Two Pomeranchuk-Regge Trajectories*

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The existence of two Pomeranchuk-Regge trajectories is conjectured. It then follows, with a single numerical coincidence, that at high energies the scattering in each angular momentum state is dominated by inelastic processes. This reinstitutes the physical plausible semiclassical explanation of the origin of diffraction scattering. The width of the diffraction peak still shrinks with energy. It also follows that both total and elastic cross sections increase logarithmically, with a limiting ratio of approximately four. The real part of each phase shift approaches an integral multiple of π, providing a basis for a general Levinson's theorem; the approach is from above as $(\ln E)^{-1}$ for increasing energy E. The inelastic scattering in each partial wave approaches its maximum value only as $(\ln E)^{-2}$. Rough quantitative estimates indicate that the typical contribution of two Pomeranchuk trajectories to a total cross section might be quite large (of the order of 66 mb) at 20 BeV.

I. INTRODUCTION

THERE are two conflicting points of view on the mechanism of diffraction scattering. The gross features have been interpreted in terms of a simple semiclassical model based on the assumption that at high energies the scattering in each angular momentum state is predominantly absorptive. Although this approach has the advantage of being physically plausible, the simple method of summing partial waves yields predictions that are contradicted by current experimental measurements.[1] In the alternative approach it is assumed that the high-energy behavior of the elastic scattering amplitude is controlled by Regge poles.[2,3] This has the important advantage that sums and limits can be evaluated precisely in terms of trajectories $\alpha(t)$ of pole singularities in the complex angular momentum plane. The degree to which inelastic processes participate at high energies, as well as the asymptotic behavior of total and differential cross sections and each (complex) phase shift, is predicted in terms of Regge trajectories. Furthermore, the hypothesis of Regge poles is probably consistent with the S-matrix theory and eventually this form of diffraction theory may follow from first principles.

In order to fulfill Pomeranchuk's condition[4] of constant total cross sections at high energies, Chew and Frautschi[3] postulated the existence of a trajectory $\alpha_P(t)$ of even signature which has $\alpha_P(0)=1$, and is called the Pomeranchuk-Regge trajectory.[5] It then follows that the elastic cross section and the absorption in each angular momentum state vanish slowly at high energy. This has the semiclassical interpretation of an interaction region that increases slowly in size, while becoming more transparent. Such a behavior appears to be consistent, but experiment doesn't yet dictate the details of any single trajectory.[6] It is worthwhile, therefore, to point out that even within the framework of Regge poles there is a possibility that at high energies inelastic processes saturate the unitarity condition (maximum absorption in each partial wave). This possibility exists provided that it is consistent for trajectories to cross and, furthermore, provided that two happen to cross $\alpha=1$ at $t=0$. That is, we conjecture the existence of two Pomeranchuk-Regge trajectories $\alpha_P(t)$, $\alpha_F(t)$ having $\alpha_P(0)=\alpha_F(0)=1$. In this note we examine the consequences of such a coincidence.

In a previous letter,[7] it was argued that a constant behavior of the total cross section is inconsistent with the dominance of inelastic processes. It was inferred that both total and elastic cross sections must increase logarithmically with energy, and have a limiting ratio $(\sigma_{\text{tot}}/\sigma_{\text{el}})=4$. The dynamical origin, and in particular the interpretation in terms of singularities in the complex angular momentum plane, was obscure. These same conclusions now follow from the above conjecture. The asymptotic behavior of the real and imaginary parts of each phase shift are evaluated. The results are of importance for partial-wave dispersion relations because they show that the so-called[8] N/D method is susceptible[9] to classical Fredholm theory.

II. FORM OF THE SCATTERING AMPLITUDE

Let t be the momentum transfer invariant and s the square of the barycentric total energy. Let us define $A(s,t)$ to be that part of the elastic scattering amplitude that arises from two Pomeranchuk-Regge trajectories.[5] In order to construct an expression for $A(s,t)$, we consider the partial-wave amplitude $f(J,t)$ of the crossed channel in which t is the square of the total energy and

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J the angular momentum. For fixed t, we display two poles as a function of J,

$$f(J,t)=B(J,t)[J-\alpha(t)]^{-1}[J-\alpha_1(t)]^{-1}, \quad (2.1)$$

and assume that any singularities of B are unimportant for the present discussion. If z is the cosine of the scattering angle in the t channel, the two pole terms in the Somerfeld-Watson transformation are given by

$$f(z,t)=-\frac{\beta(t)P_{\alpha(t)}(-z)}{\Delta\alpha(t)\sin\pi\alpha(t)}+\frac{\beta_1(t)P_{\alpha_1(t)}(-z)}{\Delta\alpha(t)\sin\pi\alpha_1(t)}, \quad (2.2)$$

