agreement with the present value. The preliminary value was obtained before the rectangular glass cells were fabricated, and was based on spectra obtained with a single glass cell of circular cross section. The circular cell has a filling factor of about $\frac{1}{8}$ and a correspondingly small signal-to-noise ratio.

The present value is in good agreement with the value of 1.54 D obtained by Madden and Benedict[15] from the infrared emission spectrum of an oxy-acetylene flame and estimates of the $OH:H_2O$ concentration ratio. Although the concentration ratio estimates now seem to have been quite accurate, this was not known at the time they were made, and Madden and Benedict were not able to assign an experimental error to their value of dipole moment.

The present value is also in good agreement with the value of 1.65 ± 0.25 D recently obtained by Meyer and Myers, using microwave spectroscopy.[16] They used square-wave Zeeman modulation to facilitate detection and observed the Stark shift due to the application of a dc electric field between two parallel aluminum plates. Their observations were made on the $^2\pi_{3/2}$, $J=9/2$ Λ-doubling lines. The experimental error they quote is somewhat greater than ours, probably due to the fact that their maximum observed Stark shifts were about 1.2 Mc/sec—somewhat smaller than ours (see Fig. 5).

[15] R. P. Madden and W. S. Benedict, J. Chem. Phys. **23**, 408 (1955).

[16] Richard T. Meyer and Rollie J. Myers, J. Chem. Phys. **34**, 1074L (1961).

Meyer and Myers also point out that their experimental results indicate that the OH dipole moment is close to the OH bond moment found in water, 1.53 D, but outside the range of the most complete theoretical calculations, which give values between 2.1 and 2.7 D. This is confirmed by the present results.

V. CONCLUSIONS

The electric dipole moment for OH has been found to be 1.60 ± 0.12 D.

Four main lines (two doublets) of the $O^{17}H$ hyperfine structure for the $^2\pi_{3/2}$, $J=7/2$ Λ-doubling transition have been observed. The doublet splittings are in agreement with doublet splittings computed from hyperfine structure constants obtained from experiments on $O^{16}H$. The separation between the doublets gives a value of $d=-415.3\pm2.0$ Mc/sec, in good agreement with a simple model that assumes one unpaired $p\pi$ electron about the O^{17} nucleus.

ACKNOWLEDGMENTS

I would like to thank Professor C. H. Townes for his valuable advice and encouragement, Harold Lecar for his help in carrying out the $O^{17}H$ experimental runs, the IBM Watson Laboratory for the use of a 650 computer, and the entire staff of the Columbia Radiation Laboratory for a wide range of helpful services.

PHYSICAL REVIEW VOLUME 130, NUMBER 2 15 APRIL 1963

Theory of Cavity Masers

D. E. McCumber
Bell Telephone Laboratories, Murray Hill, New Jersey
(Received 6 August 1962; revised manuscript received 16 January 1963)

Simple interference experiments identify correlation functions which describe the response of maser amplifiers and the output of maser oscillators. General operator techniques are used to evaluate these functions in cavity maser systems for which the coupling of the maser electromagnetic field to external energy sources or sinks is mediated by systems whose short-term response depends primarily upon average or macroscopic system properties. The dielectric theory which results includes only those short-term nonlinearities associated with the external pumping fields. In the long term both the external and the cavity fields can modify the susceptibility functions. The techniques utilized permit a detailed analysis of approximations and suggest generalizations for increased precision. As one important result of this type the theory indicates that macroscopic rate equations are valid if modulation frequencies are much less than the width of the narrowest coupling system spectral line.

I. INTRODUCTION

As refined experimental procedures continue to probe more deeply the mechanisms underlying maser action[1] in spectral regions ranging from microwave to optical frequencies, it has become increasingly more important to scrutinize the simple approximations which underlie the existing mathematical analyses[2] and to develop generalizations which are valid for broad classes of maser systems. In this context the present paper has three basic objectives: to determine un-

[1] J. P. Gordon, H. J. Zeiger, and C. H. Townes, Phys. Rev. **95**, 282 (1954); **99**, 1264 (1955).

[2] A survey of a number of approaches with an extensive annotated bibliography has been given by W. E. Lamb, Jr., in *Lectures in Theoretical Physics II, University of Colorado Summer School 1959*, edited by W. E. Britten, B. W. Downs, and J. Downs (Interscience Publishers, Inc., New York, 1960), p. 435.

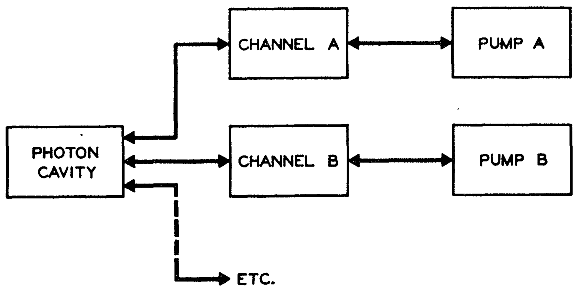

FIG. 1. Schematic diagram of typical maser systems. Direct interchannel interactions have been excluded for simplicity.

ambiguously by the analysis of simple experiments which mathematical functions describe properties of maser systems; to commence a general systematic computation of the functions of interest in terms of universal functions which characterize (perhaps phenomenologically) specific maser components; and to present a number of results which pertain to a particularly important approximation, the dielectric approximation. Our emphasis shall be directed primarily towards the energy spectrum of the free-running maser oscillator and the frequency dependence of the linear response of the maser amplifier. By a simple extension of our steady-state arguments, however, we shall also be able to determine an ultimate upper limit on the range of validity of thermodynamic energy-rate analyses of maser relaxation which do not take into account detailed spectral features.[3] In all cases by focusing our attention on properties of the electromagnetic field rather than on the quantized subsystems which generate that field, we are able to achieve a unity of description which emphasizes similarities between different maser realizations.

The maser systems to be considered consist of a confined electromagnetic photon field dynamically coupled to various external pumps—that is, energy sources or sinks—which drive and dissipate the photon field through different coupling channels. A typical system is illustrated schematically in Fig. 1. In the first (microwave) maser[1] the excitation "channel" consisted of a beam of excited NH_3 molecules which had been selected ("pumped") prior to their traversal of the maser cavity. The cavity walls and the load constituted the dissipation "channels."

The electromagnetic fields generated by maser oscillators are noteworthy for their coherence—that is, for narrowness of the band of frequencies contained in their Fourier decompositions. Similarly, maser amplifiers are noted for their narrow amplification band, for their low effective noise temperature, and for their consequent low total noise power. As we shall establish by the analysis of rudimentary experiments in Sec. II, these spectral features manifest themselves in the Fourier-integral representations of certain photon-amplitude correlation functions. Since the formal structure of the photon functions is independent of the specific details of the coupling channels, they are convenient functions upon which to base a general maser analysis.

To avoid unnecessarily restrictive assumptions about the details of the pump-photon coupling channels, we develop mathematical representations of the photon correlation functions from integrated rather than differential equations of motion. This leads us in a completely rigorous but straightforward fashion to isolate channel correlation functions which embody the relevant features of the pump-coupled channel systems. The most important set of such correlation functions are the analogues for the driven maser system of familiar dielectric or susceptibility functions. Such functions can be parametrized phenomenologically or, if a model for the pump-coupled channel systems is assumed, mathematically. We shall not pursue this latter aspect of maser analysis in this paper.

