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Commutation Relations and Conservation Laws

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The response of a physical system to external electromagnetic and gravitational fields, as embodied in the electric current and stress tensor conservation laws, is used to derive the equal-time commutation relations for charge density and energy density.

INTRODUCTION

AMONG the more important physical properties in relativistic quantum field theory are the conserved local quantities, such as the electric charge flux vector $j^\mu(x)$ and the stress tensor $T^{\mu\nu}(x)$. In order to answer questions about the simultaneous measurability of these quantities one needs the commutation relations of the operators on a space-like surface or, more specifically, at a common time. The physical independence of different points on a space-like surface guarantees the compatability of any associated localized physical properties. That is a general assertion of commutability under such circumstances. A complete treatment of equal-time commutators has been lacking, however. Thus, although it has long been remarked that the electric charge density at all spatial points obeys

$$x^0=x^{0\prime}:\quad [j^0(x),j^0(x')]=0,$$

a corresponding statement about the energy density had not been recorded until it was observed,[1] for a particular system, that

$$x^0=x^{0\prime}:\quad -i[T^{00}(x),T^{00}(x')] = -(T^{0k}(x)+T^{0k}(x'))\partial_k\delta(\mathbf{x}-\mathbf{x}').$$

It is our intention to supply a general basis for this and other equal-time commutators.

The measurement theory of the electric current vector and the stress tensor is founded upon the specific dynamical nature of these properties as the sources of the electromagnetic and gravitational fields, respectively. More precisely, we exploit the reciprocal dynamical aspect of j^μ and $T^{\mu\nu}$ whereby they determine the response of a system to external electromagnetic and gravitational fields. What is characteristic of these dynamical agencies, and equivalent to the existence of the local conservation laws for the properties of interest, is the freedom in description associated with gauge and coordinate transformations.

ELECTRIC CURRENT

The electric current provides the simpler illustration of the method. Let W be the action operator of all charge-bearing fields $\chi(x)$, excluding the purely electromagnetic action term. The vector potential $A_\mu(x)$ appears as an external quantity in this action operator,

$$W=\int (dx)\mathfrak{L}_{\mathrm{ch}}(\chi,A_\mu),$$

and the infinitesimal numerical variation

$$\delta_A W=\int (dx) j^\mu(x)\delta A_\mu(x)$$

defines the electric current vector. The requirement of gauge invariance, applied to the infinitesimal gauge transformation

$$\delta A_\mu(x)=-\partial_\mu\delta\lambda(x),$$

yields the charge conservation equation

$$\partial_\mu j^\mu(x)=0.$$

The gravitational potential $g_{\mu\nu}$ replaces A_μ in the analogous discussion of $T^{\mu\nu}$. For that circumstance the use of an external field is quite justified by the weak dynamical influence of the gravitational field in a special relativistic context. This argument does not apply to the electromagnetic field, of course, and we must remove the implication that a weak-coupling treatment of the electromagnetic field is necessarily involved. To do that we have only to rephrase our procedure by replacing W with the total action operator

$$W=\int (dx)[\mathfrak{L}_{\mathrm{ch}}(\chi,\, A_\mu+A_\mu')+\mathfrak{L}_{\mathrm{em}}(A_\mu,F_{\mu\nu})]$$

in which $A_\mu'(x)$ is an arbitrary numerical external potential. Infinitesimal variations of the latter can now be used to define the electric current vector while incorporating the full dynamical effect of the electromagnetic field.

