak{f}_e(\lambda,k)/\mathfrak{f}_e(\lambda,-k). \quad (2.6)$$

The Regge poles are therefore the *zeros* of $\mathfrak{f}(\lambda,-k)$. For $E>0$ we choose $k>0$ and thus are looking for the zeros of $\mathfrak{f}(\lambda,k)$ with $k<0$; for $E<0$ the k on the "physical sheet" is that for which $\mathrm{Im}k>0$, and hence the poles of S there are the zeros of $\mathfrak{f}(\lambda,k)$ with $\mathrm{Im}k<0$.

$\mathfrak{f}(\lambda,k)$ satisfies the symmetry relation

$$e^{i\pi\lambda}\mathfrak{f}(\lambda,-k)\mathfrak{f}(-\lambda,k)-e^{-i\pi\lambda}\mathfrak{f}(\lambda,k)\mathfrak{f}(-\lambda,-k)=2i\sin\pi\lambda. \quad (2.7)$$

It is understood that both here and in (2.6)

$$\mathfrak{f}(\lambda,-k)=\mathfrak{f}(\lambda,ke^{-i\pi}).$$

3. THE POLES FOR $E<0$

It was mentioned in reference 3 that for $E<0$ a zero of $\mathfrak{f}$ in the region $\mathrm{Re}\lambda<0$ need not lie on the real axis (as it must when $\mathrm{Re}\lambda>0$) but that if it does not, there must be *two* zeros at complex conjugate points. This possibility was there dismissed as unlikely to be realized because a zero on the real axis would then have to "split in two" in order to get off. Meanwhile numerical computations by Barut and Calogero[8] for the square well potential, have shown that complex Regge poles do indeed occur for $\mathrm{Re}l<-\frac{1}{2}$ (although the authors appear to be unaware of that fact). Figure 2 of reference 8 shows clearly that for a repulsive square well, poles of S at $E=0$ (and hence zeros of $\mathfrak{f}$), as functions of the potential strength, move toward one another in pairs and, after coincidence, disappear from the real axis. The only possible interpretation of this result is that for greater potential strength the poles are to be found in the complex plane. Their disappearance in pairs makes it possible for them to go into complex plane in conjugate pairs.

The $E=0$ zeros of $\mathfrak{f}$ for the square-well potential are determined by[8]

$$J_{\lambda+1}[R(-V)^{1/2}]=0.$$

The question therefore is simply if the Bessel function has any zeros when its argument is purely imaginary and its order *complex*, i.e., $\mathrm{Im}\lambda\neq0$. We have not been able to find a general proof that such zeros exist, but it should be a relatively simple matter to answer the question numerically. Figure 2 of reference 8 does seem to give a definitely affirmative answer.[12]

If such nonreal negative energy Regge poles can occur for the square well potential, then there is, of course, no reason to suppose that they cannot occur in other, more realistic cases.[12a]

4. THE BEHAVIOR OF $\mathfrak{f}(\lambda,k)$ AS $|\lambda|\to\infty$

In order to find the asymptotic form of $f(\lambda,k;r)$ as $|\lambda|\to\infty$ we use the integral equation (2.3) for f and the asymptotic form of the Green's function (2.4). For fixed r and r' we have for $|\lambda|\to\infty$

$$g_\lambda(k;r,r')\sim(rr')^{1/2}(2\lambda)^{-1}[(r/r')^\lambda-(r'/r)^\lambda]. \quad (4.1)$$

[12] It happens that the same equation arises in the context of an exponential potential, except that the order of the Bessel function there is proportional to k; see Eq. (10.6) of R. G. Newton, J. Math. Phys. 1, 319 (1960). The existence of $E=0$ zeroes of the Jost function for the square well off the real l axis is therefore equivalent to the existence of $l=0$ zeros for the exponential potential off the positive imaginary k axis.

[12a] It is shown in reference 11a that they occur in general.

Consequently, for $r'>r$ and $\mathrm{Re}\lambda>0$

$$g\sim-(2\lambda)^{-1}r^{\frac{1}{2}-\lambda}r'^{\frac{1}{2}+\lambda},$$

insertion of which in the integral equation for f yields

$$f(\lambda,k;r)\sim f_0(\lambda,k;r)+(2\lambda)^{-1}\int_r^\infty dr'\,r'^{\frac{1}{2}+\lambda}r^{\frac{1}{2}-\lambda}V(r')f(\lambda,k;r').$$

Setting

$$r^\lambda f(\lambda,k;r)\equiv h(\lambda,k;r)$$

we get

$$h(\lambda,k;r)\sim h_0(\lambda,k;r)+r^{1/2}(2\lambda)^{-1}\int_r^\infty dr'\,r'^{1/2}V(r')h(\lambda,k;r').$$

Therefore,

$$h(\lambda,k;r)\simeq h_0(\lambda,k;r)$$

and

$$f(\lambda,k;r)\simeq f_0(\lambda,k;r) \tag{4.2}$$

as $|\lambda|\to\infty$ with $\mathrm{Re}\lambda>0$. The same is easily seen to hold when $\mathrm{Re}\lambda=0$, and since f is an even function of λ, it holds in all directions.

Now it should be noticed immediately that the foregoing argument is in no sense rigorous. The asymptotic value of g is not approached uniformly in r and r'. The larger r or r', the larger must $|\lambda|$ be for the estimate to hold.[13] It is, therefore, not clear precisely what the conditions on the potential are so that the result (4.2) holds. All we can say is that it will hold provided that the potential decreases *sufficiently* rapidly as $r\to\infty$.[14] *Uniform* estimates on g appear to be hard to come by.