where $\Delta\alpha(t)=\alpha(t)-\alpha_1(t)$ and, except for trivial factors, $\beta(t)=B(\alpha(t),t)$ and $\beta_1(t)=B(\alpha_1(t),t)$. Since $\alpha(t)$ and $\alpha_1(t)$ are to be Pomeranchuk trajectories, $f(z,t)$ must be symmetrized with respect to z (even signature[5]) and $\alpha(0)=\alpha_1(0)=1$; then $\Delta\alpha(0)=0$ and $\beta(0)=\beta_1(0)$. For fixed t, z is proportional to s and for large s, z is large and negative and the "pole terms" $f(z,t)$ dominate the residual Sommerfeld-Watson integral. As customary, we identify $A(s,t)$ with $f(z,t)$ and assume this $A(s,t)$ is the exact dominant term of the complete elastic amplitude, any "unitarity corrections" being supposedly additive corrections to $A(s,t)$ that have a weaker behavior in s as s increases and t remains finite. The expression for $A(s,t)$ is then

$$A(s,t)=-\frac{\beta(t)[P_{\alpha(t)}(-z)+P_{\alpha(t)}(z)]}{\Delta\alpha(t)\sin\pi\alpha(t)} +\frac{\beta_1(t)[P_{\alpha_1(t)}(-z)+P_{\alpha_1(t)}(z)]}{\Delta\alpha(t)\sin\pi\alpha_1(t)}. \quad (2.3)$$

If we use unequal mass kinematics where m and μ are the two masses in the s channel, we have $z=z(s,t)=(s+p^2+q^2)/(2pq)$, where $p=p(t)=(\frac{1}{4}t-m^2)^{1/2}$ and $q=q(t)=(\frac{1}{4}t-\mu^2)^{1/2}$. For fixed $t<4\mu^2$, we have $pq<0$ and $z\lesseqgtr\mp1$ for $s\gtreqless-(p\pm q)^2$.

Now for α not an integer, $P_\alpha(z)$ has a branch point at $z=-1$ and the cut may be taken along the real axis, $z\leqslant-1$. The discontinuity across the cut is

$$P_\alpha(x-i0)-P_\alpha(x+i0)=-2\pi i\sin\pi\alpha P_\alpha(-x),$$

where x is real, $x<-1$. Thus, as a function of s, for $t<4\mu^2$, $A(s,t)$ has a cut from the terms $P_\alpha(-z)$ in Eq. (2.3) from $s=-(p-q)^2$ to $s=+\infty$. The absorptive part $A_s(s,t)=(1/2i)[A(s+i0,t)-A(s-i0,t)]$ is

$$A_s(s,t)=[\beta(t)P_{\alpha(t)}(-z)-\beta_1(t)P_{\alpha_1(t)}(-z)]/\Delta\alpha(t) \quad (2.4)$$

for $t<0$, $s\geqslant-(p+q)^2$. The forward scattering amplitude is obtained by taking the limit $t\to0$; for the absorptive part we have

$$A_s(s,0)=\beta(0)CP_1(-z)+\beta(0)\frac{d}{d\alpha}P_\alpha(-z)\big|_{\alpha=1}$$
$$=\beta(0)\{-zC-z\ln[\tfrac{1}{2}(1-z)]-1-z\}, \quad (2.5)$$

where[10] $\beta(0)C=[\beta'(0)-\beta_1'(0)][\alpha'(0)-\alpha_1'(0)]^{-1}$. We normalize $A(s,t)$ so that the optical theorem reads $\sigma_{\text{tot}}=4\pi(ks^{1/2})^{-1}A_s(s,0)$, where k is the magnitude of the barycentric three-momentum, $4sk^2=[s-(m+\mu)^2]\times[s-(m-\mu)^2]$. It is useful to use the variables p and E, the momentum and total energy of the particle of mass μ in the laboratory system of the other particle. Then $ks^{1/2}=mp$, $z(s,0)=-E/\mu$ and the contribution to the total cross section from the two Pomeranchuk lines σ_{PF} is

$$\sigma_{PF}(s)=4\pi\beta(0)(m\mu p)^{-1}\times\{CE+E\ln[(E+\mu)/2\mu]+E-\mu\}. \quad (2.6)$$

Thus, $4\pi\beta(0)/m\mu$, estimated to be 10 mb in Sec. III, is the coefficient of the logarithmic increase. The scale factor E_0 in the behavior $\sigma\sim(10\text{ mb})\ln(E/E_0)$, is determined by C, which is roughly the strength of the first-order Regge pole that is concealed under the second-order pole. The rough linear approximation of Sec. III indicates that $C\approx+1$, yielding $\sigma_{PF}(20\text{ BeV})\approx66$ mb and $E_0\approx35$ MeV. This estimate gives only a preliminary orientation and should not be taken too seriously. If it is roughly correct and if there are two Pomeranchuk-Regge trajectories, the contributions to σ_{tot} of the remaining trajectories must be large and negative.