The charge conservation equation

$$\partial_0 j^0(x)=-\partial_k j^k(x)$$

is an example of a relationship between operators, of the type

$$\partial_0 A(x)=B(x),$$

that is maintained for arbitrary values of certain parameters—the external potentials, in this example. Now, the quantum action principle,

$$\delta\langle\sigma_1|\sigma_2\rangle=i\langle\sigma_1|\delta W_{12}|\sigma_2\rangle,$$

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[1] J. Schwinger, Phys. Rev. **127**, 324 (1962).

applies, in particular, to infinitesimal alterations in the structure of the Lagrange function, as realized by variations of numerical parameters. It is a corollary of the action principle that

$$\delta\langle\sigma_1|F(x)|\sigma_2\rangle=\langle\sigma_1|\delta'F(x)+i\int_{\sigma_2}^{\sigma_1}(dx')(F(x)\delta\mathfrak{L}(x'))_+|\sigma_2\rangle,$$

where $\delta'F(x)$ refers to an explicit dependence of the operator $F(x)$ upon these parameters. To maintain the relationship between $A(x)$ and $B(x)$ then requires that

$$\partial_0\left[\delta'A(x)+i\int(dx')(A(x)\delta\mathfrak{L}(x'))_+\right]$$
$$=\delta'B(x)+i\int(dx')(B(x)\delta\mathfrak{L}(x'))_+.$$

But, the time derivative of the ordered product is given by

$$\partial_0(A(x)\delta\mathfrak{L}(x'))_+=(\partial_0A(x)\delta\mathfrak{L}(x'))_+ +\delta(x^0-x^{0\prime})[A(x),\delta\mathfrak{L}(x')],$$

and therefore,

$$-i\int(d\mathbf{x}')[A(x),\delta\mathfrak{L}(x')]|_{x^0=x^{0\prime}}=\partial_0\delta'A(x)-\delta'B(x).$$

This statement supplies a general foundation for equal-time commutation relations. Note, incidentally, that $A(x)$ cannot depend explicitly upon the parameters unless $B(x)$ correspondingly involves the time derivative of these parameters. In the absence of such a dependence, the right-hand side of the above equation is just $-\delta'B(x)$.

When a number of parameters are involved, the explicit dependence upon the parameters is subject to certain integrability conditions or reciprocity relations. We illustrate this with the continuum of parameters constituted by the external potential $A_\mu(x)$. The calculation of the second variation for a transformation function proceeds from the action principle as

$$\delta^2\langle\sigma_1|\sigma_2\rangle=\delta\left[i\int(dx)\delta A_\mu(x)\langle\sigma_1|j^\mu(x)|\sigma_2\rangle\right]$$
$$=\int(dx)(dx')\delta A_\mu(x)\delta A_\nu(x')[-\langle\sigma_1|(j^\mu(x)j^\nu(x'))_+|\sigma_2\rangle +i\langle\sigma_1|\delta'j^\mu(x)/\delta A_\nu(x')|\sigma_2\rangle],$$

and the necessary symmetry of this result supplies the reciprocity relation

$$\delta'j^\mu(x)/\delta A_\nu(x')=\delta'j^\nu(x')/\delta A_\mu(x).$$

In order to obtain explicit equal-time commutation relations for components of the electric current vector, we must be somewhat more specific about the dependence of the current upon the external potential. The major consideration here is locality. The current usually does not involve the potential at relatively space-like points, but we only insist here that $j_\mu(x)$ does not refer to the potential at neighboring times, which is to say that it does not contain the time derivative of the potential. That restriction defines a certain class of electric charge-bearing physical systems (which may well be without exception). The immediate implication from the conservation equation is that $j^0(x)$ cannot be an explicit function of the external potential. The reciprocity relation then asserts that

$$\delta'j^k(x)/\delta A_0(x')=\delta'j^0(x')/\delta A_k(x)=0.$$

The equal-time commutation relation supplied by the conservation equation for charge now reads

$$-i\int(d\mathbf{x}')[j^0(x),j^\mu(x')]\delta A_\mu(x')$$
$$=\partial_k\int(d\mathbf{x}')[\delta_3'j^k(x)/\delta A_l(x')]\delta A_l(x'),$$

and therefore, $(x^0=x^{0\prime})$

$$[j^0(x),j^0(x')]=0,$$
$$-i[j^0(x),j^l(x')]=\partial_k[\delta_3'j^k(x)/\delta A_l(x')]$$
$$=\partial_k[\delta_3'j^l(x')/\delta A_k(x)].$$

The variational derivatives that appear here are the three-dimensional ones defined by

$$\delta'j^k(x)/\delta A_l(x')=\delta(x^0-x^{0\prime})\delta_3'j^k(x)/\delta A_l(x').$$

Despite the use of an external potential, these commutators are assertions about an isolated physical system, if the potential is set equal to zero after differentiation.

