Born Cross Sections for Inelastic Scattering of Electrons by Hydrogen Atoms. IV. Approximate Values for Allowed Transitions up to $n=10$*†

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The Born total cross section is calculated for the inelastic scattering of electrons by hydrogen atoms undergoing the transitions: $10s \to 11p$ and $10,9 \to 11,10$. By means of interpolation and extrapolation the cutoff momentum $K_c a_0$ is estimated for all transitions of the type $\Delta n=+1$, $+2$, $\Delta l=+1$, for $n=1$ through 10. These are used to construct tables which provide a simple estimate of the Bethe total cross section.

INTRODUCTION

THIS paper extends the previous calculations[1–3] of the total cross section in the Born approximation for inelastic scattering of electrons by hydrogen atoms to the transitions $n=10$, $l=9 \to n'=11$, $l'=10$ and $n=10$, $l=0 \to n'=11$, $l'=1$. It also contains an estimate of the Bethe (dipole) approximation cross sections for all transitions of the type $\Delta n=+1$, $+2$, $\Delta l=+1$ arising from those states $n=1$ through 10 for which Born cross sections are not available. These approximate cross sections are presented in view of the current need for such cross sections in investigations of the solar chromosphere and thermonuclear and other plasmas. It should be noted that the validity of the Born approximation for transitions between highly excited states and at low incident electron energies has not been established.

FORMULATION

In Paper I the total Born cross section for the transition n, $l \to n'$, l' was given as

$$\sigma(k)=B(ka_0)^{-2}\{a_{-1}I_q^{-1}+a_1I_q^1+a_3I_q^3+\cdots\}\pi a_0^2,$$

where $E=13.605(ka_0)^2$ is the incident electron energy in electron volts and a_0 is the radius of the first Bohr orbit. The constants B and a_i are determined by the particular transition and

$$I_q^r=\int_{\gamma_1}^{\gamma_2}\frac{\gamma^r}{[1+\gamma^2]^q}d\gamma;\quad q=2(n+n'),$$

$$\gamma_1\gamma_2=(n'-n)/(n'+n);\quad ka_0=(\gamma_1+\gamma_2)(n'+n)/2nn'.$$

The above cross section is obtained by averaging over the magnetic quantum number m and summing over m'.

By means of these formulas it is possible to evaluate the total Born cross section for any transition. However, the labor involved increases so rapidly with n that it is desirable to develop simpler, even if more approximate, methods. One such method has been discussed recently by Milford and Scanlon.[4,5]

This paper considers another method of estimating the cutoff momentum for the Bethe (dipole) approximation. The Bethe total cross section for the case $\Delta l=+1$ is given by

$$\sigma^D=\frac{1}{3}\left(\frac{l+1}{2l+1}\right)\left|\frac{I_{n,l\to n',l+1}}{a_0}\right|^2\frac{54.42}{E}\times\ln\left[\frac{54.42(K_ca_0)^2E}{(\Delta E_{nn'})^2}\right]\pi a_0^2,\quad(1)$$

where $|I_{n,l\to n',l+1}/a_0|^2$ are the squares of the radial integrals in the dipole moment matrix elements and are tabulated by Green *et al.*[6]; $\Delta E_{nn'}$ is the change in energy level in electron volts between the states n and n', and $\hbar K_c$ is the cutoff momentum. The quantity K_ca_0 for any given transition is usually determined by equating the Bethe expression to the Born cross section for the transition at the highest energy for which the latter is known.

When the known values[7] of K_ca_0 for the case $\Delta n=+1$ are plotted against n, a definite pattern is revealed (Fig. 1), and it is expected that the pattern will continue to higher values of n, providing an estimate of K_ca_0 for these transitions. In order to improve this estimate the Born cross sections for the transitions $n=10$, $l=0 \to n'=11$, $l'=1$ and $n=10$, $l=9 \to n'=11$, $l'=10$ were calculated. The constants B and a_i for these transitions

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[7] $1 \to 2$ and $1 \to 3$ calculated on the basis of cross sections given by H. S. W. Massey, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1956), Vol. **36**, p. 354; $2 \to 3$ reference 5; $3 \to 4$ and $3 \to 5$ reference 1; $4 \to 5$ and $4 \to 6$ reference 2; $5 \to 6$ reference 3.

