Quantum Electrodynamics without Electromagnetic Field*

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It is shown that in a nonlinear Heisenberg-type theory a bound state appears which has all the properties of a neutral vector particle. From such a model all the results of standard electrodynamics follow in the limit of infinite bare coupling constant. The method used is that of Schwinger's external sources.

1. INTRODUCTION

In the last few years several attempts[1–3] have been made to derive quantum electrodynamics from a field theory with four-fermion coupling. In the present paper we discuss this problem with the use of Schwinger's equations for Green's functions.[4] It will be shown that in theories with a four-fermion coupling a bound state appears when the bare coupling constant is sufficiently large. The formation of such a bound state changes the whole discussion of renormalizability. All interactions between the fermions can then be viewed as mediated by a boson and the theory turns out to be renormalizable in the perturbation expansion. In this paper, we restrict ourselves to quantum electrodynamics, although the method we use seems to be rather general and can be applied to mesodynamics as well. There are two reasons which make quantum electrodynamics more attractive in this respect than other theories. First, the use of the perturbation expansion is justified and second, the gauge invariance leads to several simplications. In particular, the limit of an infinite bare coupling constant can easily be found.

2. EQUATIONS FOR THE GENERATING FUNCTIONAL

We shall discuss the theory of a self-interacting spinor field ψ, with the Lagrangian density $\mathcal{L}$ chosen in the form

$$\mathcal{L}=-\bar{\psi}D_x\psi-(2\mu_0^2)^{-1}j_\mu j^\mu, \tag{1}$$

where

$$D_x=m_0-i\gamma^\mu\partial_\mu, \tag{2}$$

and

$$j_\mu=-e_0\bar{\psi}\gamma_\mu\psi. \tag{3}$$

All properties of such a theory can be obtained from the time-ordered Green functions. To deal with all the Green functions at the same time we introduce the generating functional,

$$\begin{aligned}\tau\{\eta,\bar{\eta}\} &=\left\langle 0\left|T\exp\left[i\int(\bar{\psi}\eta+\bar{\eta}\psi)dx\right]\right|0\right\rangle\\ &=\sum_{n=0}^{\infty}\frac{i^n}{(n!)^2}\int\bar{\eta}(x_1)\cdots\bar{\eta}(x_n)G(x_1\cdots x_n,y_1\cdots y_n)\\ &\qquad\times\eta(y_1)\cdots\eta(y_n)dx_1\cdots dy_n,\end{aligned} \tag{4}$$

$$G(x_1,\cdots,x_n,y_1,\cdots,y_n)=(-i)^n\frac{\delta^n}{\delta\bar{\eta}(x_1)\cdots\delta\bar{\eta}(x_n)}\frac{\delta^n}{\delta\eta(y_1)\cdots\delta\eta(y_n)}\tau\bigg|_{\eta=0=\bar{\eta}}.$$

From the field equations

$$(D_x+e_0^2/\mu_0^2\gamma^\mu j_\mu)\psi=0, \tag{5}$$

and from the equal-time commutation relations it follows that the functional τ obeys the following equations in functional derivatives[5]:

$$\begin{aligned}&\left[D_x-\frac{e_0^2}{\mu_0^2}\gamma^\mu\frac{\delta}{\delta\eta}\gamma_\mu\frac{\delta}{\delta\bar{\eta}}\right]\frac{1}{i}\frac{\delta}{\delta\bar{\eta}}\tau=\eta\tau,\\ &\frac{1}{i}\frac{\delta}{\delta\eta}\left[\overleftarrow{D}_x-\frac{e_0^2}{\mu_0^2}\gamma^\mu\frac{\delta}{\delta\eta}\gamma_\mu\frac{\delta}{\delta\bar{\eta}}\right]\tau=\bar{\eta}\tau.\end{aligned} \tag{6}$$

For reasons which will be clear later, we replace Eqs. (6) by a new set of equations for a certain more general functional Z.