One should recognize that an explicit dependence of the current upon an external potential occurs for all physical systems. Let us use the conservation equation again, and convert the second commutator into

$$[j^0(x),\,-i\partial_0j^0(x')]=-\partial_k\partial_l'[\delta_3'j^l(x')/\delta A_k(x)],$$

which is symmetrical in the two points x and x'. A contradiction to the hypothetical vanishing of the right-hand member of this equation arises from the positiveness exhibited by the vacuum expectation value of the left-hand member. Thus, if $\varphi(x)$ is an arbitrary real function, with which one forms the Hermitian operator

$$J(x^0)=\int(d\mathbf{x})\varphi(x)j^0(x),$$

the equation of motion

$$i\partial_0j^0(x)=[j^0(x),P^0],$$

combined with the null energy of the vacuum, yields

$$\int (d\mathbf{x})(d\mathbf{x}')\varphi(x)\langle[j^0(x), -i\partial_0 j^0(x')]\rangle\varphi(x') = 2\langle JP^0J\rangle \geq 0.$$

Now it is essential to call upon the relativistic principle that any sufficiently localized act must excite the vacuum, which implies that functions $\varphi(x)$ surely exist for which the states $\langle J$ have an energy expectation value greater than zero. It can be shown that the explicit dependence of the current on the potential is completely local,

$$-\delta_3' j^k(x)/\delta A_l(x') = \delta(\mathbf{x}-\mathbf{x}')j^{kl}(x).$$

The expectation value of the symmetrical tensor $j^{kl}(x)$ in the invariant vacuum state is of the form

$$\langle j^{kl}(x)\rangle = \delta^{kl}C.$$

Accordingly,

$$2\langle JP^0J\rangle = C\int (d\mathbf{x})[\nabla\varphi(x)]^2$$

and the constant C must be a positive number,

$$C>0,$$

which shows, incidentally, that a positive energy expectation value is realized for every nonconstant function $\varphi(x)$.

STRESS TENSOR

Through the agency of an external gravitational field, the stress-energy-momentum tensor $T^{\mu\nu}(x)$ is defined by the variational equation

$$\delta_g W = \int (dx)(-g)^{1/2}\tfrac{1}{2}T^{\mu\nu}\delta g_{\mu\nu},$$

in which

$$g = \det g_{\mu\nu}.$$

The role formerly played by gauge invariance is now taken over by the requirement of general coordinate invariance. The infinitesimal coordinate transformation

$$\bar{x}^\mu = x^\mu + \delta x^\mu(x)$$

induces

$$\delta g_{\mu\nu} = \delta x^\lambda \partial_\lambda g_{\mu\nu} + g_{\lambda\nu}\partial_\mu \delta x^\lambda + g_{\mu\lambda}\partial_\nu \delta x^\lambda,$$

from which we infer the extended conservation equations

$$\partial_\mu[(-g)^{1/2}g_{\lambda\nu}T^{\mu\nu}] = \tfrac{1}{2}(-g)^{1/2}T^{\mu\nu}\partial_\lambda g_{\mu\nu}.$$