TABLE I. Born differential cross section coefficients.[a]

			10s–11p				
B	3.219115(−1)	a_{17}	−2.371797(18)	a_{37}	−1.959354(22)	a_{57}	−3.076656(17)
a_{-1}	1.354414(4)	a_{19}	1.527320(19)	a_{39}	1.621771(22)	a_{59}	2.862436(16)
a_1	−6.046104(6)	a_{21}	−7.781460(19)	a_{41}	−1.106791(22)	a_{61}	−1.974931(15)
a_3	9.782609(8)	a_{23}	3.171434(20)	a_{43}	6.216096(21)	a_{63}	9.720597(13)
a_5	−7.446208(10)	a_{25}	−1.043077(21)	a_{45}	−2.864655(21)	a_{65}	−3.257791(12)
a_7	3.279561(12)	a_{27}	2.787756(21)	a_{47}	1.078847(21)	a_{67}	7.177471(10)
a_9	−9.365089(13)	a_{29}	−6.087329(21)	a_{49}	−3.302542(20)	a_{69}	−9.491418(8)
a_{11}	1.855909(15)	a_{31}	1.090499(22)	a_{51}	8.160921(19)	a_{71}	6.067548(6)
a_{13}	−2.668819(16)	a_{33}	−1.607483(22)	a_{53}	−1.613632(19)		
a_{15}	2.873785(17)	a_{35}	1.953599(22)	a_{55}	2.524081(18)		
			10,9–11,10				
B	1.038006(−1)	a_7	4.372861(10)	a_{17}	−7.126736(12)	a_{27}	7.590207(12)
a_{-1}	1.482250(5)	a_9	−2.545677(11)	a_{19}	8.061616(12)	a_{29}	−3.699650(12)
a_1	−1.013859(7)	a_{11}	9.877582(11)	a_{21}	−7.401545(12)	a_{31}	1.087545(13)
a_3	3.045321(8)	a_{13}	−2.636172(12)	a_{23}	7.228830(12)	a_{33}	−1.698126(12)
a_5	−4.738796(9)	a_{15}	5.050873(12)	a_{25}	−5.440500(12)	a_{35}	2.329609(13)

[a] The numbers in parentheses represent powers of 10.

are given in Table I. They are accurate to within an error of 1 in the last place. In computing the integrals $I_q^r(r>0)$ the following limit was employed:

$$\int_{\gamma_1}^{\gamma_2} \frac{\gamma^r}{(1+\gamma^2)^q} d\gamma \underset{E\to\infty}{\longrightarrow} \int_0^\infty \frac{\gamma^r}{(1+\gamma^2)^q} d\gamma = \frac{\Gamma\left(\frac{r+1}{2}\right)\Gamma\left(q-\frac{r+1}{2}\right)}{2\Gamma(q)}.$$

The use of this approximation is justified by the fact that for sufficiently high energies these integrals are very insensitive to further increase in the energy. The observation was made in reference 2 that for $r\geq 3$ this approximation was in agreement with accurate results to at least two significant figures at energies as low as 10 eV for the 4–5 and 4–6 transitions. Also, the use of this approximation here is consistent with the fact that, for the determination of $K_c a_0$, only a high-energy Born cross section is required. The integral I_q^{-1} was evaluated

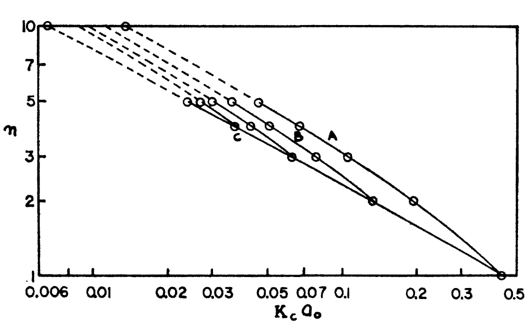

FIG. 1. The cutoff momentum $K_c a_0$ plotted against n for transitions of the type $\Delta n=+1$, $\Delta l=+1$. The circled points represent values of $K_c a_0$ based on known values of the Born cross section. A, $l=n-1 \rightarrow l'=n$; B, $l=n-2 \rightarrow l'=n-1$; C, $l=0 \rightarrow l'=1$.

in the usual manner and is the only integral which is energy-dependent in this approximation.

