Non-Abelian Gauge Fields. Lorentz Gauge Formulation

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Non-Abelian vector gauge theory is given a first-order Lorentz gauge formulation and then transformed into the radiation gauge. The result agrees with the independently constructed radiation gauge theory. There is a brief discussion of the axial gauge.

THE major purpose of this note is to prove the formal equivalence between a manifestly covariant Lorentz gauge formulation of non-Abelian vector gauge field theory and the independently devised radiation gauge formulation.[1] The Lorentz gauge version is analogous to that introduced by Fermi for the electromagnetic field, in which a supplementary condition on states is used.

LORENTZ GAUGE

Let us consider the following scalar Lagrange function:

$$\mathcal{L}=-\tfrac{1}{2}G^{\mu\nu}.[\partial_\mu\phi_\nu-\partial_\nu\phi_\mu+i(\phi_\mu t\phi_\nu)] +\tfrac{1}{4}f^2G^{\mu\nu}G_{\mu\nu}-G\partial_\mu\phi^\mu+\phi_\mu k^\mu+\mathcal{L}(\psi),$$

where

$$\mathcal{L}(\psi)=\tfrac{1}{2}i\psi.(\alpha^\mu\partial_\mu+\beta m)\psi$$

and

$$k_a{}^\mu(x)=\tfrac{1}{2}\psi(x)\alpha^\mu T_a\psi(x)$$

refer to a Dirac field. The notational conventions of reference 1 are employed. The response of this Lagrange function to the numerical infinitesimal gauge transformation

$$G_{\mu\nu}\rightarrow(1+i\text{‘}t\delta\lambda\text{’})G_{\mu\nu},\qquad G\rightarrow(1+i\text{‘}t\delta\lambda\text{’})G,$$
$$\phi_\mu\rightarrow(1+i\text{‘}t\delta\lambda\text{’})\phi_\mu+\partial_\mu\delta\lambda,\quad \psi\rightarrow(1+i\text{‘}T\delta\lambda\text{’})\psi,$$

is given by

$$\mathcal{L}\rightarrow\mathcal{L}-G(\partial_\mu-i\text{‘}t\phi_\mu\text{’})\partial^\mu\delta\lambda.$$

One should resist the impulse to conclude that the Lagrange function would be invariant were the infinitesimal gauge function to obey

$$(\partial_\mu-i\text{‘}t\phi_\mu\text{’})\partial^\mu\delta\lambda=0,$$

for this could not be a numerical gauge transformation. The Dirac part of the Lagrange function is gauge invariant, which implies the differential conservation law

$$(\partial_\mu-i\text{‘}t\phi_\mu\text{’})k^\mu=0.$$

The field equations derived from the action principle are

$$\partial_\mu\phi_\nu-\partial_\nu\phi_\mu+i(\phi_\mu t\phi_\nu)=f^2G_{\mu\nu},$$
$$(\partial_\nu-i\text{‘}t\phi_\nu\text{’}).G^{\mu\nu}-\partial^\mu G=k^\mu,$$
$$\partial_\mu\phi^\mu=0,$$

and

$$[\alpha^\mu(\partial_\mu-i\text{‘}T\phi_\mu\text{’})+\beta m]\psi=0.$$

Note that the Lorentz condition $\partial_\mu\phi^\mu=0$ is an operator equation. This is no source of difficulty in a theory based on first-order differential equations, as contrasted with the more usual procedure employing second-order differential equations.[2] Apart from the explicit construction of G_{kl} in terms of ϕ_k and ϕ_l, all the field equations are equations of motion. This is emphasized by the structure of the time derivative term in $\mathcal{L}$,

$$-G^{0k}\partial_0\phi_k-G\partial_0\phi^0+\tfrac{1}{2}i\psi\partial_0\psi,$$

which also exhibits the pairs of complementary canonical variables. The nonvanishing equal time commutators are

$$i[\phi_{ka}(x),G_b{}^{0l}(x')]=\delta_{ab}\delta_k{}^l\delta(\mathbf{x}-\mathbf{x}'),$$
$$i[\phi_a{}^0(x),G_b(x')]=\delta_{ab}\delta(\mathbf{x}-\mathbf{x}'),$$

while

$$\{\psi_\alpha(x),\psi_\beta(x')\}=\delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{x}').$$