Alternative forms are

$$\partial_\mu(g_{\lambda\nu}T^{\mu\nu}) = \tfrac{1}{2}T^{\mu\nu}(\partial_\lambda g_{\mu\nu} - g_{\lambda\nu}g^{\alpha\beta}\partial_\mu g_{\alpha\beta})$$

and

$$\partial_\mu[(-g)^{1/2}T^{\mu\nu}] = -(-g)^{1/2}T^{\alpha\beta}\Gamma_{\alpha\beta}{}^\nu$$

where, of course,

$$\Gamma_{\alpha\beta}{}^\nu = \tfrac{1}{2}g^{\nu\lambda}[\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\alpha\lambda} - \partial_\lambda g_{\alpha\beta}].$$

As a first application, consider an infinitesimal deviation, $\delta g_{\mu\nu}(x)$, from the Minkowski metric. The extended conservation equations can then be presented as

$$\partial_\lambda \mathbf{T}^{\lambda\nu} = -\tfrac{1}{2}\delta g_{\lambda\kappa}\partial^\nu T^{\lambda\kappa},$$

where

$$\mathbf{T}^{\mu\nu} = (-g)^{1/2}T^{\mu\nu} - g^{\mu\nu}\tfrac{1}{2}\delta g_{\lambda\kappa}T^{\lambda\kappa} + T^{\mu\lambda}\delta g_{\lambda\kappa}g^{\nu\kappa}.$$

Let us also observe that

$$\partial_\lambda(x^\mu \mathbf{T}^{\lambda\nu} - x^\nu \mathbf{T}^{\lambda\mu}) = -\tfrac{1}{2}\delta g_{\lambda\kappa}(x^\mu\partial^\nu - x^\nu\partial^\mu)T^{\lambda\kappa} + \mathbf{T}^{\mu\nu} - \mathbf{T}^{\nu\mu},$$

in which

$$\mathbf{T}^{\mu\nu} - \mathbf{T}^{\nu\mu} = \tfrac{1}{2}\delta g_{\lambda\kappa}(T^{\mu\lambda}g^{\nu\kappa} - T^{\nu\lambda}g^{\mu\kappa} + T^{\mu\kappa}g^{\nu\lambda} - T^{\nu\kappa}g^{\mu\lambda}).$$

An integration over all three-dimensional space removes the space derivative terms and yields

$$\partial_0\int (d\mathbf{x})\mathbf{T}^0{}_\nu = -\int (d\mathbf{x})\tfrac{1}{2}\delta g_{\lambda\kappa}\partial_\nu T^{\lambda\kappa},$$

$$\partial_0\int (d\mathbf{x})(x_\mu \mathbf{T}^0{}_\nu - x_\nu \mathbf{T}^0{}_\mu) = -\int (d\mathbf{x})[\tfrac{1}{2}\delta g_{\lambda\kappa}(x_\mu\partial_\nu - x_\nu\partial_\mu)T^{\lambda\kappa} - \mathbf{T}_{\mu\nu} + \mathbf{T}_{\nu\mu}].$$

These forms lead immediately to commutation relations between the components of the stress tensor and the generators of the special relativistic infinitesimal coordinate transformations,

$$P_\nu = \int (d\mathbf{x})T^0{}_\nu, \quad J_{\mu\nu} = \int (d\mathbf{x})(x_\mu T^0{}_\nu - x_\nu T^0{}_\mu),$$

namely,

$$[T^{\lambda\kappa}(x), P_\nu] = -i\partial_\nu T^{\lambda\kappa}(x)$$

and

$$[T^{\lambda\kappa}, J_{\mu\nu}] = -i(x_\mu\partial_\nu - x_\nu\partial_\mu)T^{\lambda\kappa} + i(\delta_\nu{}^\lambda T_\mu{}^\kappa - \delta_\mu{}^\lambda T^\kappa{}_\nu + \delta_\nu{}^\kappa T_\mu{}^\lambda - \delta_\mu{}^\kappa T^\lambda{}_\nu).$$

These commutators, representing the transformation properties of the stress tensor, produce, through integration, the commutation relations of the ten infinitesimal generators of the inhomogeneous Lorentz group. In this way special relativistic kinematics emerges from gravitational dynamics.