It is sometimes convenient to express the Bethe cross section (1) in terms of the oscillator strength f:

$$\sigma^D = \frac{740.38}{E(\Delta E_{nn'})} f_{if} \ln\left[\frac{54.42(K_c a_0)^2 E}{(\Delta E_{nn'})^2}\right] \pi a_0^2, \qquad (2)$$

$$f_{if} = (\Delta E_{nn'}/13.605)\,|\chi_{if}^{(1)}/a_0|^2,$$

$$\chi_{if}^{(1)} = \int \psi_i(\mathbf{r}) Z \psi_f^*(\mathbf{r}) d^3r,$$

where the oscillator strength f_{if} has been averaged over the initial magnetic quantum number m, and the subscripts i, f denote the initial and final states, respectively.

RESULTS

The total Born cross sections and corresponding cutoff momenta for the transitions $n=10$, $l=0 \rightarrow n'=11$, $l'=1$ and $n=10$, $l=9 \rightarrow n'=11$, $l'=10$ are listed in Table II. These numbers should be accurate to at least two significant figures.

When this information is added to Fig. 1, it is possible to complete the pattern as shown by the broken lines. The portion of Fig. 1 corresponding to $n=5$ through 10 has been magnified in Fig. 2 for purposes of clarity. It

TABLE II. Born total cross sections and momentum cutoff values for 10–11 transitions.

	E (eV)	$\sigma/\pi a_0^2$	$K_c a_0$
10s–11p	1124.4	223.0	6.5×10^{-3}
	17990.	18.51	6.5×10^{-3}
10,9–11,10	1124.4	923.5	0.0136
	17990.	73.85	0.0136

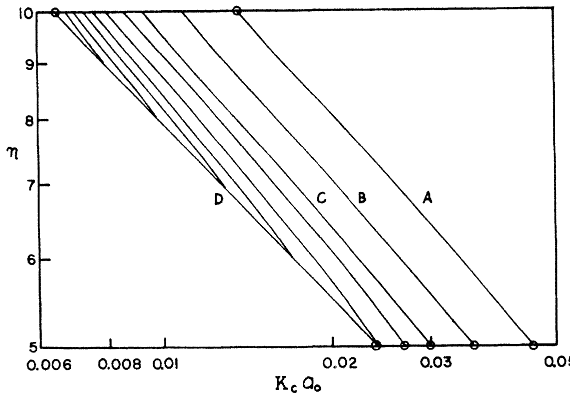

FIG. 2. The cutoff momentum $K_c a_0$ plotted against n for transitions of the type $\Delta n=+1$, $\Delta l=+1$. This figure is an enlarged version of the upper left-hand portion of Fig. 1. A, $l=n-1 \rightarrow l'=n$; B, $l=n-2 \rightarrow l'=n-1$; C, $l=n-3 \rightarrow l'=n-2$; D, $l=0 \rightarrow l'=1$.