Steady-state susceptibility functions contain the coupling channel information most relevant to the photon-field analysis when the short-term dielectric response of the channel is a macroscopic property of the channel system. Assuming that the channel susceptibility is indeed independent of the instantaneous photon field strength in the short term—in the long term the field strength affects the channel "populations" and, consequently, the response characteristics—we consider in some detail the calculation of various expressions of relevance to masers: the energy spectrum of the maser oscillator, the frequency dependence of the maser amplifier, and rates of energy transfer between photon and channel systems. It is an interesting result that the free-running maser oscillator is in many respects a noise amplifier. The noise signal results from spontaneous emission of radiation from excited channel states. For passive or dissipative channels that excitation is determined by the usual temperature-dependent population factor, a manifestation of the fact that so-called thermal noise is simply spontaneous-emission noise from systems that are thermally excited.

Following our discussion in Sec. II of rudimentary experiments which serve to identify relevant photon correlation functions, we outline in Sec. III the properties of the statistical ensemble which characterizes the steady-state maser. In Sec. IV we define with somewhat greater precision than in Sec. II the operators which describe the photon field within the maser cavity and indicate a general form for the photon-channel coupling. In the next section we introduce two mathematically convenient photon correlation functions and indicate their connection to the physical observables of Sec. II. We also indicate symmetry properties of these functions and present the first two steps in their computation. In Sec. VI we discuss the factorization leading to the dielectric approximation. In Sec. VII we use this approximation to complete the computation of one of

[3] H. Statz and G. de Mars, in *Quantum Electronics*, edited by C. H. Townes (Columbia University Press, New York, 1960), p. 530.

the photon correlation functions. The other correlation function, since it depends upon unknown ensemble expectation values, cannot conveniently be calculated by the same technique. Instead, we use the stationary property of the ensemble describing steady-state maser operation to establish a rudimentary fluctuation-dissipation theorem by means of which the second correlation function may be computed. In Sec. IX we derive expressions for the rate of energy transfer between photon and channel systems in the dielectric approximation. These are used in Sec. X to draw certain conclusions about thermodynamic energy-balance treatments (rate equations[3]) of masers. In Sec. XI we consider an especially simple two-channel system, indicate typical maser properties, and conclude by illustrating graphically how the output spectrum of a maser oscillator and the response of a maser amplifier vary as the operating conditions of the maser change. In the final section we briefly recapitulate the important features of our presentation.

II. CORRELATION FUNCTIONS FOR THE ELECTROMAGNETIC FIELD

As we have remarked, the distinctive aspects of maser systems manifest themselves in the properties of the electromagnetic (photon) field generated or amplified by the maser. These properties are in turn reflected in correlation functions of photon operators. Those correlation functions which are most relevant can easily be seen if we consider a few simple experiments appropriate to an optical maser.[4]

In Fig. 2(a) we have indicated a simple interference experiment to measure the spectral properties of the output of an optical maser oscillator. The output beam is split by a mirror arrangement into two beams which are again brought together on an interference plane. If $\bar{a}$ is an operator representing the total photon amplitude at the interference plane, then $\bar{a}$ is equal to the sum of the amplitudes from the two different paths. Because the path lengths may be different, those two amplitudes involve the cavity photon amplitudes $a_\lambda(t)$ at different times:

$$\bar{a}(\mathbf{x},t)=\sum_\lambda \left[\alpha_{1\lambda}(\mathbf{x})a_\lambda(t-d_1(\mathbf{x})/c) +\alpha_{2\lambda}(\mathbf{x})a_\lambda(t-d_2(\mathbf{x})/c)\right]. \quad (2.1)$$

As we shall discuss further in Sec. IV, the parameter λ distinguishes the modes of the radiation field within the cavity. For convenience the phases of the coefficients $\alpha_{1\lambda}$, $\alpha_{2\lambda}$ are chosen such that $\alpha_{1\lambda}{}^*\alpha_{2\lambda'}$ is real for $\lambda=\lambda'$.

The detected *intensity* $I(\mathbf{x})$ of the light at the interference plane is proportional to the average number of photons reaching the plane—that is, to the expectation value $\bar{a}^\dagger\bar{a}$ at the point $\mathbf{x}$:

$$I(\mathbf{x})=\langle\bar{a}^\dagger(\mathbf{x},t)\,\bar{a}(\mathbf{x},t)\rangle. \quad (2.2)$$

Since the emission from each cavity mode is characterized by a definite radiation pattern, the interferometer geometry will usually insure that all but a few $\alpha_{i\lambda}$ in (2.1) are zero. If, in fact, only one mode λ is relevant, we find upon substituting (2.1) into (2.2) that

$$I(\mathbf{x})=(|\alpha_{1\lambda}|^2+|\alpha_{2\lambda}|^2)\langle a_\lambda{}^\dagger a_\lambda\rangle+|\alpha_{1\lambda}{}^*\alpha_{2\lambda}| \times\left\langle a_\lambda{}^\dagger\left(\frac{d_1-d_2}{c}\right)a_\lambda+a_\lambda{}^\dagger a\left(\frac{d_1-d_2}{c}\right)\right\rangle. \quad (2.3)$$

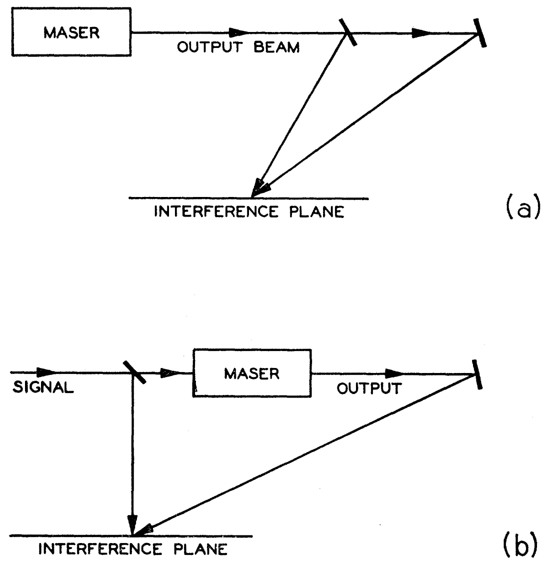

FIG. 2. Schematic diagrams of simple optical interferometers by means of which radiation coherence may be measured. (a) Arrangement for determining the energy spectrum of a free-running optical maser (oscillator). (b) Arrangement for measuring the amplitude response characteristics of a driven optical maser (amplifier). The short solid bars in each diagram represent mirrors.