We shall from now on take it for granted that V decreases sufficiently rapidly with increasing r that $f(\lambda,k;r)$ approaches $f_0(\lambda,k;r)$, its "unperturbed" value. We then get from (2.1) as $|\lambda|\to\infty$ in any direction

$$\mathfrak{f}(\lambda,k)\sim1-\tfrac{1}{2}\pi i\int_0^\infty dr\,rV(r)H_\lambda^{(2)}(kr)J_\lambda(kr);$$

that is to say, $\mathfrak{f}(\lambda,k)$ approaches its first Born approximation. Since we have already made use of such arguments anyway, we do not hesitate again to apply asymptotic formulas in order to evaluate $\mathfrak{f}$ for large $|\lambda|$. The fact that in some directions the first Born approximation to $\mathfrak{f}$ tends to infinity should not be regarded as evidence that the argument breaks down because higher orders become important. If the potential vanishes identically beyond a certain distance then our argument will certainly be correct, even though the first approximation tends to infinity. In many other, more realistic cases, it will no doubt also still hold, but it is not clear whether exponential decrease of V is sufficient to assure it.[14]

Let us first assume that $\mathrm{Re}\lambda\to+\infty$. Then

$$H_\lambda^{(2)}(x)=[e^{i\pi\lambda}J_\lambda(x)-J_{-\lambda}(x)]/i\sin\pi\lambda$$
$$\sim i\frac{J_{-\lambda}(x)}{\sin\pi\lambda}\sim i\frac{(\frac{1}{2}x)^{-\lambda}}{\Gamma(1-\lambda)\sin\pi\lambda}=\frac{i}{\pi}\Gamma(\lambda)(\tfrac{1}{2}x)^{-\lambda}, \tag{4.3}$$

when λ is not an integer; when $\lambda=n$, then

$$H_n^{(2)}=(-)^nJ_n+\frac{1}{i\pi}\frac{\partial J_\lambda}{\partial\lambda}\bigg|_{\lambda=n}+\frac{(-)^n}{i\pi}\frac{\partial J_\lambda}{\partial\lambda}\bigg|_{\lambda=-n}$$
$$\underset{n\to\infty}{\sim}\frac{(-)^n}{i\pi}\frac{\partial J_\lambda}{\partial\lambda}\bigg|_{\lambda=-n}\sim\frac{(-)^n}{i\pi}\frac{\partial}{\partial\lambda}\frac{(\frac{1}{2}x)^\lambda}{\Gamma(1+\lambda)}\bigg|_{\lambda=-n}$$
$$=\frac{(-)^n}{i\pi}\left[\frac{\partial}{\partial\lambda}\frac{(\frac{1}{2}x)^\lambda\sin\pi\lambda}{\pi\lambda}\Gamma(1-\lambda)\right]_{\lambda=-n}$$
$$=\frac{i}{\pi}(\tfrac{1}{2}x)^{-n}\Gamma(n).$$

Hence the asymptotic form (4.3) holds for integers, too. Thus as $\mathrm{Re}\lambda\to+\infty$

$$H_\lambda^{(2)}(x)J_\lambda(x)\sim(i/\pi)\Gamma(\lambda)/\Gamma(1+\lambda)=i/\pi\lambda$$

and so

$$\mathfrak{f}(\lambda,k)\sim1+(1/2\lambda)\int_0^\infty dr\,rV(r). \tag{4.4}$$

For $\mathrm{Re}\lambda$ fixed and $\mathrm{Im}\lambda\equiv\nu\to\pm\infty$ we have

$$|J_\lambda(x)|\sim|2\pi\lambda|^{-1/2}e^{\frac{1}{2}\pi|\nu|}(2|\lambda|/ex)^{-\mathrm{Re}\lambda}$$

and thus

$$|H_\lambda^{(2)}(x)|\sim(2/\pi)^{1/2}|\lambda|^{-1/2}e^{-\frac{1}{2}\pi\nu}(2|\lambda|/ex)^{(\nu/|\nu|)\mathrm{Re}\lambda},$$
$$|H_\lambda^{(2)}(x)J_\lambda(x)|\sim|\pi\lambda|^{-1}e^{\frac{1}{2}\pi(|\nu|-\nu)}(2|\lambda|/ex)^{(1-\nu/|\nu|)\mathrm{Re}\lambda}.$$

Consequently (4.4) holds also when $\mathrm{Im}\lambda\to+\infty$, $\mathrm{Re}\lambda=\text{constant}$; but when $\nu=\mathrm{Im}\lambda\to-\infty$ while $\mathrm{Re}\lambda$ is bounded, then[14a]

$$\mathfrak{f}(\lambda,k)=O(\lambda^{-1}e^{-\pi\nu}). \tag{4.5}$$

Now we let $\mathrm{Re}\lambda\to-\infty$. Then J_λ dominates in $H_\lambda^{(2)}$ and we get[15]

$$\mathfrak{f}(\lambda,k)\sim1-\tfrac{1}{2}\pi\int_0^\infty dr\,rV(r)[J_\lambda(kr)]^2(e^{i\pi\lambda}/\sin\pi\lambda). \tag{4.6}$$

[13] For example, $J_\nu(\nu)$ decreases very much less rapidly than $J_\nu(x)$ for fixed x; whereas the latter goes like $(\frac{1}{2}x)^\nu/\Gamma(1+\nu)$, the former goes only as $\nu^{-1/3}$; see G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, New York, 1958), p. 232.