It is useful to have a more explicit expression for the asymptotic behavior of $A(s,t)$. For $\alpha>0$, we have $P_\alpha(z)=D(\alpha)z^\alpha+\mathcal{O}(z^{\alpha-2})$ for large z, where $D(\alpha)=(1/\pi)\times\int_0^\pi d\theta(1+\cos\theta)^\alpha=\pi^{-1/2}2^\alpha\Gamma(\alpha+\frac{1}{2})/\Gamma(\alpha+1)$. It is convenient to introduce a scale factor $\lambda>0$ and define auxiliary coefficients $b(t)$ and $b_1(t)$ by

$$b(t)=\beta(t)D(\alpha(t))(-2pq\lambda)^{-\alpha(t)}t(\Delta\alpha(t))^{-1} \quad (2.7)$$

and a similar expression for $b_1(t)$. After detailed calculations the dependence on λ can be removed with the help of the following formulas:

$$2m\mu\Delta\alpha'(0)\,\lambda b(0)=\beta(0), \quad (2.8)$$

and

$$2m\mu\Delta\alpha'(0)\,\lambda b'(0)$$
$$=\beta'(0)+\beta(0)(1-\ln2)\alpha'(0)-\tfrac{1}{2}\beta(0)\Delta\alpha''(0)[\Delta\alpha'(0)]^{-1}$$
$$+\beta(0)(m^2+\mu^2)(4m^2\mu^2)^{-1}-\beta(0)\alpha'(0)\ln(2m\mu\lambda). \quad (2.9)$$

The asymptotic form of $A(s,t)$ can now be written as

$$A(s,t)=b(t)t^{-1}(s\lambda)^{\alpha(t)}[i-\cot\tfrac{1}{2}\pi\alpha(t)]$$
$$-b_1(t)t^{-1}(s\lambda)^{\alpha_1(t)}[i-\cot\tfrac{1}{2}\pi\alpha_1(t)]+\mathcal{O}(s^{-1}). \quad (2.10)$$

The ratio of real to imaginary parts is interesting. If $\alpha(t)>\alpha_1(t)$ for some small t, the term s^α dominates and one easily sees that

$$\text{Re}A(s,t)/\text{Im}A(s,t)\sim-\cot\tfrac{1}{2}\pi\alpha(t)\approx\tfrac{1}{2}\pi t\alpha'(0).$$

For the forward scattering amplitude we have

$$\text{Im}A(s,0)\sim\lambda s[b'(0)-b_1'(0)+b(0)\Delta\alpha'(0)\ln\lambda s]$$

and

$$\text{Re}A(s,0)\sim\pi\beta(0)s/4m\mu.$$

So the imaginary part dominates only by a logarithmic factor.

III. PARTIAL-WAVE AMPLITUDES

The task of this section is to examine the structure of the angular momentum components of the elastic S-matrix elements. This brings out the limitations imposed by unitarity and presence of absorptive processes. The partial-wave amplitude $A_l(s)$ is related to the complex phase shift $\delta_l(s)$ by

$$A_l(s)=(s^{\frac{1}{2}}/2ik)\{\eta_l(s)\exp[2i\,\mathrm{Re}\delta_l(s)]-1\}, \quad (3.1)$$

where

$$\eta_l(s)=\exp[-2\,\mathrm{Im}\delta_l(s)],\quad 0\leqslant\eta_l\leqslant 1,$$

is the absorption parameter. The assumption that inelastic events dominate each partial wave at large energies implies that for each l, η_l vanishes for large s. This means that any fixed l,

$$\lim_{s\to\infty}\mathrm{Im}A_l(s)=1. \quad (3.2)$$

We now show that the behavior of the two Pomeranchuk poles has a form such that Eq. (3.2) can be satisfied.

The partial-wave amplitude is expressed in terms of $A(s,t)$ by

$$A_l(s)=\frac{1}{4k^2}\int_{-4k^2}^{0}dt\,P_l[1+t(2k^2)^{-1}]A(s,t). \quad (3.3)$$

The part of $A_l(s)$ that is independent of l was identified in reference 7. An argument was given there to show that the asymptotic behavior is independent of the lower limit of integration. Inserting Eq. (2.10) for the asymptotic behavior of $A(s,t)$, we have for large s,[11]

$$\mathrm{Im}A_l(s)=\int_{-s}^{0}dt \times t^{-1}\{b(t)s^{[\alpha(t)-1]}-b_1(t)s^{[\alpha_1(t)-1]}\}+\mathcal{O}(s^{-\nu}), \quad (3.4)$$

where $\nu>0$. It is easy to show that the asymptotic expansion of the integral in Eq. (3.4) has the form

$$\mathrm{Im}A_l(s)=A+(\ln s)^{-1}B+\mathcal{O}(\ln^{-2}s), \quad (3.5)$$

and that A is given by

$$A=b(0)\int_{-\infty}^{0}dt\,t^{-1}\{s^{t\alpha'(0)}-s^{t\alpha_1'(0)}\}$$
$$=b(0)\ln[\alpha'(0)/\alpha_1'(0)]. \quad (3.6)$$