Except for the known $\mathbf{x}$ dependence of $\alpha_{1\lambda}$ and $\alpha_{2\lambda}$, the first term of (2.3) represents a constant background intensity upon which the physically interesting modulations of the second "cross" term are imposed. That term depends upon the pathlength difference (d_1-d_2), a function of position on the interference plane. Note that the modulation expression is symmetrized: The first term within the expectation value is the Hermitian conjugate of the second, and the two are joined by a plus sign. We shall identify the positive-definite even Fourier transform of that modulation term with the *energy spectrum*[5] $\mathcal{H}_\lambda(\omega)$ of the λ mode of the free-running maser:

$$\mathcal{H}_\lambda(\omega)\equiv\omega_\lambda\int_{-\infty}^{\infty}d\tau\; e^{i\omega\tau}\langle a_\lambda{}^\dagger(t+\tau)a_\lambda(t) +a_\lambda{}^\dagger(t)a_\lambda(t+\tau)\rangle. \quad (2.4\text{a})$$

The normalization is such that

$$\int_0^\infty \frac{d\omega}{2\pi}\mathcal{H}_\lambda(\omega)=\omega_\lambda\langle a_\lambda{}^\dagger a_\lambda\rangle=\langle H_{0\lambda}{}^p\rangle-\tfrac{1}{2}\omega_\lambda, \quad (2.4\text{b})$$

[4] M. Born and E. Wolf, *Principles of Optics* (Pergamon Press, New York, 1959), Chap. X.

[5] N. Wiener, Acta Math. **55**, 117 (1930); S. O. Rice, Bell System Tech. J. **23**, 282 (1944); **24**, 46 (1945).

the total λ-mode cavity energy exclusive of the inaccessible zero-point energy.

A different experimental situation is illustrated in Fig. 2(b). We again have an interference experiment, but only one of the interfering beams is from the maser output. The other derives from a signal which is amplified by the maser. We are measuring in this case the correlation between the input and the output amplitudes of a maser amplifier. The intensity relations are essentially as before: The intensity at any point $\mathbf{x}$ is proportional as in (2.2) to the number of photons reaching that point; the total amplitude is the sum of two beam amplitudes,

$$\bar{a}(\mathbf{x},t)=a_s(t-d_s(\mathbf{x})/c)+\sum_\lambda \alpha_{0\lambda}(\mathbf{x})a_\lambda(t-d_0(\mathbf{x})/c); \quad (2.5)$$

and the resulting expression contains constant terms plus a symmetrized interference term. If we indicate the signal-perturbed maser ensemble by the s-indexed single brackets $\langle\cdots\rangle_s$ and the normalized stationary signal ensemble[5] by the double brackets $\langle\langle\cdots\rangle\rangle$, then

$$I(\mathbf{x})=\langle\langle I_s(\mathbf{x})\rangle\rangle, \quad (2.6a)$$

where

$$I_s(\mathbf{x})=a_s^\dagger(t-d_s/c)a_s(t-d_s/c)+\sum_\lambda |\alpha_{0\lambda}(x)|^2\langle a_\lambda^\dagger a_\lambda\rangle_s +\sum_\lambda \{\alpha_{0\lambda}a_s^\dagger(t-d_s/c)\langle a_\lambda(t-d_0/c)\rangle_s +\alpha_{0\lambda}^*\langle a_\lambda^\dagger(t-d_0/c)\rangle_s a_s(t-d_s/c)\}. \quad (2.6b)$$

It remains to determine how the cavity expectation values $\langle a_\lambda(t)\rangle_s$ are perturbed by the incident signal field.

If the signal is sufficiently weak, so that the response of the amplifier is linear, that expectation value may be represented as the sum of two terms:

$$\langle a_\lambda(t)\rangle_s=\langle a_\lambda(t)\rangle_{s=0}-i\sum_{\lambda'}\int_{-\infty}^{t} dt'\,\langle a_\lambda(t)a_{\lambda'}^\dagger(t') -a_{\lambda'}^\dagger(t')a_\lambda(t)\rangle_{s=0}\beta_{s\lambda'}a_s(t'). \quad (2.7)$$

The first term is the average amplitude when the signal field $a_s(t)$ is absent. In all situations of interest to us here it will be zero. The important contributions to (2.6) will derive from the second or linear-response term of (2.7). Since it is not an object of this paper to analyze the coupling of the electromagnetic field within the maser cavity to that outside the cavity, we shall not discuss in detail the simple but plausible model upon which the specific linear-response term (2.7) is based. For different models that term will take slightly different forms. However, all forms will involve an integration kernel closely related to the function

$$R(\lambda\lambda';t-t')=\frac{1}{i}\langle a_\lambda(t)a_{\lambda'}^\dagger(t')-a_{\lambda'}^\dagger(t')a_\lambda(t)\rangle =\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-t')}r(\lambda\lambda';\omega), \quad (2.8)$$

which appears in (2.7). We shall call the amplitude correlation function (2.8) the *linear-response function* of the maser amplifier. [Observe that the t' integral of the linear-response term of (2.7) is retarded: $t'\leq t$.] The commutator structure displayed in (2.8) is a general feature of linear-response functions.[6–8]

In summary, we have indicated by the analysis of two elementary interference experiments that two functions of paramount importance for the description of masers (however they are constructed) are the elementary amplitude correlation functions (2.4) and (2.8). With little further difficulty one can envisage other experiments in which radiation intensity at one time is correlated with intensity at another time. The relevant correlation functions involve symmetrized and antisymmetrized products of the photon intensity (number) operator $I=a^\dagger a$. For example, if one were to measure with one photosensitive device the λ-mode emission of an optical maser at time t, measure with another similar device the λ′-mode emission at the time t', and pass the sum of the two measurements through a square-law device, one would record the average output

$$\langle[I_\lambda(t)+I_{\lambda'}(t')]^2\rangle=\langle I_\lambda^2\rangle+\langle I_{\lambda'}^2\rangle +\langle I_\lambda(t)I_{\lambda'}(t')+I_{\lambda'}(t')I_\lambda(t)\rangle. \quad (2.9)$$

That most experimental observables can be identified with properties of correlation functions is an important point to note, since such identifications often eliminate interpretive difficulties in other approaches and, moreover, make contact with an extensive body of (field) theory originally developed for other purposes.

III. THE MASER ENSEMBLE

In the preceding section we utilized without definition a normalized single-bracketed expectation value $\langle\cdots\rangle$ based implicitly upon an ensemble of states characterizing the maser system. In this paper we restrict ourselves primarily to steady-state maser operation and do not consider the relaxation transients which accompany the initial preparation of our system. The present state of the systems we treat derives from the continued stationary operation of the various channel pumps over a time interval T commencing in the distant past. The interval T is sufficiently large so that the initial transients have disappeared but not so large that the storage systems supplying power to the various pumps exhibit measurable depletion. With this upper limit upon T implicitly understood, we define in the usual manner for an arbitrary operator $\mathcal{O}$ and a finite

[6] H. B. Callen and T. R. Welton, Phys. Rev. **83**, 34 (1951); H. B. Callen and R. F. Greene, *ibid.* **86**, 702 (1952); H. Ekstein and N. Rostoker, *ibid.* **100**, 1023 (1955); J. Weber, *ibid.* **101**, 1620 (1956).

[7] W. Bernard and H. B. Callen, Rev. Mod. Phys. **31**, 1017 (1959).

[8] R. Kubo, Can. J. Phys. **34**, 1274 (1956); J. Phys. Soc. Japan **12**, 570 (1957); W. Kohn and J. M. Luttinger, Phys. Rev. **108**, 590 (1957); H. Nakano, Progr. Theoret. Phys. (Kyoto) **15**, 77 (1956); M. Lax, Phys. Rev. **109**, 1921 (1958); P. C. Martin and J. Schwinger, *ibid.* **115**, 1342 (1959).