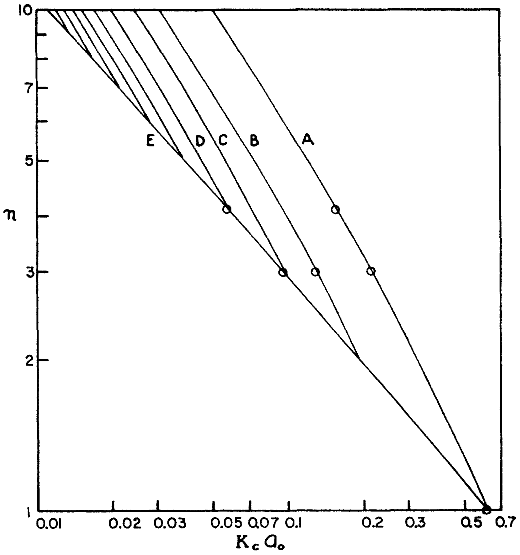

FIG. 3. The cutoff momentum $K_c a_0$ plotted against n for transitions of the type $\Delta n=+2$, $\Delta l=+1$. The circled points represent values of $K_c a_0$ based on known values of the Born cross section. A, $l=n-1 \rightarrow l'=n$; B, $l=n-2 \rightarrow l'=n-1$; C, $l=n-3 \rightarrow l'=n-2$; D, $l=n-4 \rightarrow l'=n-3$; E, $l=0 \rightarrow l'=1$.

is clear that for a given value of $(n,\Delta n)$, there must be n values of $K_c a_0$ which in descending order of magnitude correspond respectively to the initial states: $l=n-1$, $n-2 \cdots 1,0$. In addition, values of $K_c a_0$ corresponding to all transitions of the type $\Delta n=+2$, $\Delta l=+1$ for which Born cross sections have been calculated are plotted against n in Fig. 3; the available information in Fig. 3 is consistent with the pattern of Fig. 1. Figure 3 has been completed by extrapolation on the assumption that the case $\Delta n=+2$ has qualitatively the same behavior as the case $\Delta n=+1$.

These graphs then provide a simple method of estimating the cutoff momentum for certain transitions.

TABLE III. Values of the momentum cutoff, $K_c a_0$.[a]

$\Delta n=+1$: $l-l'$ \ $n-n'$	1–2	2–3	3–4	4–5	5–6	6–7	7–8	8–9	9–10	10–11
0–1	4.43(−1)	1.331(−1)	6.31(−2)	3.70(−2)	2.4(−2)	1.7(−2)	1.3(−2)	1.0(−2)	8(−3)	6.5(−3)
1–2		1.951(−1)	7.82(−2)	4.32(−2)	2.7(−2)	1.9(−2)	1.4(−2)	1.0(−2)	8(−3)	7(−3)
2–3			1.059(−1)	5.08(−2)	3.0(−2)	2(−2)	1.5(−2)	1.1(−2)	9(−3)	7(−3)
3–4				6.68(−2)	3.60(−2)	2(−2)	1.6(−2)	1.2(−2)	9(−3)	7(−3)
4–5					4.61(−2)	3(−2)	1.7(−2)	1.3(−2)	1.0(−2)	8(−3)
5–6						3(−2)	2(−2)	1.4(−2)	1.0(−2)	8(−3)
6–7							3(−2)	1.6(−2)	1.1(−2)	9(−3)
7–8								2(−2)	1.3(−2)	9(−3)
8–9									1.6(−2)	1.1(−2)
9–10										1.36(−2)
Approx.[b]	5.6(−1)	1.91(−1)	9.6(−2)	5.8(−2)	3.9(−2)	2.8(−2)	2.1(−2)	1.62(−2)	1.29(−2)	1.06(−2)

$\Delta n=+2$: $l-l'$ \ $n-n'$	1–3	2–4	3–5	4–6	5–7	6–8	7–9	8–10	9–11	10–12
0–1	6.16(−1)	1.9(−1)	9.48(−2)	5.76(−2)	4(−2)	3(−2)	2(−2)	1.7(−2)	1.3(−2)	1.1(−2)
1–2		3(−1)	1.292(−1)	7(−2)	5(−2)	3(−2)	2(−2)	1.8(−2)	1.5(−2)	1.2(−2)
2–3			2.15(−1)	9(−2)	6(−2)	4(−2)	3(−2)	2(−2)	1.6(−2)	1.3(−2)
3–4				1.552(−1)	7(−2)	5(−2)	3(−2)	2(−2)	1.7(−2)	1.4(−2)
4–5					1.2(−1)	6(−2)	4(−2)	3(−2)	1.9(−2)	1.5(−2)
5–6						1.0(−1)	5(−2)	3(−2)	2(−2)	1.7(−2)
6–7							8(−2)	4(−2)	3(−2)	2(−2)
7–8								7(−2)	4(−2)	2(−2)
8–9									6(−2)	3(−2)
9–10										5(−2)
Approx.[b]	7.0(−1)	2.8(−1)	1.52(−1)	9.6(−2)	6.6(−2)	4.8(−2)	3.7(−2)	2.9(−2)	2.3(−2)	1.94(−2)

[a] The numbers in parentheses represent powers of 10.
[b] See reference 4.