Condition (3.2) can be achieved by requiring the numerical coincidence that A is unity. Using Eq. (2.8), we write this in terms of the original parameters:

$$\beta(0)=2m\mu\Delta\alpha'(0)\{\ln[\alpha'(0)/\alpha_1'(0)]\}^{-1}$$
$$\approx 2m\mu\alpha'(0), \quad (3.7)$$

for small $\Delta\alpha'(0)$. The coefficient B can be evaluated with the help of l'Hospital's rule. We find

$$B=\varphi\int_{-\infty}^{0}dt\,\{b'(0)e^{t\alpha'(0)\varphi}-b_1'(0)e^{t\alpha_1'(0)\varphi}\}$$
$$+\lim_{\varphi\to\infty}-b(0)\varphi^2\frac{d}{d\varphi}\int_{-\infty}^{0}dt\,t^{-1}\{e^{[\alpha(t)-1]\varphi}-e^{[\alpha_1(t)-1]\varphi}\}$$

or

$$B=b'(0)/\alpha'(0)-b_1'(0)/\alpha_1'(0)$$
$$-\tfrac{1}{2}b(0)\{\alpha''(0)[\alpha'(0)]^{-2}-\alpha_1''(0)[\alpha_1'(0)]^{-2}\}.$$

Finally B is expressed in terms of the original parameters by

$$4m\mu\alpha'(0)\alpha_1'(0)B$$
$$=-\beta'(0)-\beta_1'(0)+\beta(0)[\alpha'(0)\alpha_1'(0)]^{-1}$$
$$\times[\alpha''(0)\alpha_1'(0)+\alpha_1''(0)\alpha'(0)]$$
$$+[\alpha'(0)+\alpha_1'(0)]\beta(0)C$$
$$-\beta(0)(m^2+\mu^2)(4m^2\mu^2)^{-1}. \quad (3.8)$$

Before making an estimate of the magnitude of B, let us display its significance. For this we evaluate the leading term in the asymptotic expansion of $\mathrm{Re}A_l(s)$. Using Eqs. (2.10) and (3.3), we have

$$\mathrm{Re}A_l(s)=-\int_{-s}^{0}dt\,t^{-1}\{b(t)\cot\tfrac{1}{2}\pi\alpha(t)s^{[\alpha(t)-1]}$$
$$-b_1(t)\cot\tfrac{1}{2}\pi\alpha_1(t)s^{[\alpha_1(t)-1]}\}+\mathcal{O}(s^{-\nu}), \quad (3.9)$$

where $\nu>0$. Since $\cot\frac{1}{2}\pi\alpha(t)\approx-\frac{1}{2}\pi\alpha'(0)t$, there is no singularity in either part of the integral near $t=0$, and integration by parts suffices; we use[10] $s^\alpha=[\alpha'(t)\ln s]^{-1}\times\{\exp[\alpha(t)\ln s]\}'$. The first term ($\sim\ln^{-1}s$) vanishes and we find

$$\mathrm{Re}A_l(s)=-\tfrac{1}{2}\pi B(\ln s)^{-2}+\mathcal{O}(\ln^{-3}s),$$

where B is the same B as given by Eq. (3.8). Now from Eq. (3.1) we have

$$\tan 2\,\mathrm{Re}\delta_l(s)\sim\tfrac{1}{2}\pi(\ln s)^{-1}$$

or

$$\mathrm{Re}\delta_l(s)=\tfrac{1}{2}n\pi+\tfrac{1}{4}\pi(\ln s)^{-1}+\mathcal{O}(\ln^{-2}s), \quad (3.11)$$

where n is an integer. Hence, $\eta_l(s)$ is given by

$$\eta_l(s)=-B\cos n\pi(\ln s)^{-1}+\mathcal{O}(\ln^{-2}s). \quad (3.12)$$

Since $\eta_l\geqslant 0$, it follows that n is even if B is negative. To estimate B, we use Eq. (3.7) and assume that $\alpha'(0)\approx\alpha_1'(0)$ and $\beta'(0)\approx\beta_1'(0)$. Then B is approximately

$$B\approx\alpha''(0)[\alpha'(0)]^{-2}-(m^2+\mu^2)[8m^2\mu^2\alpha'(0)]^{-1}$$
$$-\beta'(0)[2m\mu\alpha'(0)\alpha_1'(0)]^{-1}+C. \quad (3.13)$$

Now $\alpha(t)$ presumably vanishes near $t=-50\mu^2$, where μ is the pion mass. In order to avoid a ghost, $\beta(-50\mu^2)$ must also vanish. In the linear approximation we, thus, have $\alpha'(0)\approx(50\mu^2)^{-1}$, $\beta'(0)\approx\beta(0)\alpha'(0)$. This gives

[10] Prime denotes differentiation with respect to t.
[11] We set λ equal to unity.