"present" time t

$$\langle \mathcal{O}(t)\rangle = \operatorname*{Av}_{\rho^0} \lim_{T\to\infty} \sum_{jj'} \rho_{jj'}{}^0 \langle j' | \mathcal{O}(t+T) | j\rangle, \qquad (3.1)$$

where ρ^0 is a normalized density matrix which describes the starting ensemble and where the average is over all such admissible ensembles. We assume that the expectation value (3.1) exists and is unique. It is, of course, independent of any specific starting ensemble. It is also independent of any finite t.

We wish to emphasize that the maser ensemble defined by Eq. (3.1) is not normally a thermal equilibrium ensemble. This is an important point because for equilibrium ensembles there exist general fluctuation-dissipation theorems[6,7] which provide mathematical connections between symmetric and antisymmetric correlation functions, only one of which is usually easy to calculate directly. For the maser ensemble such general theorems are not known; however, we shall be able in Sec. VIII to exploit the t independence of (3.1) to derive a limited fluctuation-dissipation theorem applicable to amplitude correlation functions of the *driven* maser ensemble.

IV. THE CAVITY ELECTROMAGNETIC FIELD

The narrow line shapes characteristic of the electromagnetic field in cavity maser systems suggests that we describe the field within the cavity in terms of the denumerable eigenmodes of an ideal resonant cavity of finite volume V rather than in terms of the damped modes of the loaded but undriven cavity. Doing this, we treat those perturbations which damp the photon field and those which drive it on equal footing. We view both as resulting from the interaction of the photon field with the active media of appropriate energy-transfer channels. The frequency-independent mirror losses, for example, associated with the Fabry-Perot etalon of an optical cavity maser can easily be incorporated into such a scheme by the simple expedient of a broadband loss channel.

For the electromagnetic field within the cavity we assume that there exists a complete set of orthogonal normal modes, each indicated by an index λ, say, and each having its characteristic spatial configuration $\mathbf{u}_\lambda(\mathbf{r})$. The observable electric and magnetic fields are representable as a sum over modes:

$$\mathbf{B}(\mathbf{r},t) = \sum_\lambda \nabla\times\mathbf{u}_\lambda(\mathbf{r}) P_\lambda(t);$$

$$\boldsymbol{\varepsilon}(\mathbf{r},t) = -\frac{1}{c}\frac{\partial}{\partial t}\sum_\lambda \mathbf{u}_\lambda(\mathbf{r}) P_\lambda(t) = \sum_\lambda \frac{\omega_\lambda{}^2}{c}\mathbf{u}_\lambda(\mathbf{r}) Q_\lambda(t). \qquad (4.1)$$

In a perfect cavity the time dependence in the Heisenberg representation of the conjugate variables $P_\lambda(t)$ and $Q_\lambda(t)$ is harmonic, a fact which leads one to identify these variables with the momentum and displacement, respectively, of a harmonic oscillator[9]:

$$\begin{aligned} H_0{}^p &= \tfrac{1}{2}\textstyle\sum_\lambda (P_\lambda{}^2 + \omega_\lambda{}^2 Q_\lambda{}^2); \\ P_\lambda(t)^0 &= P_\lambda \cos\omega_\lambda t - \omega_\lambda Q_\lambda \sin\omega_\lambda t; \\ Q_\lambda(t)^0 &= Q_\lambda \cos\omega_\lambda t + (1/\omega_\lambda) P_\lambda \sin\omega_\lambda t. \end{aligned} \qquad (4.2)$$

Quantizing the oscillator in the familiar manner, we have the canonical commutation relations ($\hbar\equiv 1$)

$$[Q_\lambda, Q_{\lambda'}] = [P_\lambda, P_{\lambda'}] = 0, \quad [Q_\lambda, P_{\lambda'}] = i\delta_{\lambda\lambda'}. \qquad (4.3)$$

It is convenient for certain purposes to introduce the linear combinations

$$\begin{aligned} a_\lambda &= \frac{1}{(2\omega_\lambda)^{1/2}}(\omega_\lambda Q_\lambda + iP_\lambda), \\ a_\lambda{}^\dagger &= \frac{1}{(2\omega_\lambda)^{1/2}}(\omega_\lambda Q_\lambda - iP_\lambda), \end{aligned} \qquad (4.4)$$

which are, in fact, the amplitude operators we introduced in analyzing the interference experiments of Sec. II. Their commutation relations follow immediately from Eqs. (4.3):

$$[a_\lambda, a_{\lambda'}] = [a_\lambda{}^\dagger, a_{\lambda'}{}^\dagger] = 0, \quad [a_\lambda, a_{\lambda'}{}^\dagger] = \delta_{\lambda\lambda'}. \qquad (4.5)$$

For the perfect cavity the eigenvalues ω_λ and the eigenvectors $\mathbf{u}_\lambda(\mathbf{r})$ are determined by the Maxwell equations

$$\begin{aligned} \nabla\cdot\boldsymbol{\varepsilon}(\omega_\lambda)\cdot\mathbf{u}_\lambda(\mathbf{r}) &= 0, \\ \nabla\times\boldsymbol{\mu}(\omega_\lambda)^{-1}\cdot[\nabla\times\mathbf{u}_\lambda(\mathbf{r})] - (\omega_\lambda{}^2/c^2)\boldsymbol{\varepsilon}(\omega_\lambda)\cdot\mathbf{u}_\lambda(\mathbf{r}) &= 0, \end{aligned} \qquad (4.6)$$

and by appropriate boundary and normalization conditions. In Eqs. (4.6) $\boldsymbol{\varepsilon}(\omega)$ is a frequency-dependent dielectric tensor and $\boldsymbol{\mu}(\omega)$ a corresponding permeability tensor for the cavity interior exclusive of the active channel systems to be considered shortly.[10] For mathematical simplicity we assume that the real cavity and the perfect cavity [upon which (4.6) is based] have the same spatial boundary conditions so that the eigenvectors $\mathbf{u}_\lambda(\mathbf{r})$ of the perfect cavity form a complete set of basis vectors for the real cavity. Likewise, in order not to obscure our analysis by extraneous geometric complications, we assume that the channel interactions with the radiation field are uniformly distributed throughout the maser cavity and that the eigenvectors of the real maser cavity are identical with the $\mathbf{u}_\lambda(\mathbf{r})$ of the perfect cavity. The precise mathematical meaning of these last statements will become clearer later. They may be applied to cavity wall losses only if the latter are so small that in the modes of interest many "reflections" are required before the undriven field is

[9] P. A. M. Dirac, *Quantum Mechanics* (Clarendon Press, New York, 1947), 3rd ed., Chap. X.