Infinitesimal numerical gauge transformations are generated by the operator

$$G_{\delta\lambda}=\int(d\mathbf{x})[G\partial_0\delta\lambda-\delta\lambda\partial_0G-Gi\text{‘}t\phi_0\text{’}\delta\lambda]$$

in the sense illustrated by

$$-i[\phi_\mu,G_{\delta\lambda}]=(\partial_\mu-i\text{‘}t\phi_\mu\text{’})\delta\lambda,\quad -i[\psi,G_{\delta\lambda}]=i\text{‘}T\delta\lambda\text{’}\psi.$$

The composition law of successive infinitesimal gauge transformations is expressed by the group commutation property

$$-i[G_{\delta\lambda_1},G_{\delta\lambda_2}]=G_{\delta\lambda_{12}},$$

where

$$\delta\lambda_{12}=-i(\delta\lambda_1 t\delta\lambda_2).$$

The following equal time commutators can be regarded as specific implications of this group property:

$$[G(x),G(x')]=[G(x),\partial_0G(x')]=0,$$

and

$$[\partial_0G(x),\partial_0G(x')]=-\text{‘}t\partial_0G(x)\text{’}\delta(\mathbf{x}-\mathbf{x}').$$

A particular solution of the commutation equations obeyed by the gauge group generators is $G_{\delta\lambda}'=0$, for all $\delta\lambda$. Hence, there should exist special gauge invariant

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[1] J. Schwinger, Phys. Rev. **125**, 1043 (1962); **127**, 324 (1962).
[2] C. N. Yang and R. Mills, Phys. Rev. **96**, 191 (1954).

states that obey

$$G_{\delta\lambda}\Psi=0$$

or

$$G(x)\Psi=0, \quad \partial_0 G(x)\Psi=0.$$

Furthermore, this property is not confined to one time. It is a consequence of the extended current conservation law and the field equations that

$$(\partial_\mu - i^\epsilon t\phi_\mu{}')\partial^\mu G(x)=0.$$

Accordingly, the second, and every higher time derivative of $G(x)$ also vanishes when applied to a gauge invariant state, and the latter are characterized by the eigenvector equations

$$G(x)\Psi=0$$

for all space-time points x. It is these gauge invariant states with which we are concerned. Note, incidentally, that if Ψ is a gauge-invariant state so also is $F\Psi$, where F is a gauge-invariant operator, for

$$G_{\delta\lambda}F\Psi=[G_{\delta\lambda},F]\Psi=0.$$

The energy density operator of the system is given by

$$\Theta^{00}=T^{00}-\phi^0[(\partial_k-i^\epsilon t\phi_k{}')G^{0k}-k^0]-\phi^k\partial_k G,$$

in which $T^{00}(x)$ is the gauge invariant operator

$$T^{00}=\tfrac{1}{2}f^2[(G^{0k})^2+\tfrac{1}{2}(G_{kl})^2]-\tfrac{1}{2}i\psi.[\alpha^k(\partial_k-i^\epsilon T\phi_k{}')+\beta m]\psi.$$

Similarly, the momentum density is

$$\Theta^0{}_k=T^0{}_k-\phi_k[(\partial_l-i^\epsilon t\phi_l{}')G^{0l}-k^0]-\phi^0\partial_k G,$$

where

$$T^0{}_k=f^2G^{0l}.G_{kl}-\tfrac{1}{2}i\psi.(\partial_k-i^\epsilon T\phi_k{}')\psi+\tfrac{1}{2}\partial^l\tfrac{1}{4}\psi\sigma_{kl}\psi$$

is gauge invariant. It is not difficult to verify the fundamental equal-time commutators

$$-i[\Theta^{00}(x),\Theta^{00}(x')]=-(\Theta^{0k}(x)+\Theta^{0k}(x'))\partial_k\delta(\mathbf{x}-\mathbf{x}')$$

and

$$-i[T^{00}(x),T^{00}(x')]=-(T^{0k}(x)+T^{0k}(x'))\partial_k\delta(\mathbf{x}-\mathbf{x}').$$