$C\approx+1$. To estimate $\alpha''(0)$, we represent $\alpha(t)$ by a single pole formula with pole at $t=T$, then $\alpha''(0)\approx 2T^{-1}\alpha'(0)$. Another way to estimate $\beta'(0)$ is to suppose (*ad hoc*) that $\beta'(0)\approx 2m\mu\alpha''(0)$, since $\beta(0)=2m\mu\alpha'(0)$. The two estimates agree for $T\approx 100\mu^2$, which seems somewhat small, but is adequate for determining the sign of B. The respective terms in B are

$$B\approx 1-6-1+1\approx -5.$$

There doesn't seem to be any reason to doubt the conclusion that B is negative. Therefore, $\mathrm{Re}\delta_l(s)$ approaches an integral multiple of π, slowly from above, as s tends to infinity. This provides a basis for a generalized Levinson's theorem.[12]

The results of Eqs. (3.11) and (3.12) have an important application to partial-wave dispersion relations. First, arguments of the Pomeranchuk type used by Chew and Mandelstam[8] to infer the asymptotic behavior of the discontinuity across the "left-hand cut" are invalid because they depend on the existence of integrals of the form

$$\int^{\infty} ds\, s^{-1}\eta_l(s),$$

which diverges in our model. In a more detailed analysis,[9] it was shown that the so-called N/D method is susceptible to classical Fredholm theory provided, principally, that the integral

$$\int^{\infty} ds\, s^{-1}\eta_l(s)\sin^2\mathrm{Re}\delta_l(s)$$

exists, which it does in the present theory. The existence of two Pomeranchuk-Regge trajectories, therefore, gives a fairly complete and useful picture of the diffraction limit.

IV. ELASTIC SCATTERING

Further insight into the behavior of two Pomeranchuk trajectories can be achieved by evaluating the two leading terms in the high-energy behavior of the total elastic cross section. First, since the total cross section increases with energy, a theorem of Martin[13] ensures the shrinking of the width of the diffraction peak.

We use Eq. (2.10) for $A(s,t)$. It is easy to see that $\mathrm{Re}\,A(s,t)$ contributes only to order $\ln^{-1}s$ for large s. We neglect terms of this order. Keeping only $\mathrm{Im}A(s,t)$, we have

$$\sigma_{\mathrm{el}}(s)=4\pi s^{-1}\int_{-s}^{0} dt \times t^{-2}\{b(t)s^{\alpha(t)}-b_1(t)s^{\alpha_1(t)}\}^2+\mathcal{O}(\ln^{-1}s). \quad (4.1)$$

[12] N. Levinson, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **25**, 9 (1949); R. Haag, Nuovo Cimento **5**, 203 (1957).

[13] A. Martin, Annual International Conference on High-Energy Physics at CERN, July, 1962 (to be published).

This can be written in the form

$$\sigma_{\mathrm{el}}(s)=4\pi b(0)\{b(0)I(s)+[b'(0)+b_1'(0)]J +[b'(0)-b_1'(0)]K\}+\mathcal{O}(\ln^{-1}s), \quad (4.2)$$

where

$$I(s)=\int_{-\infty}^{0} dt\, t^{-2}\{s^{[\alpha(t)-1]}-s^{[\alpha_1(t)-1]}\}^2, \quad (4.3)$$

$$J=\int_{-\infty}^{0} dt\, t^{-1}\{s^{t\alpha'(0)}-s^{t\alpha_1'(0)}\}^2, \quad (4.4)$$

and

$$K=\int_{-\infty}^{0} dt\, t^{-1}\{s^{2t\alpha'(0)}-s^{2t\alpha_1'(0)}\}. \quad (4.5)$$

The integrals J and K are independent of s and can be evaluated with the change of variables

$$\theta=\exp\{[\alpha'(0)+\alpha_1'(0)]\ln s\}$$

and

$$\xi=\Delta\alpha'(0)/[\alpha'(0)+\alpha_1'(0)].$$

The result is

$$J=\ln(1-\xi^2),$$
$$K=\ln[(1+\xi)/(1-\xi)]=\ln[\alpha'(0)/\alpha_1'(0)].$$

We now construct the asymptotic expansion of $I(s)$, finding

$$I(s)=\ln s I_1+I_2+\mathcal{O}(\ln^{-1}s), \quad (4.6)$$

where

$$I_1=\varphi^{-1}\int_{-\infty}^{0} dt\, t^{-2}\{e^{t\alpha'(0)\varphi}-e^{t\alpha_1'(0)\varphi}\}^2 =[\alpha'(0)+\alpha_1'(0)] \times\{\ln(1-\xi^2)+\xi\ln[(1+\xi)/(1-\xi)]\} \quad (4.7)$$