TABLE IV. Values of $D=54.42(K_ca_0)^2/(\Delta E_{nn'})^2$.[a]

					$\Delta n=+1$					
$l-l'$ \ $n-n'$	1–2	2–3	3–4	4–5	5–6	6–7	7–8	8–9	9–10	10–11
0–1	1.03(−1)	2.7(−1)	5.0(−1)	8.0(−1)	1.14	1.6	2	3	3	4
1–2		5.8(−1)	7.6(−1)	1.08	1.44	1.9	3	3	3	5
2–3			1.39	1.50	1.77	2	3	3	4	5
3–4				2.6	2.6	2	4	4	4	5
4–5					4.2	5	4	5	5	6
5–6						5	5	5	5	6
6–7							1.2(1)	8	6	8
7–8								1.1(1)	9	8
8–9									1.6(1)	1.2(1)
9–10										1.8(1)

					$\Delta n=+2$					
$l-l'$ \ $n-n'$	1–3	2–4	3–5	4–6	5–7	6–8	7–9	8–10	9–11	10–12
0–1	1.41(−1)	3(−1)	5.2(−1)	8.1(−1)	1.2	1.8	1.8	3	3	4
1–2		8(−1)	9.7(−1)	1.2	2	1.8	1.8	3	4	4
2–3			2.7	2	3	3	4	4	5	5
3–4				5.9	4	5	4	4	5	6
4–5					1.1(1)	8	7	8	7	6
5–6						2(1)	1.3(1)	8	7	9
6–7							3(1)	1.5(1)	1.6(1)	1.3(1)
7–8								5(1)	3(1)	1.3(1)
8–9									7(1)	3(1)
9–10										9(1)

[a] The numbers in parentheses represent powers of 10.

These estimated values of K_ca_0 and all other known values[1–5] are listed in Table III. (The values of K_ca_0 which are estimated are given to only one significant figure.) The estimates for the case $\Delta n=+1$ are more reliable than those for $\Delta n=+2$ since the former are based on more complete data. Other than this no measure of accuracy can be given. In the last line of each table are listed the corresponding values of the cutoff momenta as determined by the approximate formula suggested in reference 4. This formula depends only on the values of n and n'. Note that for all values of n and n' except $n=1$ this simple formula predicts a

TABLE V. Values of $C=18.14[(l+1)/(2l+1)]|I/a_0|^2$.[a]

					$\Delta n=+1$					
$l-l'$ \ $n-n'$	1–2	2–3	3–4	4–5	5–6	6–7	7–8	8–9	9–10	10–11
0–1	3.0(1)	1.70(2)	5.4(2)	1.32(3)	2.7(3)	5.0(3)	8.4(3)	1.34(4)	2.0(4)	3.0(4)
1–2		2.7(2)	6.9(2)	1.47(3)	2.8(3)	4.8(3)	7.8(3)	1.20(4)	1.76(4)	2.5(4)
2–3			1.14(3)	2.2(3)	3.8(3)	6.2(3)	9.5(3)	1.42(4)	2.0(4)	2.8(4)
3–4				3.3(3)	5.3(3)	8.2(3)	1.22(4)	1.76(4)	2.5(4)	3.4(4)
4–5					7.5(3)	1.10(4)	1.58(4)	2.2(4)	3.0(4)	4.1(4)
5–6						1.48(4)	2.1(4)	2.8(4)	3.7(4)	4.9(4)
6–7							2.7(4)	3.5(4)	4.6(4)	5.9(4)
7–8								4.4(4)	5.7(4)	7.2(4)
8–9									7.0(4)	8.7(4)
9–10										1.05(5)