The latter is the reduced version of the Θ^{00} commutation relation for gauge invariant states. This follows from

$$(\Theta^{00}-T^{00})\Psi=0, \quad (\Theta^0{}_k-T^0{}_k)\Psi=0$$

and the remark that $T^{00}\Psi$ is also a gauge invariant state, so that

$$\Theta^{00}(x)\Theta^{00}(x')\Psi=T^{00}(x)T^{00}(x')\Psi.$$

The equation of motion and Lorentz transformation properties of any gauge invariant operator F can be calculated from T^{00}, for gauge invariant states, since

$$[F,\Theta^{00}]\Psi=[F,T^{00}]\Psi.$$

This remark does not apply to any field operator, however, since none of these is gauge invariant, in a non-Abelian gauge theory.

RADIATION GAUGE

The energy operator constructed from Θ^{00} is a linear functional of the complementary field variables G and ϕ^0. If no restrictions are imposed on the vector space, the energy spectrum ranges continuously from $+\infty$ to $-\infty$, since these variables can be subjected to arbitrary linear displacements. That freedom of translation must be suppressed to form a subspace of physically admissible states, and this is accomplished by considering only the gauge-invariant states. But such states, as eigenvectors of operators with continuous spectra, have no finite norm. Only by eliminating the field variables that are superfluous in the physical subspace can vectors of finite norm be obtained.

It is convenient for that purpose to decompose the complementary fields $\phi_k(x)$ and $G^{0k}(x)$ into longitudinal and transverse parts, as in

$$\phi_k(x)=\phi_k(x)^L+\phi_k(x)^T, \quad \phi_k(x)^L=\partial_k\lambda(x), \quad \partial_k\phi^k(x)^T=0.$$

The canonical commutators decompose correspondingly,

$$i[\phi_k(x)^L,G^{0l}(x')^L]=(\delta_k{}^l\delta(\mathbf{x}-\mathbf{x}'))^L=\partial_k\partial^{l'}(4\pi|\mathbf{x}-\mathbf{x}'|)^{-1},$$

$$i[\phi_k(x)^T,G^{0l}(x')^T]=(\delta_k{}^l\delta(\mathbf{x}-\mathbf{x}'))^T$$
$$=\delta_k{}^l\delta(\mathbf{x}-\mathbf{x}')-\partial_k\partial^{l'}(4\pi|\mathbf{x}-\mathbf{x}'|)^{-1}.$$

We also adopt a partial representation of the gauge invariant states, which is labeled at a particular time by the eigenvalues of $\phi^0(x)$ and $\phi_k(x)^L$. The complementary variables are represented by three-dimensional functional differential operators

$$G_a(x)\rightarrow i(\delta_3/\delta\phi_a{}^0(x)'), \quad G_a{}^{0k}(x)^L\rightarrow i(\delta_3/\delta\phi_{ka}(x)^{L'}).$$

Thus, the supplementary condition $G(x)\Psi=0$ becomes the wave functional equation

$$(\delta_3/\delta\phi_a{}^0(x)')\Psi=0,$$

and Ψ is independent of the eigenvalues $\phi_a{}^0(x)'$.