[10] In a ruby maser, for example, the susceptibilities are those appropriate to the Al_2O_3 host lattice; the Cr^{3+} impurities constitute an active channel medium to be treated in further detail.

appreciably attenuated. If $\tau_{w\lambda}$ is the lifetime of the λ mode of the empty cavity when dissipation results only from wall losses (and wall loading) and if D_λ is the characteristic cavity dimension appropriate to the λ mode, the systems for which our assumptions will apply are such that

$$\tau_{w\lambda} \gg D_\lambda / c. \tag{4.7}$$

This condition is clearly met for axial modes in optical masers of the Fabry-Perot type which have end reflectivities $R \approx 1$, since in the Fabry-Perot masers $\tau_{w\lambda} c / D_\lambda \approx (1-R)^{-1}$. The condition (4.7) is also met for lightly-loaded high-Q microwave maser cavities, since in the microwave cavity D_λ is of the order of the radiation wavelength and consequently $\tau_{w\lambda} c / D_\lambda \approx Q_{w\lambda} \gg 1$. The condition is not met by a long traveling-wave maser amplifier for which the end surfaces are only weakly reflecting. It is primarily for this reason that we have restricted ourselves in the title of this paper to cavity masers.

The interaction of the electromagnetic field with a particle source field may be described by the interaction Hamiltonian

$$\begin{aligned} H_I &= H_I^{(1)} + H_I^{(2)} \\ &= \left\{ \frac{1}{c} \int_V (dr) \mathbf{j}(\mathbf{r}) \cdot \mathbf{A}(\mathbf{r}) - g\beta \int_V (dr) \mathbf{S}(\mathbf{r}) \cdot \nabla \times \mathbf{A}(\mathbf{r}) \right\}_{(1)} \\ &\quad + \left\{ \frac{1}{2c} \frac{q}{m} \int_V (dr) \rho(\mathbf{r}) \mathbf{A}(\mathbf{r}) \cdot \mathbf{A}(\mathbf{r}) \right\}_{(2)}, \end{aligned} \tag{4.8}$$

where for the source field $\rho(\mathbf{r})$ is the charge density operator, $\mathbf{j}(\mathbf{r})$ is the charge-current density operator (for $\mathbf{A}=0$), $\mathbf{S}(\mathbf{r})$ is the spin density operator, $g\beta$ is the gyromagnetic conversion factor, and q/m is the particle charge-to-mass ratio. The operator

$$\mathbf{A}(\mathbf{r},t) = \sum_\lambda \mathbf{u}_\lambda(\mathbf{r}) P_\lambda(t) \tag{4.1'}$$

is the vector potential of the cavity electromagnetic field in the Coulomb gauge for which in the absence of a real net charge density the scalar potential vanishes. In maser applications channel systems composed of many weakly interacting identical quantized subsystems excited or de-excited by an external pump are of interest only if the subsystems have large radiative transition probabilities (relative to other decay probabilities) between modes whose energy differences lie in the neighborhood of the relevant cavity frequencies ω_λ. In such cases $H_I^{(2)}$ does not play a fundamental role, and we may include its effects in the "passive" cavity tensors $\boldsymbol{\mu}$, $\boldsymbol{\varepsilon}$ previously introduced. The important components are those of $H_I^{(1)}$ which, by using (4.1'), we may express in the form

$$H_I^{(1)} = \sum_\lambda \gamma_\lambda P_\lambda, \tag{4.9}$$

where the Hermitian operator γ_λ typically has the structure

$$\gamma_\lambda = \frac{1}{c} \int_V (dr) \mathbf{j}(\mathbf{r}) \cdot \mathbf{u}_\lambda(\mathbf{r}) - g\beta \int_V (dr) \mathbf{S}(\mathbf{r}) \cdot \nabla \times \mathbf{u}_\lambda(\mathbf{r}).$$

It measures the projection onto the λ mode of the electromagnetic source densities $\mathbf{j}(\mathbf{r})$, $\mathbf{S}(\mathbf{r})$.

In Raman-effect masers[11] the situation is only slightly different. From one viewpoint Eq. (4.8) obtains essentially as before. It describes the coupling of the cavity field $\mathbf{A}$ with the operators $\mathbf{j}(\mathbf{r})$, $\mathbf{S}(\mathbf{r})$, and $\rho(\mathbf{r})$ of the pump-channel coupling system. As before, the second-order components $H_I^{(2)}$ in the cavity photon fields are negligible and H_I may be written in the form (4.9). The channel system is also coupled by an interaction similar in structure to that in Eq. (4.8) to a pumping or source photon field $\mathbf{A}_s$. The nonlinear interaction of this field with the channel will introduce modifications in the time dependence of γ_λ such that γ_λ will display sum or difference frequency components in the neighborhood of $\omega = \pm\omega_\lambda$ which were not present either in the undriven channel or in the driving field $\mathbf{A}_s$. In a somewhat more phenomenological approach to the Raman effect one would alternatively assume the structure (4.9) and define γ_λ to be an effective source operator containing the photon field of the pump as a factor.

In what follows we shall assume that all photon-channel interactions are of the type (4.9). We may readily demonstrate by combining the interaction (4.9) with the noninteracting Hamiltonian (4.2) that

$$\frac{\partial}{\partial t} Q_\lambda = P_\lambda + \gamma_\lambda, \quad \frac{\partial}{\partial t} P_\lambda = -\omega_\lambda^2 Q_\lambda. \tag{4.10}$$

Integrating these Heisenberg equations of motion, we have

$$\begin{aligned} P_\lambda(t) &= P_\lambda(t)^0 - \omega_\lambda \int_0^t dt' \sin\omega_\lambda(t-t') \gamma_\lambda(t'), \\ Q_\lambda(t) &= Q_\lambda(t)^0 + \int_0^t dt' \cos\omega_\lambda(t-t') \gamma_\lambda(t'). \end{aligned} \tag{4.11}$$

The first term on the right-hand side of each of Eqs. (4.11) has previously been defined in (4.2). Those first terms describe the time evolution of the noninteracting cavity fields; the second terms describe the effect of the photon-channel interactions.

V. THE PHOTON CORRELATION FUNCTIONS $\mathcal{P}_\pm(\tau)$

Rather than directly compute the amplitude correlation functions of Sec. II, we find it somewhat more

[11] A. Javan, Bull. Am. Phys. Soc. **3**, 213 (1958); J. Weber, Rev. Mod. Phys. **61**, 681 (1959); G. Eckhardt, R. W. Hellwarth, F. J. McClung, S. E. Schwarz, D. Weiner, and E. J. Woodbury, Phys. Rev. Letters **9**, 455 (1962).

convenient to compute the anticommutator-commutator functions

$$\mathcal{P}_{\pm}(\lambda\lambda';\tau)=\langle[P_\lambda(t+\tau),P_{\lambda'}(t)]_{\pm}\rangle. \quad (5.1)$$

The algebraic relations (4.4) and the equations of motion (4.10) insure that our results will be experimentally applicable. In fact, if we define the Fourier transforms $p_{\pm}(\omega)$ such that

$$\mathcal{P}_{\pm}(\lambda\lambda';\tau)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}p_{\pm}(\lambda\lambda';\omega)e^{-i\omega\tau}, \quad (5.2)$$

then it follows from Eqs. (2.4), (2.8), (4.4), and (4.10) after some manipulation that

$$\mathcal{H}_\lambda(\omega)=\frac{1}{2\omega_\lambda^2}\{(\omega^2+\omega_\lambda^2)p_+(\lambda\lambda;\omega)-2\omega\omega_\lambda p_-(\lambda\lambda;\omega)\} \quad (5.3a)$$

and

$$r(\lambda\lambda';\omega)=\frac{1}{2i}\frac{(\omega+\omega_\lambda)(\omega+\omega_{\lambda'})}{(\omega_\lambda\omega_{\lambda'})^{3/2}}p_-(\lambda\lambda';\omega). \quad (5.3b)$$

Since the operators P_λ are Hermitian, the anticommutator function $\mathcal{P}_+(\tau)$ is pure real and the commutator function $\mathcal{P}_-(\tau)$ is pure imaginary. If we explicitly restrict ourselves to "finite" times in the sense of Sec. III, then the functions (5.1) are independent of t and

$$\mathcal{P}_{\pm}(\lambda\lambda';\tau)=\pm\mathcal{P}_{\pm}(\lambda'\lambda;-\tau). \quad (5.4)$$

Observe that this symmetry obtains only for the t-independent stationary ensemble.