and where

$$I_2=\lim_{\varphi\to\infty}-\varphi^2\frac{d}{d\varphi}\left(\frac{I[\exp(\varphi)]}{\varphi}\right) =\frac{1}{2}\left(\frac{\alpha''(0)}{\alpha'(0)}+\frac{\alpha_1''(0)}{\alpha_1'(0)}\right)-\frac{\alpha''(0)+\alpha_1''(0)}{\alpha'(0)+\alpha_1'(0)}. \quad (4.8)$$

The final result for $\sigma_{\mathrm{el}}(s)$ can be written in terms of the original parameters β with the help of Eqs. (2.8) and (2.9). We find

$$\sigma_{\mathrm{el}}(s)=4\pi[\beta(0)/2m\mu\Delta\alpha'(0)]^{-2}\{I_1\ln(s/4m\mu)+I_2 +K\Delta\alpha'(0)(1+C)+J[(\beta'(0)+\beta_1'(0))\beta^{-1}(0) +\alpha'(0)+\alpha_1'(0)+(m^2+\mu^2)(2m\mu)^{-1} -\Delta\alpha''(0)/\Delta\alpha'(0)]\}+\mathcal{O}(\ln^{-1}s). \quad (4.9)$$

Therefore, both elastic and total cross sections increase as $\ln s$; the limiting ratio is

$$\lim_{s\to\infty}\frac{\sigma_{\rm tot}(s)}{\sigma_{\rm el}(s)}=2\{1+\ln(1-\xi^2)[\xi\ln((1+\xi)/(1-\xi))]^{-1}\}^{-1}\approx 4(1+\xi^2/12)\quad(4.10)$$

for small ξ. If we again make the approximations employed following Eq. (3.12), the expression for $\sigma_{\rm el}(s)$ becomes

$$\sigma_{\rm el}(s)\approx 2\pi\alpha'(0)\{\ln(s/4m\mu)+1+2C-\beta'(0)/\beta(0)\alpha'(0)-[8\mu^2\alpha'(0)]^{-1}+\Delta\alpha''(0)[2\alpha'(0)\alpha'(0)]^{-1}\}.\quad(4.11)$$

In a preliminary estimate based on a single-pole model for $\alpha(t)$, it seems to us that $\Delta\alpha''(0)/\Delta\alpha'(0)$ should be positive and less than about $(25\mu^2)^{-1}$. With this and previously estimated values, $\sigma_{\rm el}(s)$ becomes

$$\sigma_{\rm el}(s)\approx(2.5\text{ mb})[\ln(E/2\mu)-3],$$

which is positive above $E\approx 6$ BeV. In this rough estimate, the constant terms in $\sigma_{\rm tot}$ and $\sigma_{\rm el}$ are not in the ratio four, and in any case the corrections to Eq. (4.9) are small only of order $\ln^{-1}s$; therefore there is no simple explanation of the empirical ratio $(\sigma_{\rm tot}/\sigma_{\rm el})\approx 4$ at energies 3–30 BeV.

V. FORWARD DISPERSION RELATIONS

The pion-nucleon forward dispersion relation[14,15] can be used to test for the existence of two Pomeranchuk trajectories. Consider the symmetric amplitude defined in terms of the notation of Chew, Goldberger, Low, and Nambu[16] by

$$f^{(+)}(E)=f^{(+)}(-E)=(4\pi)^{-1}[A^{(+)}(s,t=0)+EB^{(+)}(s,t=0)],\quad(5.1)$$

where E is the total energy of the pion in the laboratory system of the nucleon. The usual subtracted dispersion relation can be rewritten in terms of the pion momentum p, $p^2=E^2-\mu^2$, as follows:

$$f^{(+)}(p)=(1+\mu/m)a^{(+)}+f^2p^2m^{-1}(1-\mu^2/4m^2)^{-1}(E^2-\mu^4/4m^2)^{-1}+\frac{p^2}{2\pi^2}\int_0^\infty\frac{\sigma_{\rm tot}{}^{(+)}(p')dp'}{p'^2-p^2},\quad(5.2)$$

where $a^{(+)}$ is the S-wave $(+)$ amplitude scattering length and

$$\sigma_{\rm tot}{}^{(+)}(p)=\tfrac{1}{2}[\sigma_{\rm tot}{}^{\pi^+p}(p)+\sigma_{\rm tot}{}^{\pi^-p}(p)].\quad(5.3)$$