The introduction of three-dimensional vector notation ($G^{0k}\rightarrow\mathbf{G}$) permits us to write $\partial_0 G$ as

$$-(\nabla-i^\epsilon t\boldsymbol{\phi}')\cdot\mathbf{G}+k^0=(\nabla-i^\epsilon t\boldsymbol{\phi}')\cdot\mathbf{L},$$

where $\mathbf{L}(x)$ is a longitudinal vector,

$$\mathbf{L}=-\mathbf{G}^L+\nabla\mathfrak{D}_\phi((\nabla-i^\epsilon t\boldsymbol{\phi}')\cdot\mathbf{G}^T-k^0).$$

This is presented in a symbolic notation, with $\mathfrak{D}_\phi$ defined by

$$-(\nabla-i^\epsilon t\boldsymbol{\phi}(x)')\cdot\nabla\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}')=\delta(\mathbf{x}-\mathbf{x}').$$

It is taken for granted in this work that $\mathfrak{D}_\phi$ exists and is unique. With the aid of the equal-time commutator

$$-i[(\nabla-i^\epsilon t\boldsymbol{\phi}(x)')\cdot\mathbf{L}(x),\boldsymbol{\phi}(x')]$$
$$=-(\nabla-i^\epsilon t\boldsymbol{\phi}(x)')\delta(\mathbf{x}-\mathbf{x}'),$$

we find that

$$[(\nabla-i^\epsilon t\boldsymbol{\phi}(x)')\cdot\mathbf{L}(x),\ (\nabla'-i^\epsilon t\boldsymbol{\phi}(x')')\cdot\mathbf{L}(x')]$$
$$=-{}^\epsilon t(\nabla-i^\epsilon t\boldsymbol{\phi}(x)')\cdot\mathbf{L}(x)'\delta(\mathbf{x}-\mathbf{x}')$$
$$-(\nabla-i^\epsilon t\boldsymbol{\phi}(x)')\cdot[\mathbf{L}(x),\mathbf{L}(x')]\cdot(-\nabla'-i^\epsilon t\boldsymbol{\phi}(x')'),$$

where the last differential operator acts to the left. A comparison with the commutation relations obeyed by $\partial_0 G$ shows that they are replaced by

$$[\mathbf{L}(x),\mathbf{L}(x')]=0,$$

which facilitates the use of the second supplementary condition in the form

$$\mathbf{L}(x)\Psi=0.$$

The longitudinal variables can be eliminated from the physical quantities T^{00} and T^{0k}. This is carried out in several stages. Evidently,

$$\mathbf{G}\Psi=(\mathbf{G}^T+\mathbf{G}^L)\Psi=\mathbf{G}_1\Psi$$

with

$$\mathbf{G}_1=[1+\nabla\mathfrak{D}_\phi(\nabla-i^\varepsilon t\boldsymbol{\phi}')]\cdot\mathbf{G}^T-\nabla\mathfrak{D}_\phi k^0,$$

but this does not suffice to eliminate $\mathbf{G}^L$ from

$$G^{0m}(x).G^k{}_m(x)\Psi=\tfrac{1}{2}[G^{0m}(x)G^k{}_m(x)+G^k{}_m(x)G^{0m}(x)]\Psi,$$

for example. We must also include a commutator term:

$$G^{0m}.G^k{}_m\Psi=G_1{}^{0m}.G^k{}_m\Psi+\tfrac{1}{2}[G^{0m}-G_1{}^{0m},\,G^k{}_m]\Psi.$$

Now

$$\mathbf{G}-\mathbf{G}_1=-\nabla\mathfrak{D}_\phi[(\nabla-i^\varepsilon t\boldsymbol{\phi}')\cdot\mathbf{G}-k^0]$$

and

$$[G_{km}(x),\,(\nabla-i^\varepsilon t\boldsymbol{\phi}(x')')\cdot\mathbf{G}(x')]=i^\varepsilon tG_{km}(x)'\delta(\mathbf{x}-\mathbf{x}'),$$

so that

$$[G^{0m}(x)-G_1{}^{0m}(x),\,G^k{}_m(x)]\Psi=-\mathrm{tr}(t\partial^m\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}))G^k{}_m(x)\Psi,$$

where

$$\partial^m\mathfrak{D}_\phi(\mathbf{x},\mathbf{x})=\lim_{\mathbf{x}'\to\mathbf{x}}\partial^m\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}').$$

This gives the replacement

$$G^{0m}.G^k{}_m\Psi=G_2{}^{0m}.G^k{}_m\Psi$$

with

$$\mathbf{G}_2(x)=\mathbf{G}_1(x)-\tfrac{1}{2}\,\mathrm{tr}(t\nabla\mathfrak{D}_\phi(\mathbf{x},\mathbf{x})).$$