[14] It will be noticed that all we really need is the weaker statement that as $|\lambda|\to\infty$, f approaches a constant multiple of f_0 or at worst approaches f_0 within a factor of the form $|\lambda|^{a|\lambda|}$. So long as that is true, the statements below will hold.

[14a] From the relation $\mathfrak{f}(\lambda,-k)=\mathfrak{f}^*(\lambda^*,k)$ for real k and (4.5) it follows that for fixed $\mathrm{Re}\lambda$ and $k>0$, $\mathfrak{f}(\lambda,-k)$ blows up as $\nu\to+\infty$. Consequently, (4.4) for fixed $\mathrm{Re}\lambda$ and $\nu\to+\infty$ has no bearing on the question of whether there can be far away Regge poles in the vertical direction. It does *not* follow, moreover, that $S\to1$ as $\nu\to+\infty$.

[15] The argument following (4.3) shows that this holds, in general, for the integers too.

Let us now assume that the potential is a superposition of Yukawa potentials:

$$rV(r)=\gamma\int_{\mu_0}^{\infty}d\mu\,\sigma(\mu)e^{-\mu r}.$$

Then the integral can be carried out and we get[16]

$$\mathfrak{f}(\lambda,k)\sim 1-\frac{\gamma e^{i\pi\lambda}}{2k\sin\pi\lambda}\int_{\mu_0}^{\infty}d\mu\,\sigma(\mu)Q_{\lambda-\frac{1}{2}}\left(1+\frac{\mu^2}{2k^2}\right), \quad (4.7)$$

Q being the Legendre function of the second kind. We may now insert the asymptotic value of Q as $\mathrm{Re}\lambda\to-\infty$. The result is[17]

$$\mathfrak{f}(\lambda,k)\sim 1+\frac{1}{2}\frac{\gamma e^{i\pi\lambda}}{\cos\pi\lambda}\int_{\mu_0}^{\infty}d\mu\,\sigma(\mu)\left(\frac{\pi}{-2\lambda\mu k}\right)^{1/2}\times\left(1+\frac{\mu^2}{4k^2}\right)^{-1/4}\Lambda^{-2\lambda}, \quad (4.8)$$

where

$$\Lambda=\frac{\mu}{2k}+\left(1+\frac{\mu^2}{4k^2}\right)^{1/2}>1.$$

Since this has poles at the negative half-integral values of λ, we examine $\mathfrak{f}_e$ of (2.5):

$$\mathfrak{f}_e(\lambda,k)=\mathfrak{f}(\lambda,k)\Gamma(\tfrac{1}{2}-\lambda)\cos\pi\lambda/\pi. \quad (4.9)$$

In order to avoid the more than exponential increase of the Γ function, we consider

$$g(\lambda)\equiv\mathfrak{f}_e(\lambda,k)\mathfrak{f}_e(-\lambda,k)=\mathfrak{f}(\lambda,k)\mathfrak{f}(-\lambda,k)\cos\pi\lambda/\pi, \quad (4.10)$$

which is an even function of λ. As $\mathrm{Re}\lambda\to+\infty$ we get

$$g(\lambda)\sim(\cos\pi\lambda/\pi)+e^{-i\pi\lambda}(\gamma/2\pi)\times\int_{\mu_0}^{\infty}d\mu\,\sigma(\mu)\left(\frac{\pi}{2\lambda\mu k}\right)^{1/2}\left(1+\frac{\mu^2}{4k^2}\right)^{-1/4}\Lambda^{2\lambda}, \quad (4.11)$$

while on the imaginary axis, $g=O(\lambda^{-1}e^{\pi|\lambda|})$.

Suppose first that V is a simple Yukawa potential. Then $g(\lambda)$ grows exponentially for large λ. It is of order[18,19] $\rho=1$ and type $\tau<\infty$; thus it is "of exponential type." The same is true if V is a finite sum of Yukawa potentials, or if it is a *proper* integral of Yukawa potentials. We can then draw a number of conclusions about the asymptotic distribution of its zeros in the λ plane. Since there can be no distant zeros of $\mathfrak{f}$ in the right half plane, this tells us the zero distribution of $\mathfrak{f}$ on the left, and hence that of the Regge poles there.

For large $|\lambda|$, $g(\lambda)$ grows exponentially, with different rates, depending on the phase of λ. Application of Jensen's theorem[20] tells us that

$$N(r)=\int_0^r dt\,t^{-1}n(t),$$

grows linearly with r, where $n(t)$ is the number of zeros in a circle of radius t about the origin. Therefore, the average density $N'(r)$ of the magnitudes of zeros is asymptotically constant, which implies that the magnitudes of the zeros are asymptotically on the average evenly spaced:

$$|\lambda_n|\propto n. \quad (4.12)$$

We now apply Carleman's theorem[20] to $g(i\lambda)$. For imaginary λ, g grows exponentially. Hence,

$$\sum_{|\lambda_n|\leq R}\frac{|\mathrm{Re}\lambda_n|}{|\lambda_n|^2}\propto\ln R. \quad (4.13)$$

Next we apply the same theorem to $g(\lambda)$. For real λ, too, g grows exponentially. Thus, we find that

$$\sum_{|\lambda_n|\leq R}\frac{|\mathrm{Im}\lambda_n|}{|\lambda_n|^2}\sim c\ln R, \quad (4.14)$$

where c tends to naught as $k\to\infty$ (since then $\Lambda\to 1$). These results, together with (4.12), imply that both the real and the imaginary parts of the zeros are asymptotically on the average evenly spaced, and that as the energy increases, they tend to remain closer to the real axis. We already know that in the limit as $k\to\infty$ they move to the real axis, with exactly even spacing.[21] The approach to the negative integral l values is clearly non-uniform. The larger $-l=n$, the higher we must make the energy in order to get the zero close to $l=-n$.