$$\begin{aligned}&\left[D_x+e_0\gamma^\mu\frac{1}{i}\frac{\delta}{\delta J^\mu}\right]\frac{1}{i}\frac{\delta}{\delta\bar{\eta}}Z=\eta Z,\\ &\frac{1}{i}\frac{\delta}{\delta\eta}\left[\overleftarrow{D}_x+e_0\gamma^\mu\frac{1}{i}\frac{\delta}{\delta J^\mu}\right]Z=\bar{\eta}Z,\\ &\left[\mu_0^2\frac{1}{i}\frac{\delta}{\delta J^\mu}+e_0\frac{\delta}{\delta\eta}\gamma_\mu\frac{\delta}{\delta\bar{\eta}}\right]Z=-J_\mu Z.\end{aligned} \tag{7}$$

The new functional Z depends on η, $\bar{\eta}$ and on an auxiliary source function J_μ. When $J_\mu=0$, Z reduces to the initial

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functional τ, i.e.,

$$Z\{\eta,\bar{\eta},J_\mu=0\}=\tau\{\eta,\bar{\eta}\}. \quad (8)$$

Equations (7) are very similar to the equations for the generating functional in quantum electrodynamics in the Gupta-Bleuler gauge.[6] The only difference is the appearance of μ_0^2 in the third equation, instead of the operator $\Box$.

Current conservation leads to an additional equation for Z. This equation may be called the functional Ward identity.[7] It has the form

$$\mu_0^2\partial^\mu\frac{1}{i}\frac{\delta}{\delta J^\mu}Z=\left(-J^\mu{}_{,\mu}+e_0\frac{\delta}{\delta\eta}\eta-e_0\bar{\eta}\frac{\delta}{\delta\bar{\eta}}\right)Z. \quad (9)$$

3. EQUATIONS FOR THE PROPAGATORS

It will be sufficient for our purpose to investigate the simplest Green functions: the electron propagator G and the photon[8] propagator $\mathcal{G}_{\mu\nu}$, both in the presence of an external current J_μ.

$$G(x,y)=Z^{-1}\frac{1}{i}\frac{\delta^2 Z}{\delta\bar{\eta}(x)\delta\eta(y)}\bigg|_{\eta=0=\bar{\eta}},\quad \mathcal{G}_{\mu\nu}(z,z')=\frac{\delta\mathcal{A}_\mu(z)}{\delta J^\nu(z')}, \quad (10)$$

where

$$\mathcal{A}_\mu(z)=Z^{-1}\frac{1}{i}\frac{\delta Z}{\delta J^\mu(z)}\bigg|_{\eta=0=\bar{\eta}}. \quad (11)$$

All the Green functions with either no or two external spinor lines can be derived from G and $\mathcal{G}_{\mu\nu}$ by differentiating with respect to J_μ or $\mathcal{A}_\mu$. Let us introduce also the inverse propagators G^{-1} and $\mathcal{G}_{\mu\nu}{}^{-1}$ which are defined through the relations

$$\int G^{-1}(x,z)dz\,G(z,y)=\delta(x-y),$$
$$\int \mathcal{G}^{-1}{}_{\mu\nu}(z,y)dy\,\mathcal{G}^{\nu\lambda}(y,z')=\delta_\mu{}^\lambda\delta(z-z'). \quad (12)$$

Equations for G^{-1} and $\mathcal{G}_{\mu\nu}{}^{-1}$, derived from Eqs. (7), have the form

$$G^{-1}(x,y)=[D_x+e_0\gamma^\mu\mathcal{A}_\mu(x)]\delta(x-y)+ie_0^2\gamma^\mu\int G(x,x')dx'$$
$$\times\Gamma^\lambda(x',y',z)dz\,\mathcal{G}_{\mu\lambda}(x,z), \quad (13)$$

$$\mathcal{G}^{-1}{}_{\mu\nu}(z,z')=-\mu_0^2 g_{\mu\nu}\delta(z-z')-ie_0^2\,\mathrm{Tr}\gamma_\mu\int G(z,x)dx$$
$$\times\Gamma_\nu(x,y,z')dy\,G(y,z), \quad (14)$$

where

$$\Gamma^\mu(x,y,z)=(1/e_0)[\delta G^{-1}(x,y)/\delta\mathcal{A}_\mu(z)]. \quad (15)$$

Current conservation leads to two simple equations for the propagators G and $\mathcal{G}_{\mu\nu}$:

$$\mu_0^2\partial^\mu\mathcal{G}_{\mu\nu}(z,z')=-\partial_\nu\delta(z-z'), \quad (16)$$

and

$$i\partial_\mu{}^z\Gamma^\mu(x,y,z)=[\delta(x-z)-\delta(y-z)]G^{-1}(x,y). \quad (17)$$

The easiest way to derive them is to differentiate Eq. (9) with respect to η, $\bar{\eta}$, and J_μ.