The analogous elimination process for T^{00} involves

$$(G^{0k})^2\Psi=(G_1{}^{0k})^2\Psi+[G^{0k},G^0{}_{1k}]\Psi$$

and

$$[G^{0k}(x),G_1{}^0{}_k(x)]\Psi=-\mathrm{tr}(t\partial^k\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}))G_1{}^0{}_k(x)\Psi.$$

The result can be written as

$$[G^{0k}(x)]^2\Psi=[G_2{}^{0k}(x)-\tfrac{1}{2}\,\mathrm{tr}t\partial^k\mathfrak{D}_\phi(\mathbf{x},\mathbf{x})]\times[G_2{}^0{}_k(x)+\tfrac{1}{2}\,\mathrm{tr}t\partial_k\mathfrak{D}_\phi(\mathbf{x},\mathbf{x})]\Psi.$$

The second supplementary condition supplies the dependence of Ψ upon the longitudinal field eigenvalues $\boldsymbol{\phi}^{L'}$ through the integrable functional differential equation,

$$\left\{i(\delta_3/\delta\boldsymbol{\phi}^L(x)')-\nabla\int(d\mathbf{x}')\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}')\times[(\nabla'-i^\varepsilon t\boldsymbol{\phi}(x')')\cdot\mathbf{G}^T(x')-k^0(x')]\right\}\Psi=0,$$

in which

$$\boldsymbol{\phi}(x)=\boldsymbol{\phi}(x)^T+\boldsymbol{\phi}(x)^{L'}.$$

We shall write the solution as

$$\Psi=V(\boldsymbol{\phi}^{L'})|\quad\rangle_1$$

where the operator V obeys the initial conditions

$$V(0)=1,$$

and $|\quad\rangle_1$ is independent of the longitudinal eigenvalues. The isolation of all $\boldsymbol{\phi}^{L'}$ dependence in the operator V is characteristic of any gauge invariant state and, therefore, applies also to $T^{00}\Psi$ and $T^{0k}\Psi$. Thus,

$$T^{00}(\phi^{L'})\Psi=V(\phi^{L'})T^{00}(0)|\quad\rangle_1$$

which states, in effect, that $\boldsymbol{\phi}^L$ can be set equal to zero in T^{00} and T^{0k} after $\mathbf{G}^L$ has been eliminated.

The transformation to the subspace of physical states, and the radiation gauge, has now been made. One problem remains, however. The operator $\mathbf{G}_2$ is not Hermitian. Indeed,

$$\mathbf{G}_2-\mathbf{G}_2{}^\dagger=-[1+\nabla\mathfrak{D}_\phi(\nabla-i^\varepsilon t\boldsymbol{\phi}')]\cdot\mathrm{tr}t\nabla\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}),$$

which can be exhibited in the structure of $\mathbf{G}_2$ by writing

$$\mathbf{G}_2=[1+\nabla\mathfrak{D}_\phi(\nabla-i^\varepsilon t\boldsymbol{\phi}')]:(\mathbf{G}^T-\tfrac{1}{2}\,\mathrm{tr}t\nabla\mathfrak{D}_\phi)-\nabla\mathfrak{D}_\phi k^0.$$

The non-Hermitian term is removed by the transformation

$$|\quad\rangle_1=\exp[\tfrac{1}{2}v(\boldsymbol{\phi}^T)]|\quad\rangle,$$

where $v(\boldsymbol{\phi})$ is defined for arbitrary $\boldsymbol{\phi}$ by

$$\delta_3 v/\delta\phi_{ka}(x)=-\mathrm{tr}it_a\partial_k\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}),$$

provided these are integrable differential equations. The required integrability conditions are valid,

$$[\delta_3/\delta\phi_{lb}(x')][\delta_3/\delta\phi_{ka}(x)]v=-\mathrm{tr}t_a\partial_k\mathfrak{D}_\phi(\mathbf{x},\mathbf{x}')t_b\partial_l{}'\mathfrak{D}_\phi(\mathbf{x}',\mathbf{x})=[\delta_3/\delta\phi_{ka}(x)][\delta_3/\delta\phi_{lb}(x')]v.$$