In the more realistic case of a general superposition of Yukawa potentials we cannot say as much. If all the moments of $\sigma(\mu)$ are finite, g is an entire function.[3] If we make the stronger assumption that $\sigma(\mu)$ goes down exponentially for large μ, then the asymptotic behavior $\sigma(\mu)\sim ce^{-\alpha\mu}$ of the μ integral in $g(\lambda)$ for large λ is determined by the value of the integrand for large μ, which is

$$\int^{\infty}d\mu\,\mu^{2\lambda-1}e^{-\alpha\mu}\propto\lambda^{-1/2}\exp\{2\lambda[\ln(2\lambda/\alpha)-1]\}.$$

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[16] See G. N. Watson, reference 13, p. 389. It should be noted that the function Q_ν in formula (2) there is the Legendre function of the second kind, and not the function Q_ν defined previously in the book. The failure to notice that fact led to an error, in Grobner and Hofreiter, *Integraltafel* (Springer-Verlag, Wien and Innsbruck, 1949), p. 203, No. 10.

[17] See Bateman Manuscript Project, in *Higher Transcendental Functions*, edited by A. Erdelyi (McGraw-Hill Book Company, New York, 1953), Vol. 1, p. 143, (22). There is a typographical error in formula (22); $\Gamma(\frac{1}{2}-\mu)$ should be replaced by $\Gamma(\frac{1}{2}-\nu)$.

[18] See R. P. Boas, *Entire Functions* (Academic Press Inc., New York, 1954).

[19] $g(\lambda)$ may also be considered directly a function of λ^2. It is then of order $\frac{1}{2}$ and must, therefore, have infinitely many zeros (see reference 18). We usually take that fact for granted.

[20] See reference 18, p. 2.

[21] We neglect the possibility that some trajectories may terminate at infinity.

It follows that $g(\lambda)$ is of order $\rho=1$ and type $\tau=\infty$. Jensen's theorem then tells us that

$$N(r)\propto r\ln r,$$

and

$$n(r)\propto r\ln r.$$

Thus, the average density of the magnitudes of zeros grows logarithmically; the average spacing between the magnitudes of successive zeros decreases as $1/\ln n$ and hence

$$|\lambda_n|\propto n/\ln n. \tag{4.15}$$

On the imaginary axis, $g(\lambda)$ still grows exponentially. Carleman's theorem applied to $g(i\lambda)$, therefore implies, that

$$\sum_{|\lambda_n|<R}\frac{|\mathrm{Re}\lambda_n|}{|\lambda_n|^2}\propto\ln R, \tag{4.16}$$

and, therefore,

$$|\mathrm{Re}\lambda_n|\propto n/(\ln n)^2, \tag{4.17}$$

while

$$|\mathrm{Im}\lambda_n|\propto n/\ln n. \tag{4.18}$$

Thus, while the projections of the zeros on the real axis get more and more closely spaced, they diffuse more and more *away* from the real axis and toward the imaginary axis. Nevertheless, as the energy increases, the diffusion toward the imaginary axis is less and less rapid.

In the physically somewhat less interesting case of a square-well potential, we may insert the asymptotic form of the Bessel functions in (4.6) and get as $\mathrm{Re}\lambda\to+\infty$:

$$g(\lambda)\sim(1/\pi)\cos\pi\lambda+e^{-i\pi\lambda}(\sin2\pi\lambda/2\pi\lambda)e^{2\lambda\ln\lambda-2\lambda}\times(\tfrac{1}{2}k)^{-2\lambda}\int_0^\infty dr\,V(r)r^{1-2\lambda}$$

$$=(1/\pi)\cos\pi\lambda-e^{-i\pi\lambda}(\sin2\pi\lambda/4\pi\lambda^2)e^{2\lambda\ln\lambda-2\lambda}\times(\tfrac{1}{2}kR)^{-2\lambda}V_0R^2.$$

So again the function is of order $\rho=1$ and type $\tau=\infty$. The asymptotic zero distribution is essentially the same as in the case of a superposition of Yukawa potentials. We, therefore, expect that these results are of rather general validity.