An integration by parts served to extract $\mathrm{Im}f^{(+)}(p)$,

$$\frac{p^2}{2\pi^2}\int_0^\infty\frac{\sigma(p')dp'}{p'^2-p^2}=\frac{ip'\sigma(p)}{4\pi}+\frac{p}{4\pi^2}\int_0^\infty dp'\frac{d\sigma(p')}{dp'}\ln\left|\frac{p'+p}{p'-p}\right|.$$

Thus,

$$\mathrm{Re}f^{(+)}(p)=(1+\mu/m)a^{(+)}+f^2p^2m^{-1}(1-\mu^2/4m^2)^{-1}(E^2-\mu^4/4m^2)^{-1}+\frac{p}{4\pi^2}\int_0^\infty d\sigma_{\rm tot}{}^{(+)}(p')\ln\left|\frac{p'+p}{p'-p}\right|.\quad(5.4)$$

Let us examine the consistency of Eq. (5.4) for large p. For p greater than some value p_0, we suppose that the forward amplitude can be adequately approximated by a few Regge-pole terms: we consider a single or double Pomeranchuk term, a term with $\alpha_2=\alpha_2(t=0)$ that has $0<\alpha_2<1$ and a term with $\alpha_3=\alpha_3(0)<0$. The Regge behavior then gives the following correlation between real and imaginary parts of $f^{(+)}$:[17]

$$\sigma_{\rm tot}{}^{(+)}(p)=\mathcal{A}\ln(p/p_0)+\sigma_P(p_0)+\sigma_2(p_0)(p/p_0)^{\alpha_2-1}+\sigma_3(p_0)(p/p_0)^{\alpha_3-1}\quad(5.5)$$

for $p>p_0$, and

$$\mathrm{Re}f^{(+)}(p)=\tfrac{1}{8}\mathcal{A}|p|-p_0\sigma_2(p_0)(4\pi)^{-1}\times\cot\tfrac{1}{2}\pi\alpha_2(p/p_0)^{\alpha_2}+\mathcal{O}(p^{\alpha_3})\quad(5.6)$$

for large p. If the double Pomeranchuk behavior occurs $\mathcal{A}=8\pi\beta(0)/2m\mu$; otherwise $\mathcal{A}$ vanishes. We now split the integral in Eq. (5.4) into two parts at some $p'=p_1>p_0$, and use Eq. (5.5) for $p'>p_1$. The following behavior for large p can be evaluated with the help of l'Hospital's rule:

$$\frac{\mathcal{A}p}{4\pi^2}\int_{p_1}^\infty\frac{dp'}{p'}\ln\left|\frac{p'+p}{p'-p}\right|=\frac{\mathcal{A}|p|}{8}-\frac{\mathcal{A}p_1}{2\pi^2}+\mathcal{O}(p^{-1}),\quad(5.7)$$

$$\frac{p\sigma_2(p^0)(\alpha_2-1)}{4\pi^2p_0{}^{\alpha_2-1}}\int_{p_1}^\infty dp'\,p'^{\alpha_2-2}\ln\left|\frac{p'+p}{p'-p}\right|=-p_0\sigma_2(p_0)(4\pi)^{-1}\cot\tfrac{1}{2}\pi\alpha_2(p/p_0)^{\alpha_2}+p_0\sigma_2(p_0)(2\pi^2)^{-1}(1-\alpha_2)\alpha_2{}^{-1}(p_1/p_0)^{\alpha_2}+\mathcal{O}(p^{-1})\quad(5.8)$$

for $0<\alpha_2<1$, and

$$\frac{p\sigma_3(p_0)(\alpha_3-1)}{2\pi^2p_0{}^{\alpha_3-1}}\int_{p_1}^\infty dp'\,p'^{\alpha_3-2}\ln\left|\frac{p'+p}{p'-p}\right|=p_0\sigma_3(p_0)(2\pi^2)^{-1}(1-\alpha_3)\alpha_3{}^{-1}(p_1/p_0)^{\alpha_3}+\mathcal{O}(p^{-1})\quad(5.9)$$

[14] T. J. Devlin, B. J. Mayer, and V. Perez-Mendez, Phys. Rev. **125**, 690 (1962), contains numerous references. A misprint occurs in their dispersion relation.

[15] Keiji Igi, Phys. Rev. Letters **9**, 76 (1962).

[16] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957).

[17] The use of the asymptotic form of Legendre functions introduces a small error of order $(\mu/p_0)^2$.

for $\alpha_3<0$. One can see that the components of the right-hand side of Eq. (5.4) that increase with p are given by Eqs. (5.7) through (5.9) and that they agree with Eq. (5.6). The constant components must agree as well; this gives a sum rule,[17]

$$0=(1+\mu/m)a^{(+)}+f^2m^{-1}(1-\mu^2/4m^2)^{-1} +(2\pi^2)^{-1}\Big\{p_1\sigma_{\text{tot}}^{(+)}(p_1)-\int_0^{p_1}dp\,\sigma_{\text{tot}}^{(+)}(p) -\mathcal{a}p_1+p_0\sigma_2(p_0)(\alpha_2^{-1}-1)(p_1/p_0)^{\alpha_2} +p_0\sigma_3(p_0)(\alpha_3^{-1}-1)(p_1/p_0)^{\alpha_3}\Big\}. \quad (5.10)$$

This expression is independent of p_1 for $p_1>p_0$ where Eq. (5.5) applies. The sum rule can be generalized in an obvious way to include several terms of the type that have parameters α_2, $\sigma_2(p_0)$.