The outcome of this last stage is the replacement of $\mathbf{G}_2$ with the Hermitian operator

$$\mathbf{G}=[1+\nabla\mathfrak{D}_\phi(\nabla-i^\varepsilon t\boldsymbol{\phi}')]:\mathbf{G}^T-\nabla\mathfrak{D}_\phi k^0.$$

The final results for the Hermitian energy and momentum density operators are

$$T^{00}=\tfrac{1}{2}f^2(\mathbf{G}-\tfrac{1}{2}\,\mathrm{tr}t\nabla\mathfrak{D}_\phi)(\mathbf{G}+\tfrac{1}{2}\,\mathrm{tr}t\nabla\mathfrak{D}_\phi)+\tfrac{1}{4}f^2(G_{kl})^2-\tfrac{1}{2}i\psi.(\boldsymbol{\alpha}\cdot(\nabla-i^\varepsilon T\boldsymbol{\phi}')+\beta m)\psi$$

and

$$T^0{}_k=f^2G^{0l}\cdot G_{kl}-\tfrac{1}{2}i\psi.(\partial_k-i^\varepsilon T\phi_k')\psi+\tfrac{1}{2}\partial^l\tfrac{1}{4}\psi\sigma_{kl}\psi,$$

in complete agreement with the independent radiation gauge treatment.

AXIAL GAUGE

There is an alternative to the radiation gauge in which the decomposition into longitudinal and transverse components is replaced by one into components parallel and perpendicular to a fixed axis. We describe the latter as the axial gauge.[3] An entirely analogous elimination procedure can be used. Let the axis be labeled as the third direction. We write

$$-(\nabla-i\,{}^{\epsilon}t\boldsymbol{\phi}')\cdot\mathbf{G}+k^0=(\partial_3-i\,{}^{\epsilon}t\phi_3')A,$$

where

$$A=-G^{03}+G_1^{03}$$

and

$$G_1^{03}=\mathfrak{D}_3[k^0-(\nabla-i\,{}^{\epsilon}t\boldsymbol{\phi}')\cdot\mathbf{G}^\perp],$$
$$(\partial_3-i\,{}^{\epsilon}t\phi_3(x)')\mathfrak{D}_3(\mathbf{x},\mathbf{x}')=\delta(\mathbf{x}-\mathbf{x}'),$$

while $\mathbf{G}^\perp$ refers to the components perpendicular to the axis. The equal-time commutation relations for $\partial_0 G$ are satisfied with

$$[A(x),A(x')]=0,$$

and the second supplementary condition becomes

$$A(x)\Psi=0.$$

The consequences of eliminating G^{03} are given by

$$G^{03}.G_{k3}\Psi=G_2^{03}.G_{k3}\Psi$$

with

$$G_2^{03}(x)=G_1^{03}(x)+\tfrac{1}{2}\operatorname{tr}t\mathfrak{D}_3(\mathbf{x},\mathbf{x})$$

and

$$(G^{03})^2\Psi=[G_2^{03}+\tfrac{1}{2}\operatorname{tr}t\mathfrak{D}_3][G_2^{03}-\tfrac{1}{2}\operatorname{tr}t\mathfrak{D}_3]\Psi.$$