Substituting the first of the integrated equations of motion (4.11) into (5.1), we obtain

$$\mathcal{P}_{\pm}(\lambda\lambda';\tau)=\mathcal{P}_{\pm}(\lambda\lambda';\tau)^0-\omega_\lambda\int_0^\tau d\tau'\,\sin\omega_\lambda(\tau-\tau')\langle[\gamma_\lambda(\tau'),P_{\lambda'}]_{\pm}\rangle, \quad (5.5)$$

where

$$\mathcal{P}_{\pm}(\lambda\lambda';\tau)^0=\langle[P_\lambda(\tau)^0,P_{\lambda'}]_{\pm}\rangle. \quad (5.6)$$

We could have obtained the same result by using the operator identity

$$\mathcal{O}(t)=\mathcal{O}(t)^0-i\int_0^t dt'\,\{[\mathcal{O}(t-t')^0,H_I]\}(t'), \quad (5.7)$$

valid for an arbitrary operator $\mathcal{O}$. The time dependence of the various terms in (5.7) is governed for explicitly time-independent Hamiltonian operators H_0, H_I, and H (such that $H=H_0+H_I$) by the following conventions [consistent with Eqs. (4.2) and (4.11)] to be used throughout this paper:

$$\mathcal{O}(t)^0=e^{iH_0t}\mathcal{O}e^{-iH_0t};\quad \mathcal{O}(t)=e^{iHt}\mathcal{O}e^{-iHt};$$
$$\{\mathcal{O}(t)^0\}(t')=e^{iHt'}\{e^{iH_0t}\mathcal{O}e^{-iH_0t}\}e^{-iHt'};\ \text{etc.} \quad (5.8)$$

The identity (5.7) can be easily verified by differentiation. It is also valid with the appropriate generalizations of (5.8) when H_0, H_I, and H are explicitly time dependent.[7] In our case H_I is the photon-channel interaction (4.9) and H is the full Hamiltonian of the interacting photon, channel, and pump systems.

Applying (5.7) to the operator $\gamma_\lambda(\tau')$ in the last term of (5.5), we obtain the following integral representation of the important expectation value of that term.

$$\langle[\gamma_\lambda(\tau'),P_{\lambda'}]_{\pm}\rangle=\langle[\gamma_\lambda(\tau')^0,P_{\lambda'}]_{\pm}\rangle-i\sum_{\lambda''}\int_0^{\tau'}d\tau''\times\langle[\{[\gamma_\lambda(\tau'-\tau'')^0,\gamma_{\lambda''}]P_{\lambda''}\}(\tau''),P_{\lambda'}]_{\pm}\rangle. \quad (5.9)$$

Although the solution of appropriate differential equations of motion would provide a satisfactory alternative approach to this expectation value, we have chosen to use integrated equations of motion because they relate somewhat more directly to independently measurable channel correlation functions. Once these functions have been identified and been shown to be generally relevant, one can, if he so desires, assume specific channel models upon which to base further analysis. We wish to emphasize, however, that this subsequent analysis has less to do with maser action *per se* than it does with specific channel properties. Maser action—that is, the amplification of the electromagnetic field by the stimulated emission of radiation—depends in the abstract sense only upon properties of the relevant channel correlation functions and not upon other physical features of the channel systems. In this sense the discrete-level independent-particle model of the coupling channels which has characterized most past analyses is much too restrictive, as has been especially apparent since the discovery of Raman-effect masers.[11]

In both of Eqs. (5.5) and (5.9) the first term on the right-hand side is particularly simple in the commutator (−) case. From Eqs. (4.2) and (4.3) it follows that

$$\mathcal{P}_-(\tau)^0=-i\omega_\lambda\delta_{\lambda\lambda'}\sin\omega_\lambda t. \quad (5.10a)$$

From the intrinsic independence of the uncoupled photon and channel systems it follows that

$$\langle[\gamma_\lambda(\tau')^0,P_{\lambda'}]\rangle=0. \quad (5.10b)$$

The anticommutator case is less simple because the operators within the expectation value do not reduce to numbers as in (5.10) but remain as operators which reflect the structural properties of the expectation value itself.

VI. THE DIELECTRIC APPROXIMATION

As is clear from its definition in Sec. IV, the operator γ_λ is the source term in the Maxwell equations describing the λ-mode component of the cavity photon field. Using the identity (5.7) with $\gamma_\lambda(\tau)$, as we did to derive (5.9), we achieve a natural separation of that source term into a component $\gamma_\lambda(\tau)^0$ which reflects the intrinsic dynamics of the pumped channel systems and a second term which reflects the polarization induced into the channel systems by the photon field[12]:

$$\gamma_\lambda(\tau)=\gamma_\lambda(\tau)^0-i\sum_{\lambda'}\int_0^\tau d\tau'\,[\gamma_\lambda(\tau-\tau')^0,\gamma_{\lambda'}](\tau')\times P_{\lambda'}(\tau'). \quad (6.1)$$

The second term of (6.1) is the operator analog of the dielectric polarization familiar in classical theory. It would have the linear classical form if we could identify the operator $[\gamma_\lambda(\tau-\tau')^0,\gamma_{\lambda'}](\tau')$ with a numerical field-independent susceptibility function $Y(\lambda\lambda';\tau-\tau')_{\tau'}$.

The weak-field case has been extensively discussed in the literature for matter systems in thermal equilibrium.[6-8] If one computes for such systems the linear polarization induced by a weak electromagnetic field, one finds [omitting effects of the $H_I^{(2)}$ term of (4.8) omitted from (4.9)] that the matter susceptibility function is a multiple of

$$Y(\lambda\lambda';\tau-\tau')^0|_{\text{thermal}}=\langle[\gamma_\lambda(\tau-\tau')^0,\gamma_{\lambda'}]\rangle_{\text{th}}^0, \quad (6.2)$$

where $\langle\cdots\rangle_{\text{th}}^0$ is the thermal equilibrium ensemble average appropriate to the matter systems in the absence of photon perturbations.