5. THE ZEROS OF $\mathfrak{f}(\lambda,k)$ FOR $k\approx0$

Let us examine $\mathfrak{f}(\lambda,k)$ in the vicinity of $k=0$. We then have

$$H_\lambda^{(2)}(kr)\sim\frac{1}{i\pi\lambda}(\tfrac{1}{2}kr)^{-\lambda}\Gamma(1+\lambda)\left[e^{i\pi\lambda}(\tfrac{1}{2}kr)^{2\lambda}\frac{\Gamma(1-\lambda)}{\Gamma(1+\lambda)}-1\right]$$

and

$$H_\lambda^{(2)}(kr)J_\lambda(kr')\sim\frac{1}{i\pi\lambda}\left(\frac{r'}{r}\right)^\lambda\left[e^{i\pi\lambda}(\tfrac{1}{2}kr)^{2\lambda}\frac{\Gamma(1-\lambda)}{\Gamma(1+\lambda)}-1\right].$$

The "1" in the bracket has been kept, while other terms relatively small as $k\to0$ were discarded, because in the limit as $\lambda\to0$ it is equal to the first term. Insertion of these values into the series expansion for $\mathfrak{f}(\lambda,k)$ yields

$$\mathfrak{f}(\lambda,k)\sim h(\lambda,k,\gamma)\equiv1+\lambda^{-1}[k^{2\lambda}C(\lambda,\gamma)-C(0,\gamma)], \tag{5.1}$$

where

$$C(\lambda,\gamma)=-\tfrac{1}{2}e^{i\pi\lambda}2^{-2\lambda}\frac{\Gamma(1-\lambda)}{\Gamma(1+\lambda)}\left\{\gamma\int_0^\infty dr\,r^{1+2\lambda}V(r)+\gamma^2\int_0^\infty dr\int_0^r dr'\,V(r)V(r')r^{1+\lambda}r'^{\,1+\lambda}\times\left[\left(\frac{r}{r'}\right)^\lambda-\left(\frac{r'}{r}\right)^\lambda\right]\Big/2\lambda+\cdots\right\}, \tag{5.2}$$

and γ is the strength of the potential.

We now look for the zeros of h. Write $z=-\lambda$, $R=-\ln k^2$. Then the zeros of h are the solutions of

$$P(z)\equiv e^{zR}-[z+C(0,\gamma)]/C(-z,\gamma), \tag{5.3}$$

where R is to be large. What are the possible endpoints as $R\to\infty$ (i.e., $E\to0$) of the zero trajectories? There are evidently only two kinds of possible endpoints: (1) The zeros of $C(-z,\gamma)$, with $\mathrm{Re}z>0$ and its poles with $\mathrm{Re}z<0$ (γ is fixed now); (2) the point $z=0$.[21a]

Consider the vicinity of $z=0$. There the function $P(z)$ is approximated by

$$P(z)\sim Q(z)\equiv e^{zR}-1-az, \tag{5.4}$$

where

$$a\equiv\left[1-\frac{\partial C(-z,\gamma)}{\partial z}\bigg|_{z=0}\right]\Big/C(0,\gamma).$$

Write $z=x+iy$, $a=a_1+i\pi$. Then $Q(z)=0$ implies the two equations

$$\begin{aligned}e^{xR}\cos(yR)&=1+a_1x-\pi y,\\ e^{xR}\sin(yR)&=a_1y+\pi x.\end{aligned} \tag{5.5}$$

For each fixed R and a_1 these equations are easily seen to have infinitely many solutions. As $R\to\infty$ each of these solutions moves toward $z=0$, i.e., $\lambda=0$. Now as a function of $k=e^{-\frac{1}{2}R}$, Q has infinitely many sheets; but on the first sheet it has infinitely many zeros as a function of z, which move toward $z=0$ as $k\to0$. The zeros near $z=0$ are (approximately) zeros of P. So we conclude that specifically with k fixed on the first sheet, P has infinitely many zeros, and they all move to $\lambda=0$ as $k\to0$. There are, therefore, infinitely many zeros of $\mathfrak{f}(\lambda,k)$ which, as functions of k on the physical sheet, move toward $\lambda=0$ as $k\to0$. The restriction that forces the zeros of $\mathfrak{f}(\lambda,k)$ for negative k and $\mathrm{Re}\lambda>0$ to lie in

[21a] The Regge poles arising from the poles of C with $\mathrm{Re}z<0$ are the familiar right hand poles associated with bound states and resonances. These have already been discussed in references 1 and 3.

the upper half of the λ plane is not operative when $\mathrm{Re}\lambda<0$. Equation (4.6) of reference 3,

$$\mathrm{Im}\lambda^2\int_0^\infty dr\, r^{-2}|f|^2=-k,$$

which holds when $\mathfrak{f}(\lambda,k)=0$, cannot be analytically continued to $\mathrm{Re}\lambda<0$. Nevertheless, we may expect that for negative k its shadow, so to speak, still falls somewhat across the imaginary λ axis and makes most of the infinitely many zeros that approach $\lambda=0$ from the left as $E\to 0$ do so from *below* (since it is the sign of $\mathrm{Im}\lambda^2$ that is to be positive). Of course, this must not be taken too seriously, since clearly there are zeros which do approach $\lambda=0$ from the above left.

We have so far considered only $E>0$. But when $E<0$ then the only change in the Eq. (5.4) for the zeros of $\mathfrak{f}$ near $\lambda=0$ is that the imaginary part of a is missing. We, therefore, expect that most of the zeros which for $E\to 0$ approach $l=-\frac{1}{2}$, are *not* on the real axis for small *negative* energy.[12a]

It must not be concluded from the foregoing discussion that for fixed k, $\mathfrak{f}(\lambda,k)$ has infinitely many zeros in a neighborhood of $\lambda=0$. For fixed k, $\mathfrak{f}$ is an analytic function of λ at $\lambda=0$; hence, there can be no accumulation point of zeros there. For each fixed k there is only a finite number of zeros near $\lambda=0$, but as $k\to 0$ that number increases without limit; infinitely many zero trajectories arrive at $\lambda=0$ when $k=0$.