The elimination of ϕ_3' is accomplished by

$$\Psi=V(\phi_3')|\quad\rangle,$$

where

$$[i(\delta_3/\delta\phi_3(x)')-G_1^{03}(x)]V(\phi_3')=0,\quad V(0)=1$$

and the net effect in T^{00} and T^{0k} is to set $\phi_3'=0$. At this point the axial gauge becomes algebraically simpler than the radiation gauge, for

$$\phi_3=0:\quad \partial_3\mathfrak{D}_3(\mathbf{x},\mathbf{x}')=\delta(\mathbf{x}-\mathbf{x}')$$

and $\mathfrak{D}_3$ is just an integration operation, which is quite independent of the internal variables. Therefore,

$$\phi_3=0:\quad \operatorname{tr}t\mathfrak{D}_3(\mathbf{x},\mathbf{x})=0,$$

and the axial gauge is characterized by the direct substitution in T^{00} and T^{0k} of the conditions

$$\phi_3(x)=0,$$
$$G^{03}(x)=\int_{\pm\infty}^{x_3}dx_3'[k^0-(\nabla-i\,{}^{\epsilon}t\boldsymbol{\phi}')\cdot\mathbf{G}^\perp](x_1x_2x_3'),$$

where

$$\int_{\pm\infty}^{x}dx'\,f(x')=\tfrac{1}{2}\left(\int_{-\infty}^{x}+\int_{\infty}^{x}\right)dx'\,f(x')$$
$$=\int_{-\infty}^{\infty}dx'\,\tfrac{1}{2}\epsilon(x-x')f(x').$$

This description of the axial gauge must be incomplete, however. As x_3 approaches infinity in either direction, the operator G^{03} attains the limits

$$x_3\to\pm\infty:\quad G^{03}(x)\to\pm\tfrac{1}{2}\mathbf{T}(x_1x_2),$$

where

$$\mathbf{T}(x_1x_2)=\int_{-\infty}^{\infty}dx_3[k^0-(\nabla-i\,{}^{\epsilon}t\boldsymbol{\phi}')\cdot\mathbf{G}^\perp](x_1x_2x_3)$$

is not necessarily zero. That the resulting nonconvergence of the total energy is not merely a matter of an additive constant can be seen from the derived equation of motion,

$$\partial_0\boldsymbol{\phi}^\perp=-f^2\mathbf{G}^\perp-(\nabla-i\,{}^{\epsilon}t\boldsymbol{\phi}')^\perp.\phi^0,$$

where

$$\phi^0(x)=-f^2\int_{-\infty}^{\infty}dx_3'\,\tfrac{1}{2}\epsilon(x_3-x_3')G^{03}(x_1x_2x_3')$$
$$=f^2\int_{-\infty}^{\infty}dx_3'\,g(x_3-x_3')[k^0-(\nabla-i\,{}^{\epsilon}t\boldsymbol{\phi}')\cdot\mathbf{G}^\perp](x_1x_2x_3')$$

and

$$g(x_3-x_3')=-\int_{-\infty}^{\infty}dx_3''\,\tfrac{1}{2}\epsilon(x_3-x_3'')\tfrac{1}{2}\epsilon(x_3''-x_3')$$
$$=-\tfrac{1}{2}|x_3-x_3'|+\infty.$$

Furthermore, an element of gauge arbitrariness remains, characterized by infinitesimal gauge functions $\delta\lambda$ that are independent of x_3. Thus,

$$\phi_3(x)\to\phi_3(x)+(\partial_3-i\,{}^{\epsilon}t\phi_3(x)')\delta\lambda(x_1x_2)=0.$$

Both of the operators T^{00} and T^{0k} are invariant under such two-dimensional gauge transformations; and the generators of these transformations are just the operators $\mathbf{T}(x_1x_2)$, as illustrated by

$$-i[\boldsymbol{\phi}(x)^\perp,\Lambda]=-(\nabla-i\,{}^{\epsilon}t\boldsymbol{\phi}(x)')^\perp\delta\lambda(x_1x_2),$$
$$\Lambda=\int dx_1dx_2\sum_a\delta\lambda_a(x_1x_2)\mathbf{T}_a(x_1x_2),$$

so that

$$[T^{00}(x),\mathbf{T}(x_1'x_2')]=[T^{0k}(x),\mathbf{T}(x_1'x_2')]=0.$$

Further consideration of these points will be required before the axial gauge can be used effectively.

[3] The algebraic advantages of this gauge were pointed out by R. L. Arnowitt and S. I. Fickler, Phys. Rev. 127, 1821 (1962).