The operator relation (6.1) is general and independent of the ensemble definition. Approximations must take into account the properties of the ensemble, because an approximation valid in one expectation value is not necessarily valid in another. However, the classical-dielectric thermal equilibrium ensemble and the maser ensemble have two important common features which make the classical equilibrium remarks relevent to the maser discussion. First, both ensembles are time independent or stationary. In both cases we expect the linear polarization induced by a rapidly varying electromagnetic field to be given primarily by a response function which depends only upon the average state of the system and not upon instantaneous microscopic distributions. In such cases the linear response is a macroscopic statistical property of the system. In signalizing the function (6.2), previous authors have recognized that the operator $[\gamma_\lambda(\tau-\tau')^0,\gamma_{\lambda'}](\tau'')$ is dominated by its $(\tau-\tau')$ dependence and that the τ'' dependence enters only as a reflection of slow macroscopic changes in the system. In fact, it is the τ''-independent component which is selected in the expectation value (6.2). (Compare the Hartree treatment of multiparticle systems where one similarly replaces the particle-density operator by its average value.) Second, in both the classical and the maser ensembles it is meaningful to view the matter and radiation subsystems as interacting but physically distinct components of the total system. While the photon-matter interaction may be sufficiently strong so that the energy distribution within the two systems is appreciably perturbed over long periods of time, it is not so strong that the energy is significantly redistributed over the short correlation time (inverse linewidth) of the dielectric function.

These observations suggest that the $\gamma\gamma$ commutator in (6.1) is not appreciably affected by the nearby photon operator $P_{\lambda'}(\tau')$ and that as the first step of a systematic approximation procedure it may be replaced by its ensemble expectation value. However, this procedure is not fully satisfactory, as is apparent if we consider its application to the function $\mathcal{P}_-(\tau)$. We remarked in Sec. V that the function $\mathcal{P}_-(\tau)$ is pure imaginary and that it has the symmetry property (5.4). A naive average-value treatment of the $\gamma\gamma$ commutator will preserve the pure-imaginary character of $\mathcal{P}_-(\tau)$, but it will violate the symmetry property (5.4). Since the symmetry property is closely related to the time (t) invariance of the expression (5.1) and since we must exploit that time invariance to compute the symmetric function $\mathcal{P}_+(\tau)$, it seems reasonable to insist that at each step our approximations preserve the reality properties, the symmetry properties, and the time-invariance properties. This will be the case if we replace the operator combinations

$$[\gamma_\lambda(\tau+t)^0,\gamma_{\lambda'}(\tau'+t)^0]_\pm(t'), \quad (6.3)$$

wherever they appear in our equations by numerical functions $Y_\pm(\lambda\lambda';\tau-\tau')$ independent of t, t' and having the same reality and symmetry properties as $\mathcal{P}_\pm(\tau)$, respectively. [The anticommutator (+) function is not needed for (5.9) or (6.1), but it will appear in our subsequent treatment of $\mathcal{P}_+(\tau)$.] While these properties do obtain for the classical function (6.2) and its symmetric analog, the symmetry and invariance properties do not generally obtain for the nonequilibrium (driven) maser expectation values of the operators (6.3). For this reason we define

$$Y_\pm(\lambda\lambda';\tau)\equiv\langle[\gamma_\lambda(\tau/2)^0,\gamma_{\lambda'}(-\tau/2)^0]_\pm\rangle, \quad (6.4)$$

$$=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}y_\pm(\lambda\lambda';\omega)e^{-i\omega\tau}, \quad (6.5a)$$

with

$$y_\pm(\lambda\lambda';\omega)=y_\pm(\lambda\lambda';\omega)^*=\pm y_\pm(\lambda'\lambda;-\omega) \quad (6.5b)$$

and use these functions as the basis of a systematic dielectric-constant approximation.

Relative to making future systematic corrections to the present analysis, let us note somewhat more carefully that (6.4) derives from the following two distinct approximations of (6.3):

$$[\gamma_\lambda(\tau+t)^0,\gamma_{\lambda'}(\tau'+t)^0]_\pm(t')$$
$$\approx\left[\gamma_\lambda\left(\frac{\tau-\tau'}{2}\right)^0,\gamma_{\lambda'}\left(\frac{\tau'-\tau}{2}\right)^0\right]_\pm\left(\frac{\tau+\tau'}{2}+t+t'\right) \quad (6.6a)$$
$$\approx\left\langle\left[\gamma_\lambda\left(\frac{\tau-\tau'}{2}\right)^0,\gamma_{\lambda'}\left(\frac{\tau'-\tau}{2}\right)^0\right]_\pm\right\rangle. \quad (6.6b)$$

[12] In writing $[\gamma(\tau-\tau')^0,\gamma]_\pm(\tau')$ for $\{[\gamma(\tau-\tau')^0,\gamma]_\pm\}(\tau')$, we have eliminated a superfluous bracket from the (5.8) notation.

In (6.6a) we have assumed that, although the dynamics over the differential interval $(\tau-\tau')$ may be governed by $H_0=H-H_I$, the over-all time dependence of the commutator will be governed by H itself. Physically this is the statement that, while "internal" susceptibility dynamics may involve primarily H_0, the value of the susceptibility at any instant $[t+t'+(\tau+\tau')/2]$ will be governed by the state of the system at that instant. In (6.6b) we have additionally assumed that the susceptibility depends only upon the instantaneous average state of the system and that it is not influenced by or correlated to other operators which may appear in the expectation values under examination. If the ensemble with which we are concerned is not stationary—for example, if the operating conditions of the masers (and hence its representative ensemble) were functions of time—it would still be reasonable to use an approximation of the type (6.6b) provided the expectation value derived from the ensemble describing the maser at the instant $[t+t'+(\tau+\tau')/2]$. [This procedure is admissible only if the "state of the system" changes slowly over the intrinsic correlation interval (inverse line width) $0\leq|\tau-\tau'|\lesssim T_2$ of the dielectric function (6.6b).] Our dielectric approximation would then involve an explicitly time-dependent susceptibility. We shall not be concerned with such a situation in this paper except briefly in Sec. X.

Let us reiterate certain of the physical implications of the dielectric approximation. In making the approximation, we assume that the polarization induced into the channel systems by a cavity photon field is nominally a linear function of that field over a relatively short interval in the immediate past. The (retarded) proportionality function (susceptibility) is independent of the cavity field except insofar as that field modifies the average channel populations over relatively long periods; it depends only upon the instantaneous average state of the composite system; and it is not influenced by correlation effects (except insofar as they might be represented by an explicit time dependence in the susceptibility functions). Since the susceptibility depends only upon the average present state of the channel, a dielectric theory implicitly assumes that the channel systems are not notably coherent.[13] Coherent channel systems[14] may be properly treated only if one includes the higher order perturbations induced into the channels by the cavity photon field.

The susceptibility functions (6.4) do include in their intrinsic dynamics the complete pump-channel coupling. They are, therefore, capable of describing Raman processes by either of the mechanisms we mentioned in Sec. IV. They also can describe the nonlinear frequency mixing[15] of pump fields which would be relevant in second-harmonic masers. They do *not* include nonlinear cavity-photon frequency-mixing effects, since we have not included second or higher order cavity-photon corrections to γ_λ dynamics. Only first-order effects are included in our polarization terms.