We take $p_1=p_0=20$ BeV/c and evaluate the low-energy terms in the sum rule using the same experimental data as Igi.[15] This gives

$$-\mathcal{a}+\sum_j\sigma_j(p_0)(\alpha_j^{-1}-1)=3.2\pm1.6\text{ mb}, \quad (5.11)$$

where $j=2$, 3. If there are two Pomeranchuk trajectories, the theory given in Secs. II and III suggest (using $C=+1$) $\mathcal{a}=10$ mb, $\sum_j\sigma_j(p_0)=-41$ mb. The slope of the total cross section at 20 BeV/c can be used to obtain a second constraint on the parameters; using the experimental data of von Dardel *et al.*,[18] we find

$$\mathcal{a}+\sum_j\sigma_j(p_0)(\alpha_j-1)=p(d\sigma/dp)=-2\pm5\text{ mb} \quad (5.12)$$

at $p=20$ BeV/c. In principle, a third constraint could be obtained from a "p-wave sum rule" for $d\,\mathrm{Re}f^{(+)}(p)/dp^2$, but this is insensitive to the high-energy parameters and already satisfied well within experimental error.[19]

TABLE I. Parameters that satisfy the sum rule Eq. (5.10) for the case of a single Pomeranchuk trajectory, $\mathcal{a}=0$. The α_j are the $t=0$ intercepts of additional vacuum Regge trajectories and σ_j is the contribution of the jth trajectory to the total cross section at 20 BeV/c.

Set	α_2	α_3	σ_2 (mb)	σ_3 (mb)
I	0.3	$\cdots$	2	0
II	0.5	-0.2	3.2 ± 4.6	0 ± 0.5

TABLE II. Parameters that satisfy the sum rule for the case of two Pomeranchuk trajectories. The predicted values $\mathcal{a}=10$ mb, $\sigma_{PP'}$ (20 BeV) $=66$ mb are used.

Set	α_2	α_3	α_4	σ_2 (mb)	σ_3 (mb)	σ_4 (mb)
I	0.3	0.2	-0.2	-300	226	33
II	0.8	0.5	0.3	-131	130	-40
III	0.5	0.1	-0.1	-46	-7.7	12.7
IV	-0.5	0.1	-0.1	-123	27	55

Conditions (5.11) and (5.12) can be satisfied in a natural way if $\mathcal{a}$ vanishes and if there is no restriction on $\sum_j\sigma_j$, as shown in Table I. The predicted values of $\mathcal{a}$ and $\sum_j\sigma_j$ are rather large and give awkward results. Four typical sets of parameters are given in Table II. The contributions of the individual trajectories appear to be too large to be reasonable. Therefore, if there are two Pomeranchuk lines we are forced to one or several of the following conclusions: (i) our estimate of C may be incorrect, (ii) there may be some structure at $\alpha=0$, (iii) the structure at $\alpha=1$ may be even more complicated, or (iv) at 20 BeV/c, we are not yet in an asymptotic region.

VI. CONCLUSIONS

The model of high-energy behavior presented here, based on the assumption of the existence of two Pomeranchuk-Regge trajectories, shows that the Regge-pole hypothesis does not exclude the possibility that at high energies the unitarity condition is saturated by inelastic processes. It is, however, a powerful framework for pinpointing interesting alternatives. The preliminary quantitative estimates given in Sec. V seem to speak against the existence of two Pomeranchuk lines, but a definite conclusion must await a more reliable determination of parameters. Actually, the structure of the diffraction peak might be a good deal more complicated than envisioned here, as indicated in the work of Amati, Fubini, and Stanghellini,[20] and Polkinghorne.[21]

[18] G. von Dardel, D. Dekkers, R. Mermod, M. Vivargent, G. Weber, and K. Winter, Phys. Rev. Letters **8**, 173 (1962).

[19] The constraint on the high-energy parameters reads

$$\mathcal{a}-\sum_j\sigma_j(1-\alpha_j)(2-\alpha_j)^{-1}=0\pm100\text{ mb}.$$

[20] D. Amati, S. Fubini, and A. Stanghellini, Phys. Letters **1**, 29 (1962); also, CERN Report 3755/TH 264, 1962 (unpublished).

[21] J. C. Polkinghorne, Phys. Rev. **128**, 2459 (1962).