Observe what happens to the motion of the zeros when the potential strength is decreased. Since

$$Q=e^{tR/a_1}-1-t-i\pi t/a_1,$$

with $t=za_1$, the trajectories in the t plane are for $t\ll a_1$ independent of a_1, but the larger a_1 (i.e., the smaller the potential strength γ) the larger must R be (i.e., the smaller must E be) in order to get close to the origin. Consequently, in the λ plane the decreasing potential strength γ results (near $\lambda=0$) in a scale reduction of the geometrical shape of the trajectories and, more importantly, in a speeding up of the motion of a zero on its path near $\lambda=0$ as a function of the energy.[22] In a picturesque sort of way, one may say that, as the potential strength decreases, the trajectories near $\lambda=0$ get thinner and thinner.

Now there are other possible zero-energy zeros, as we have seen. These are the roots of $C(\lambda,\gamma)$. In general, we expect that there are infinitely many of them. However, we may consider what happens as the potential strength γ is made smaller and smaller. The zeros of

$$C'(\lambda)\equiv\lim_{\gamma\to 0} C(\lambda,\gamma)/\gamma,$$

are then the possible zero-energy zeros of f in the limit of vanishing potential strength. If $C'(\lambda)$ has *no zeros*, then the zeros of $C(\lambda,\gamma)$ must, in the limit as $\gamma\to 0$, move to infinity, and $\lambda=0$ is the only possible arrival point as $E\to 0$.[11a]

The foregoing general demonstrations may be illustrated in detail by two special examples for which "experimental" evidence is at hand. These are very useful numerical computations performed at Berkeley.

Ahmadzadeh *et al.*[11] have computed trajectories for a single Yukawa potential. Their curves show the behavior to be expected from the above discussion. In the case of a simple Yukawa potential of unit range, it is easy to see that $C'(\lambda)$ has no zeros. We find that

$$C(\lambda,\gamma)/\gamma=-\frac{1}{2}e^{i\pi\lambda}2^{-2\lambda}\frac{\Gamma(1-\lambda)\Gamma(1+2\lambda)}{\Gamma(1+\lambda)}\times\left[1+\frac{\gamma}{2\lambda}(1-2^{-2\lambda})+\cdots\right]. \quad (5.6)$$

For $\gamma\neq 0$ this has infinitely many zeros real as well as complex which move to infinity as $\gamma\to 0$.[22a] Hence, not only are there infinitely many trajectories that approach $\lambda=0$ (i.e., $l=-\frac{1}{2}$) as $E\to 0$, but as the potential strength decreases the other possible $E=0$ poles of S move to infinity. Now if a trajectory, one end of which is attached to a fixed point (a negative half-integer at $E\to\infty$), had its other end ($E=0$) attached to a point that moves to infinity as the strength of the potential decreases, it would have to get longer and longer as the potential gets weaker. The remarkable solution to this quandary was shown in the numerical computations of reference 11 to be that as, in course of the weakening of the potential, two trajectories cross, they may suddenly exchange tails. The pieces from the crossing point to a "$C=0$ zero energy pole" gets handed from one trajectory to another as it moves to $-\infty$ with decreasing potential strength, and no trajectories are forced to be stretched beyond bounds. A look at the curves also shows the expected rapidity with which the poles move near $\lambda=0$, if that is their zero-energy destination. Reference 11 did not examine the negative-energy trajectories off the real axis.

Barut and Calogero[8] have computed a number of curves for square-well potentials. In that case

$$C(\lambda,\gamma)/\gamma=-\tfrac{1}{4}e^{i\pi\lambda}2^{-2\lambda}R^{2+2\lambda}\Gamma(1-\lambda)\times\left[\frac{1}{\Gamma(2+\lambda)}+\frac{\frac{1}{4}\gamma}{\Gamma(3+\gamma)}+\cdots\right].$$

Hence, the "$C=0$ zero energy poles" move to the negative integral values of λ, starting at $\lambda=-2$; i.e., to the negative half-integral values of l, starting at

[22] In the t plane the zero moves essentially as E^{γ}.

[22a] For a weak attractive potential (i.e., for small negative γ) this statement is at once verified by taking the first two terms in the bracket of (5.6). The term outside the bracket does not vanish for any finite value of λ. For $\mathrm{Im}\lambda\to+\infty$, however, $C\to 0$. For $E<0$, where the exponential term is absent, $C\to 0$ for $\mathrm{Im}\lambda\to\pm\infty$.

$l=-5/2$. The larger the value of $-\lambda$, the slower the change with γ from $\gamma=0$, since the $\lambda=-2$ zero is shifted by the second Born approximation, that of $\lambda=-3$, by the third, etc. In addition to these trajectories, there is in the attractive case, of course, always the trajectory which at $E=0$ lies to the *right* of $\lambda=0$ and which moves toward $\lambda=0$ as $\gamma\rightarrow 0$. That was shown in general in references 3 and 4. These facts are all explicitly visible in the curves of Barut and Calogero,[8] including the fact that the trajectory whose $E=0$ position tends toward $l=-3/2$ is missing.[22b] Again the negative-energy trajectories off the real axis were not computed.