					$\Delta n=+2$					
$l-l'$ \ $n-n'$	1–3	2–4	3–5	4–6	5–7	6–8	7–9	8–10	9–11	10–12
0–1	4.8	3.0(1)	9.3(1)	2.2(2)	4.3(2)	7.7(2)	1.27(3)	1.98(3)	3.0(3)	4.2(3)
1–2		3.5(1)	1.07(2)	2.3(2)	4.4(2)	7.4(2)	1.18(3)	1.79(3)	2.6(3)	3.6(3)
2–3			1.20(2)	2.9(2)	5.5(2)	9.1(2)	1.42(3)	2.1(3)	3.0(3)	4.1(3)
3–4				2.9(2)	6.4(2)	1.10(3)	1.71(3)	2.5(3)	3.5(3)	4.8(3)
4–5					5.6(2)	1.19(3)	1.96(3)	2.9(3)	4.1(3)	5.6(3)
5–6						9.7(2)	2.0(3)	3.2(3)	4.7(3)	6.4(3)
6–7							1.54(3)	3.1(3)	4.9(3)	7.0(3)
7–8								2.3(3)	4.7(3)	7.2(3)
8–9									3.3(3)	6.6(3)
9–10										4.5(3)

[a] The numbers in parentheses represent powers of 10.

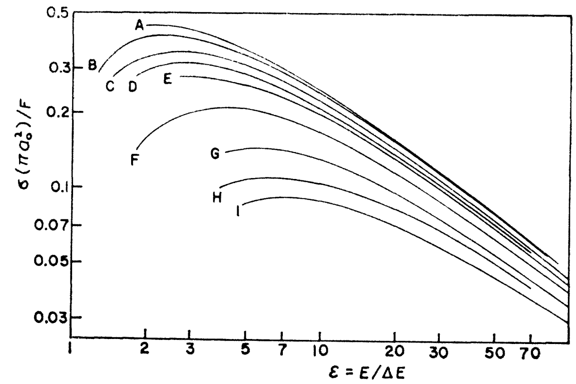

FIG. 4. Normalized Born cross section plotted against the energy expressed in threshold energy units. A, $1s$–$2p$; B, $2p$–$3d$; C, $3d$–$4f$; D, $4f$–$5g$; E, $5g$–$6h$; F, $2s$–$3p$; G, $3s$–$4p$; H, $4s$–$5p$; I, $5s$–$6p$.

value of $K_c a_0$ which lies within the range of values given by Figs. 1, 2, and 3 for the same n and n'.

The constants

$$C = 18.14\left(\frac{l+1}{2l+1}\right)\left|\frac{I}{a_0}\right|^2 \quad \text{and} \quad D = \frac{54.42(K_c a_0)^2}{(\Delta E_{nn'})^2}$$

are required in the Bethe cross section:

$$\sigma^D = (C/E)\ln(DE)\pi a_0^2. \tag{3}$$

For convenience the constants have been calculated from Table III and reference 6 and are listed in Tables IV and V. Since earlier work[1–3] has shown that, with accurate $K_c a_0$, the Bethe fits the Born cross section to within 10% down to about 15 times threshold energy, the use of the constants C,D in the Bethe formula (3) will give a reasonable estimate of the Born cross section down to very low energies for the large-n transitions.

The strong similarity among all transitions of the type $\Delta n = +1$, $\Delta l = +1$ is displayed in Fig. 4. All $s \to p$ and $l = n-1 \to l' = n'-1$ hydrogen transitions for which Born cross sections are known have been plotted here. The Born cross sections have been normalized by division by

$$F = \frac{C}{\Delta E_{nn'}} = \frac{18.14}{\Delta E_{nn'}}\left(\frac{l+1}{2l+1}\right)\left|\frac{I}{a_0}\right|^2.$$

When the cross sections are plotted against threshold energy units, $\epsilon = E/\Delta E_{nn'}$ as in Fig. 4, this normalization has the effect of equating all cross sections at $\epsilon = \infty$. The use of threshold energy units also implies that all these curves must asymptotically approach the line $\epsilon = 1$.

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