Since Eqs. (13) and (14) hold for all values of J_μ, they are equivalent to an infinite set of integral equations. The lowest three equations have the form

$$\mathcal{G}^{-1}{}_{\mu\nu}(z-z')=-\mu_0^2 g_{\mu\nu}\delta(z-z')-ie_0^2\,\mathrm{Tr}\gamma_\mu\int G(z-x)dx$$
$$\times\Gamma_\nu(x-z',\,y-z')dy\,G(y-z), \quad (18)$$

$$G^{-1}(x-y)=D_x\delta(x-y)+ie_0^2\gamma^\mu\int G(x-x')dx'$$
$$\times\Gamma^\lambda(x'-z,\,y-z)dz\,\mathcal{G}_{\mu\lambda}(x-z), \quad (19)$$

$$\Gamma^\mu(x-z,\,y-z)=\gamma^\mu\delta(x-z)\delta(y-z)+ie_0^2\gamma^\nu\int G(x-x')dx'$$
$$\times\Gamma^{\lambda\mu}(x'-z,\,y-z',\,z-z')dz'\,\mathcal{G}_{\nu\lambda}(x-z')$$
$$-ie_0^2\gamma^\rho\int G(x-x')dx'$$
$$\times\Gamma^\lambda(x'-z',\,y'-z')dy'\,G(y'-x'')dx''$$
$$\times\Gamma^\mu(x''-z,\,y-z)dz'\,\mathcal{G}_{\rho\lambda}(x-z'). \quad (20)$$

It is only the equation for the propagator $\mathcal{G}_{\mu\nu}$ which differs from the corresponding equation in quantum electrodynamics. All remaining equations, including all equations with more than two external spinor lines coincide in the four-fermion theory and in quantum electrodynamics.

4. PROPERTIES OF THE PROPAGATOR $\mathcal{G}_{\mu\nu}$

By virtue of Eq. (16) the propagator $\mathcal{G}_{\mu\nu}$ and the inverse propagator $\mathcal{G}^{-1}{}_{\mu\nu}$ can be written in the form

$$\mathcal{G}_{\mu\nu}(x-y)=-(g_{\mu\nu}-\Box^{-1}\partial_\mu\partial_\nu)\mathcal{G}(x-y)$$
$$-\mu_0^{-2}\Box^{-1}\partial_\mu\partial_\nu\delta(x-y), \quad (21)$$

$$\mathcal{G}^{-1}{}_{\mu\nu}(x-y)=-(g_{\mu\nu}-\Box^{-1}\partial_\mu\partial_\nu)\mathcal{G}^{-1}(x-y)$$
$$-\mu_0^{2}\Box^{-1}\partial_\mu\partial_\nu\delta(x-y). \quad (22)$$

From Eq. (18) we can find the equation for $\mathcal{G}$. In momentum space it reads

$$\mathcal{G}^{-1}(k^2)=\mu_0^2+\frac{ie_0^2}{3(2\pi)^4}\,\mathrm{Tr}\int\gamma_\mu G(p+k)$$
$$\times\Gamma^\mu(p+k,\,p)G(p)dp. \quad (23)$$

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[6] H. Umezawa, *Quantum Field Theory* (North-Holland Publishing Company, Amsterdam, 1956).

[7] The derivation of such an identity in quantum electrodynamics is given in another paper of the present author, Nuovo Cimento **17**, 951 (1960).

[8] It will become clear later that the name photon propagator is fully justified.

The right-hand side of Eq. (23) can be written as a spectral integral

$$\mathcal{G}^{-1}(k^2)=\mu_0^2-k^2\int_{M_0^2}^{\infty}\frac{\sigma(M^2,\Lambda)dM^2}{M^2-k^2}, \tag{24}$$

where $\sigma(M^2,\Lambda)$ is a positive spectral function and M_0 is the lowest mass of those intermediate states which contribute to the propagator $\mathcal{G}$. The factor k^2 in front of the integral appears as a result of current conservation. In order to ensure the convergence of the integral, we introduced a cutoff parameter Λ. The spectral function σ can be computed by the perturbation method.