It is now possible to understand in general terms how the S matrix approaches unity in the limit as the potential strength γ vanishes, meanwhile always maintaining poles at the negative integral l values (at $E=\infty$) and at $l=-\frac{1}{2}$, as well as at possibly other more or less fixed finite points (at $E=0$). The approach to unity is, of course, highly nonuniform in k and l. The smaller γ, the closer to their $E=\infty$ positions stay the pole trajectories for long ranges of the energy. But at smaller and smaller energies a trajectory may suddenly race toward $l=-\frac{1}{2}$. In other words, the smaller the coupling constant, the flatter are the trajectories' loops near their $E=\infty$ position, and the thinner the tails of those that stretch to $l=-\frac{1}{2}$.

6. S AS AN INFINITE PRODUCT OF ITS POLES

In Sec. 4 we found that both for a superposition of Yukawa potentials (with exponentially decreasing weight factor) and for the square well the asymptotic distribution of the zeros of $\mathfrak{f}(\lambda,k)$ for fixed k is as

$$|\lambda_n|\propto n/\ln n.$$

Consequently for such potentials (and a much wider class as well[23])

$$\sum_{n=1}^{\infty}|\lambda_n|^{-\alpha}<\infty,$$

for all $\alpha>1$. Thus, the genus[18] of the set of zeros of $\mathfrak{f}$ in the λ plane is $p=1$. Furthermore, $\mathfrak{f}_e$ is in these cases an entire function of order $\rho=1$. It can therefore be written in Hadamard's restricted form of the Weierstrass factorization[24]:

$$\mathfrak{f}_e(\lambda,-k)=\mathfrak{f}_e(0,-k)e^{a\lambda}\prod_{n=1}^{\infty}[1-(\lambda/\lambda_n)]e^{\lambda/\lambda_n},\quad(6.1)$$

[22b] If $V\sim r$ at $r=0$ then from (5.2) we find that $\lambda=-1$ is a possible solution of $C(\lambda,\gamma)=0$. In general, if at $r=0$ all the derivatives of rV exist and $V\sim r^m$ then the negative integral values of λ up to m are possible $E=0$ poles. Notice also that for precisely these potentials the negative half-integral values of λ up to $\frac{1}{2}m$ as *absent* as possible $E=\pm\infty$ poles.

[23] It is not hard to extend the investigation of Sec. 4 to cases in which $\sigma(\mu)$ decreases in a way other than exponential; say, like a Gaussian, or like an exponential of a fractional power, etc.

[24] See reference 18, p. 22.

provided that $\mathfrak{f}_e(0,-k)\neq 0$. Here a and all λ_n are, of course, functions of k. The exponential factors cannot be eliminated, since without them the product does not converge. For the S matrix we get

$$S(\lambda,k)=S(0,k)e^{\lambda[a(-k)-a(k)]}\times\prod_{1}^{\infty}\frac{\lambda_n(k)}{\lambda_n(-k)}\frac{\lambda-\lambda_n(-k)}{\lambda-\lambda_n(k)}e^{\lambda[\lambda_n(k)^{-1}-\lambda_n(-k)^{-1}]},\quad(6.2)$$

where $-k\equiv ke^{-i\pi}$. Now, for real k

$$\mathfrak{f}_e^*(\lambda^*,-k)=\mathfrak{f}_e(\lambda,k)$$

and, hence,

$$\lambda_n(-k)=\lambda_n^*(k),\qquad a_n(-k)=a_n^*(k).\quad(6.3)$$

Therefore, for $E>0$

$$S(\lambda,k)=S(0,k)e^{-2i\lambda\,\mathrm{Im}a(k)}\prod_{1}^{\infty}\frac{\lambda_n}{\lambda_n^*}\frac{\lambda-\lambda_n^*}{\lambda-\lambda_n}e^{2i\lambda\,\mathrm{Im}\lambda_n^{-1}}.\quad(6.4)$$

For $E<0$ we take $k=|k|e^{\frac{1}{2}i\pi}$; then

$$\mathfrak{f}(\lambda,-k)=\mathfrak{f}(\lambda,|k|e^{-\frac{1}{2}i\pi})$$

is real for real λ;[25] hence, $a(|k|e^{\frac{1}{2}i\pi})$ is real. The zeros λ_n which are in the left half-plane need not be real, but if they are not then they must occur in complex conjugate pairs.[3] On the other hand,

$$\mathfrak{f}(\lambda,k)=\mathfrak{f}(\lambda,|k|e^{\frac{1}{2}i\pi})$$

need not be real[25] for real λ, except for half-integral λ (i.e., in integral l). Hence, except for integral l, S is in general *not real* when $E<0$.

7. S AS A SERIES OF PARTIAL FRACTIONS

In order to write the S matrix as a series of partial fractions, i.e., a Mittag-Leffler expansion, we examine its residues S_n at the poles λ_n. According to Appendix B of reference 3 we have

$$S_n(k)=(e^{i\pi\lambda_n}k/\lambda_n)\Big/\int_0^{\infty}dr\,r^{-2}f^2(\lambda_n,-k;r).\quad(7.1)$$

The asymptotic behavior of this can be estimated rather simply by means of (4.2). As $\mathrm{Re}\lambda\rightarrow-\infty$, which is the direction in which the distant zeros of $\mathfrak{f}$ lie, we have

$$f^2(\lambda,-k;r)\sim\tfrac{1}{2}\pi ikre^{-i\pi\lambda}[H_\lambda^{(2)}(-kr)]^2\sim-i(\tfrac{1}{2}kr)^{2\lambda}(kr/\lambda)e^{-i\pi\lambda+2\lambda-2\lambda\ln(-\lambda)}.$$