To obtain more detailed information, let us choose the special gravitational field

$$g_{kl} = \delta_{kl}, \quad g_{0k} = 0, \quad -g_{00}(x) \neq 1,$$

so that properties of the energy density can be inferred by variation of $g_{00}(x)$. The extended conservation equations are

$$\partial_0[(-g_{00})T^{00}]=-\partial_k[(-g_{00})T^{0k}]+\tfrac{1}{2}T^{0k}\partial_k g_{00}$$

and

$$\partial_0[(-g_{00})^{1/2}T^{0k}]=-\partial_l[(-g_{00})^{1/2}T^{kl}]+\tfrac{1}{2}(-g_{00})^{1/2}T^{00}\partial_k g_{00},$$

where each form is chosen to avoid the explicit appearance of $\partial_0 g_{00}$. We confine our attention to the class of physical systems which are such that T^{kl} does not contain explicitly the time derivative of g_{00}, although it may be an explicit function of g_{00} at the same time.[2] It can be concluded that neither $(-g_{00})T^{00}$ nor $(-g_{00})^{1/2}T^{0k}$ are explicit functions of g_{00} for this distinguished class of material system, which is to say that these local quantities are the same functions of the fundamental dynamical variables as in the absence of an external gravitational field. The equation of motion for $(-g_{00})T^{00}$ now implies the equal-time commutator

$$-i\left[(-g_{00}T^{00})(x),\int(d\mathbf{x}')(-g_{00}T^{00})(x')\delta(-g_{00}(x'))^{1/2}\right]$$
$$=-\partial_k[(-g_{00})^{1/2}T^{0k}(x)\delta(-g_{00}(x))^{1/2}]-(-g_{00})^{1/2}T^{0k}(x)\partial_k\delta(-g_{00}(x))^{1/2},$$

where, it is noted, there is no explicit dependence upon $g_{00}(x)$, which indicates the consistency of the physical restriction. On setting $-g_{00}=1$, we obtain

$$-i[T^{00}(x),T^{00}(x')]=-[T^{0k}(x)+T^{0k}(x')]\partial_k\delta(\mathbf{x}-\mathbf{x}').$$

This derivation of the energy density commutator condition, for a class of physical systems, supplies a simple and general basis for what may well be considered the most fundamental equation of relativistic quantum field theory.

[2] In fact, T^{kl} must be an explicit local function of the second spatial derivatives of g_{00}.

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Zero-Temperature Properties of the Many-Fermion System*

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It is shown that the class of correlation graphs which arise in the calculation of thermodynamic properties in the canonical ensemble can be summed to give renormalized single-particle populations. In the limit of zero temperature the perturbation expansion of the energy then reduces to the adiabatic expansion of Goldstone about the *correct* model state. Arguments for the consistency of the expansion are developed for the case of the nonspherical Fermi surface.

I. INTRODUCTION

It is the purpose of this article to explore the physical consequences of the correlation bond graphs introduced by two of us in a previous article.[1] In the latter, it was shown that the free energy could be expressed in terms of graphs which strongly resembled the graphs of Bloch and Dominicis[2] plus graphs which arose because of correlations in the single-particle state populations, $n(\mathbf{k})$. These correlations arise because of the restraint in the trace to a summation over states with fixed number of particles.

In the limit as the number of particles goes to infinity, it was found that the only correlation graphs which arise are those which are simply connected. In this article, we exploit this property to show that the elimination of correlation bonds by summation (which is possible because of the rule of simple connectivity) results in a renormalization of the populations $\langle n(\mathbf{k})\rangle_0$. In the limit of zero temperature, one then recovers for the energy a series of terms which involves the renormalized $\langle n(\mathbf{k})\rangle$. This series is precisely the usually adiabatic series of Goldstone.[3] It is, thus, shown that

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‡ Present address: Université Libre de Bruxelles, Brussels, Belgium.
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