We shall be interested only in the limiting case $\mu_0=0$, which leads to quantum electrodynamics. The discussion could also be extended to the case $\mu_0\neq 0$ and would lead then to a vector meson theory. Although, the transition to the limit $\mu_0=0$ cannot be directly performed in the initial Lagrangian, one can do it in the equations for the propagators. The only singular term is the last term in the expression (21). It is well known, however, that this term does not affect physical results since it represents the propagation of unphysical, longitudinal photons which do not interact with electrons. We could easily eliminate this term by an appropriate gauge transformation.[9] An even more straightforward procedure which will be used in the further discussion is to keep μ_0^2 in Eq. (21) as an arbitrary parameter, the value of which does not affect gauge-independent quantities.

In the limit $\mu_0=0$, the inverse propagator $\mathcal{G}^{-1}$ has a spectral representation of the form

$$\mathcal{G}^{-1}(k^2)=-k^2\left(Z_3^{-1}+k^2\int_0^{\infty}\frac{dM^2}{M^2}\frac{\sigma}{M^2-k^2}\right), \tag{25}$$

where

$$Z_3^{-1}=\int_0^{\infty}\frac{dM^2}{M^2}\sigma. \tag{26}$$

This leads to the following representation for the propagator $\mathcal{G}$:

$$\mathcal{G}(k^2)=-\frac{Z_3}{k^2}+\int_0^{\infty}\frac{\rho dM^2}{M^2-k^2}. \tag{27}$$

In formulas (25) and (27) we made use of the fact that the pole term in (27) represents a zero-mass particle, and the existence of such a particle implies that the lowest intermediate mass M_0 is zero. Had we not put $\mu_0=0$ we would have obtained a bound state with nonzero mass.

Apart from possible differences in the spectral function and in the constant Z_3 the propagator $\mathcal{G}$ does not differ from the photon propagator in standard quantum electrodynamics. We show in the next section that after renormalization both propagators become equal.

[9] B. Zumino, J. Math. Phys. 1, 1 (1960); I. Bialynicki-Birula, *ibid.* 3, 1094 (1962).

5. PROOF OF THE EQUIVALENCE

We show now that the set of equations for the propagators yields identical renormalized solutions in the four-fermion (FF) theory and in quantum electrodynamics (QED). First, let us consider the lowest order correction to the propagator

$$\mathcal{G}^{-1}(k^2)=-k^2\int_{4m^2}^{\infty}\frac{\sigma^{(2)}dM^2}{M^2-k^2} \tag{28}$$

$$=-k^2\left[\frac{e_0^2}{12\pi^2}\ln\frac{\Lambda^2}{4m^2}+k^2\int_{4m^2}^{\infty}\frac{dM^2}{M^2}\frac{\sigma^{(2)}}{M^2-k^2}\right], \tag{29}$$

where

$$\sigma^{(2)}(M^2)=\frac{e_0^2}{12\pi^2}\left(1-\frac{4m^2}{M^2}\right)^{1/2}\left(1+\frac{2m^2}{M^2}\right). \tag{30}$$

Terms of the order k^2/Λ^2 and m^2/Λ^2 have been neglected. Expression (29) is to be compared with the unrenormalized photon propagator in quantum electrodynamics. In the same order it has the form

$$\mathcal{G}_{\mathrm{QED}}^{-1}(k^2)=-k^2\left[1+\frac{e_0^2}{12\pi^2}\ln\frac{\Lambda^2}{4m^2}+k^2\int_{4m^2}^{\infty}\frac{dM^2}{M^2}\frac{\sigma^{(2)}}{M^2-k^2}\right]. \tag{31}$$

The only difference is the value of the renormalization constant Z_3.