Of course, this cannot be used for the whole integral from zero on, since that converges only because $\mathfrak{f}(\lambda,-k)=0$ and, hence, $f(\lambda,-k;r)$ is proportional to the *regular* solution. Nonetheless, we expect that

$$\ln\int_0^{\infty}dr\,r^{-2}f^2(\lambda,-k;r)\propto-2\lambda\ln(-\lambda),$$

[25] See Appendix.

as Re$\lambda \to -\infty$, and so the residue S_n will decrease very rapidly:

$$\ln S_n = O[2\lambda_n \ln(-\lambda_n)]. \quad (7.2)$$

It must be recognized though, that there may well be exceptions to this generally expected behavior. We cannot rule out the possibility that there are large cancellations in the integral in (7.1) which may make the residue much larger.

Barring such fortuitous cancellations, the series

$$\sum S_n(\lambda-\lambda_n)^{-1}$$

converges extremely rapidly in all the cases examined in Sec. 4. We, therefore, expect that generally the simplest possible Mittag-Leffler expansion of the S matrix should be possible[26]:

$$S(\lambda,k) = P(\lambda,k) + \sum_1^\infty \frac{S_n(k)}{\lambda-\lambda_n(k)}, \quad (7.3)$$

where $P(\lambda,k)$ is for fixed k, an entire function of λ. For large $|\lambda|$, it is P that dominates, in general. But P is expected to be rather smooth and the rapid variation for finite λ should all be contained in the series.

APPENDIX

We collect some relevant remarks here about the branch point of $\mathfrak{f}(\lambda,k)$ at $k=0$. The equation

$$\mathfrak{f}^*(\lambda,k) = \mathfrak{f}(\lambda,-k) \equiv \mathfrak{f}(\lambda, ke^{-i\pi}), \quad (A1)$$

which holds for real k and λ, implies that for *negative* imaginary k, $\mathfrak{f}(\lambda,k)$ is real. But for positive imaginary k that is not necessarily true. It does show that

$$\mathfrak{f}(\lambda,|k|e^{i\frac{1}{2}\pi}) = \mathfrak{f}^*(\lambda,|k|e^{-i\frac{1}{2}\pi}). \quad (A2)$$

[26] See, for example, C. Caratheodory, *Funktionentheorie* (Birkhäuser, Basel, 1950), pp. 215 ff.

The discontinuity of $\mathfrak{f}$ across the cut along the positive imaginary axis (if that is where we choose to put it) is thus purely imaginary. The circuit relation (A8) of reference 3,

$$\mathfrak{f}(\lambda,ke^{-2\pi i}) = -e^{-2\pi i\lambda}\mathfrak{f}(\lambda,k) + (1+e^{-2\pi i\lambda})\mathfrak{f}(\lambda,-k) \quad (A3)$$

shows that for half-integral λ (i.e., integral l) $\mathfrak{f}$ is single valued and thus real on the positive imaginary axis.

Equation (A3) together with (A2) implies that

$$\mathfrak{f}(\lambda,|k|e^{-i\frac{1}{2}\pi}) = \frac{\mathfrak{f}(\lambda,|k|e^{i\frac{1}{2}\pi})e^{-2\pi i\lambda} + \mathfrak{f}^*(\lambda,|k|e^{i\frac{1}{2}\pi})}{1+e^{-2\pi i\lambda}}, \quad (A4)$$

and, hence, for *integral* λ

$$\mathfrak{f}(\lambda,|k|e^{-i\frac{1}{2}\pi}) = \mathrm{Re}\,\mathfrak{f}(\lambda,|k|e^{i\frac{1}{2}\pi}). \quad (A5)$$

Similarly, for integral λ and real k,

$$\mathfrak{f}(\lambda,ke^{-2\pi i}) - \mathfrak{f}(\lambda,k) = -4i\,\mathrm{Im}\,\mathfrak{f}(\lambda,k). \quad (A6)$$

It should also be remembered that $\mathfrak{f}(\lambda,k)$ has been defined so that for Re$\lambda>0$ it is finite at $k=0$. For Re$\lambda<0$, however, it is then *not* finite there but goes as $k^{2\lambda}$.

As for the S matrix, while it is unitary for real λ and k, it is real for real λ and purely imaginary k only when λ is a half-integer (i.e., in the "physical" case). Otherwise, we have

$$S(\lambda,|k|e^{i\frac{1}{2}\pi}) = \frac{\mathfrak{f}(\lambda,|k|e^{i\frac{1}{2}\pi})}{\mathfrak{f}(\lambda,|k|e^{-i\frac{1}{2}\pi})} = \frac{\mathfrak{f}^*(\lambda,|k|e^{-i\frac{1}{2}\pi})}{\mathfrak{f}^*(\lambda,|k|e^{-i\frac{1}{2}\pi})}$$
$$= 1/S^*(\lambda,|k|e^{-i\frac{1}{2}\pi}), \quad (A7)$$

and the combination

$$\mathcal{T}(\lambda,k) \equiv e^{-i\pi(\lambda-\frac{1}{2})}[S(\lambda,k)-1]$$

is such that

$$\mathcal{T}(\lambda,|k|e^{i\frac{1}{2}\pi}) = \mathcal{T}^*(\lambda,|k|e^{-i\frac{1}{2}\pi}). \quad (A8)$$