$$(Z_3^{\mathrm{FF}})^{-1}=(e_0^2/12\pi^2)\ln(\Lambda^2/4m^2), \tag{32}$$

$$(Z_3^{\mathrm{QED}})^{-1}=1+(e_0^2/12\pi^2)\ln(\Lambda^2/4m^2). \tag{33}$$

This difference will disappear, of course, in the renormalized expressions. In both cases, we shall have

$$\mathcal{G}_{\mathrm{ren}}^{-1}(k^2)=Z_3\mathcal{G}^{-1}(k^2)=-k^2\left[1+k^2\int_{4m^2}^{\infty}\frac{dM^2}{M^2}\frac{\sigma_{\mathrm{ren}}}{M^2-k^2}\right], \tag{34}$$

where σ_{ren} differs from σ in having e_0^2 replaced by $e^2=Z_3e_0^2$. In the lowest order we obtain

$$e^2=(1/12\pi^2)\ln(\Lambda^2/4m^2), \tag{35}$$

so that the observable charge does not depend on the bare one. Since the equations for the electron propagator and the vertex function are the same in both theories, there will be no difference (apart from the difference in factors Z_3) between the next order corrections to the electron propagator and to the vertex function. The simplest way of proving the equivalence in any order is to notice that in both theories we have the same set of Feynman diagrams and the same rules for writing the S-matrix elements. Finally, we would like to point out that the equations in the four-fermion theory can be obtained from those in quantum electrodynamics as a result of the transformation

$$e_0^2 \rightarrow \epsilon^{-1}e_0^2, \quad \mathcal{G}_{\mu\nu} \rightarrow \epsilon\mathcal{G}_{\mu\nu}, \tag{36}$$

in the limit $\epsilon \rightarrow 0$. The transformation (36) belongs to the renormalization group and leaves the physical content of the theory unchanged.

6. CONCLUSIONS

The main result of this paper, which is that particles may appear in conventional field theory without corresponding fields being introduced into the initial Lagrangian, seems to be quite general. We restricted ourselves to quantum electrodynamics, but very similar results can be obtained in mesodynamics.

There are many formal similarities between our approach and the results of Jouvet,[10] Nambu and Jona-Lasinio,[11] Bjorken,[2] and Freund.[3] It is very likely that all these, seemingly completely different, descriptions are to some extent equivalent.

We do not think that in this paper we have given a proof that the photon is a bound state of an electron-positron pair. We would rather say that our results indicate that an ambiguity exists in the relationship between physical particles and Lagrangians. There may exist several Lagrangians leading to the same set of Feynman diagrams.

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Equilibrium Configuration and Fission Barrier for Liquid Drop Nuclei with High Angular Momentum*

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The equilibrium shape and the fission barrier are calculated for the entire range of angular momenta for which a rotating droplet held together by surface tension has a stable equilibrium. A liquid drop which is originally spherical takes the shape of an oblate spheroid as the angular momentum increases. At higher angular momenta, the shape becomes concave at the poles and a ring form is created. It is shown that the equilibrium ceases to be stable at or near the critical angular momentum at which this change of topology occurs.

I. INTRODUCTION

THIS paper presents a calculation of the equilibrium configurations of rotating liquid drop nuclei and the fission barrier of such drops. The opposing effects of surface tension and centrifugal forces are considered. In this respect the calculation differs from the work of Bohr and Wheeler[1] where nonrotating nuclei were considered and the two opposing effects determining stability were the Coulomb energy and surface tension. The purpose of this paper, in which Coulomb effects are neglected, is to obtain information about the effect of angular momentum on nuclear stability. When the separate effects of angular momentum and Coulomb forces will be known, one might attempt to look at the general case where both effects exist.

The importance of nuclear states with high angular momenta was realized from the results obtained with the heavy-ion accelerators. When uranium is bombarded by 10-MeV oxygen nuclei, states with angular momenta as high as 60 units of $\hbar$ are obtained. It has been found experimentally[2] that the partial width for fission increases with increasing angular momentum.

The properties of a rotating incompressible fluid have been studied and discussed by Plateau,[3] Poincaré,[4] Rayleigh,[5] and Appel.[6] However, none of these papers contain an analytic expression for the equilibrium shape. All of the authors resort to numerical methods at one stage or another. In this paper analytic expressions for the shape of equilibrium are obtained. It is also shown that the topology of the equilibrium configuration changes when a parameter (which will be defined later) assumes the value 2.414. This value is to be compared with 2.32 and 2.4, which are the estimates of Appel and Rayleigh, respectively. The nature of those equilibria with respect to small deformations when the

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