Used with Eqs. (5.5), (5.9), and (5.10), the dielectric approximation gives

$$\mathcal{P}_-(\lambda\lambda';\tau)=-i\omega_\lambda\delta_{\lambda\lambda'}\sin\omega_\lambda\tau+i\sum_{\lambda''}\int_0^\tau d\tau'\int_0^{\tau'}d\tau''\times\omega_\lambda\sin\omega_\lambda(\tau-\tau')Y_-(\lambda\lambda';\tau'-\tau'')\mathcal{P}_-(\lambda''\lambda';\tau''). \quad (6.7)$$

Since it contains in its inhomogeneous terms unknown ensemble expectation values, the corresponding equation for $\mathcal{P}_+(\tau)$ is only of limited usefulness. One may more usefully return to (5.1) and take both the t and τ development into account. Using a straightforward extension of the previous $\mathcal{P}_-(\tau)$ techniques, we find that

$$\begin{aligned}\mathcal{P}_\pm(\lambda\lambda';\tau)=\sum_{\lambda''\lambda'''}\int_0^{t+\tau}dt''\int_0^{t}dt'''\,Z(\lambda\lambda'';\tau+t-t'')Z(\lambda'''\lambda';t'''-t)\Big[\{\langle[P_{\lambda''}(t'')^0,P_{\lambda'''}(t''')^0]_\pm\rangle\}_{(1)}\\ -\Big\{\int_0^{t''}d\bar{t}''\,\omega_{\lambda''}\sin\omega_{\lambda''}(t''-\bar{t}'')\langle[\gamma_{\lambda''}(\bar{t}'')^0,P_{\lambda'''}(t''')^0]_\pm\rangle\\ -\int_0^{t'''}d\bar{t}'''\,\langle[P_{\lambda''}(t'')^0,\gamma_{\lambda'''}(\bar{t}''')^0]_\pm\rangle\omega_{\lambda'''}\sin\omega_{\lambda'''}(\bar{t}'''-\bar{t}''')\Big\}_{(2)}\\ -\Big\{\int_0^{t''}d\bar{t}''\int_0^{t'''}d\bar{t}'''\,\omega_{\lambda''}\sin\omega_{\lambda''}(t''-\bar{t}'')Y_\pm(\lambda''\lambda''';\bar{t}''-\bar{t}''')\omega_{\lambda'''}\sin\omega_{\lambda'''}(\bar{t}'''-t''')\Big\}_{(3)}\Big], \quad (6.8)\end{aligned}$$

where $Z(\tau)$ is defined by the linear equation

$$Z(\lambda\lambda';\tau)=\delta_{\lambda\lambda'}\delta(|\tau|-0^+)+i\sum_{\lambda''}\int_0^\tau d\tau'\int_0^{\tau'}d\tau''\times\omega_\lambda\sin\omega_\lambda(\tau-\tau')Y_-(\lambda\lambda'';\tau'-\tau'')Z(\lambda''\lambda';\tau''). \quad (6.9)$$

The terms of (6.8) have been numbered to facilitate their separate discussion in Sec. VIII. The function

[13] Our implicitly incoherent dielectric theory is to be contrasted with the following maser analyses in which strong coherence plays an important role: S. Bloom, J. Appl. Phys. **27**, 785 (1956); I. R. Senitzky, Phys. Rev. Letters **1**, 167 (1958); J. R. Singer and S. Wang, *ibid.* **6**, 351 (1961); Y. Pao, J. Opt. Soc. Am. **52**, 871 (1961).

[14] S. H. Autler and C. H. Townes, Phys. Rev. **100**, 703 (1955); P. W. Anderson, J. Appl. Phys. **28**, 1049 (1957); A. M. Clogston, J. Phys. Chem. Solids **4**, 271 (1958); A. Javan, Phys. Rev. **107**, 1579 (1959).

[15] P. A. Franken, A. E. Hill, C. W. Peters, and G. Weinreich, Phys. Rev. Letters **7**, 118 (1961); D. A. Kleinman, Phys. Rev. **126**, 1977 (1962); J. A. Armstrong, N. Bloembergen, J. Ducuing, and P. S. Pershan, *ibid.* **127**, 1918 (1962); N. Bloembergen and P. S. Pershan, *ibid.* **128**, 606 (1962).

$Z(\tau)$ is such that the relation

$$\mathcal{P}_-(\lambda\lambda';\tau)=\sum_{\lambda''}\int_0^\tau d\tau'\, Z(\lambda\lambda'';\tau-\tau')\mathcal{P}_-(\tau')^0 \quad (6.10)$$

is equivalent to (6.7).[16]

In making the dielectric approximation, we have reduced the problem of computing $\mathcal{P}_\pm(\tau)$ to that of solving the linear integral equation (6.9) for $Z(\tau)$. To this extent we have linearized the maser analysis. However, it is important to note that even with this simple first-order approximation we have retained what are probably the most important nonlinearities of the maser system, the nonlinearities which govern the steady-state populations in the coupled channel systems. These populations reflect themselves in the characteristics[17] of the susceptibility spectral functions $y_\pm(\lambda\lambda';\omega)$.

VII. $\mathcal{P}_-(\tau)$ IN THE DIELECTRIC APPROXIMATION

Once the susceptibility functions $Y_\pm(\tau)$ have been determined, the problem of calculating $\mathcal{P}_\pm(\tau)$ reduces principally to the problem of solving the linear integral equation (6.10) for $Z(\tau)$. That equation is of a type conveniently treated by Laplace transformation. Since the symmetry properties of the various functions are known, we may restrict ourselves to $\tau>0$. Designating Laplace transforms by boldface characters (not to be confused with spatial vectors), we define those transforms for $\mathrm{Re}s>0$ by the typical equation[18]

$$\mathbf{P}_\pm(s)=\int_0^\infty d\tau' e^{-s\tau}\mathcal{P}_\pm(\tau). \quad (7.1a)$$

The corresponding inverse transformations are

$$\mathcal{P}_\pm(\tau)=\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i}e^{s\tau}\mathbf{P}_\pm(s). \quad (7.1b)$$

In terms of Laplace transforms Eqs. (6.9) and (6.10) become, respectively,

$$\mathbf{Z}(\lambda\lambda';s)=\delta_{\lambda\lambda'}+i\sum_{\lambda''}\frac{\omega_\lambda^2}{s^2+\omega_\lambda^2}\mathbf{Y}_-(\lambda\lambda'';s)\mathbf{Z}(\lambda''\lambda';s) \quad (7.2)$$

and

$$\mathbf{P}_-(\lambda\lambda';s)=-i\frac{\omega_\lambda^2}{s^2+\omega_\lambda^2}\mathbf{Z}(\lambda\lambda';s) =-i\mathbf{Z}(\lambda\lambda';s)\frac{\omega_{\lambda'}^2}{s^2+\omega_{\lambda'}^2}. \quad (7.3)$$

Equation (7.2) represents in essence a linear eigenvalue problem. The eigenvalues are reflected in the s-plane singularities of $\mathbf{Z}(\lambda\lambda';s)$ and the eigenvectors in the weights to be assigned those singularities. In order not to obscure our discussion by extraneous geometric considerations, we simplify the eigenvector problem by the uniformity assumption anticipated in Sec. IV. We assume that the spatial eigenvectors $\mathbf{u}_\lambda(\mathbf{r})$ of the electromagnetic field of the noninteracting cavity are also eigenvectors of the dielectric cavity. That is, we assume that

$$Y_\pm(\lambda\lambda';\tau)=Y_\pm(\lambda;\tau)\delta_{\lambda\lambda'} \quad (7.4a)$$

or, equivalently, that

$$y_\pm(\lambda\lambda';\omega)=y_\pm(\lambda;\omega)\delta_{\lambda\lambda'}. \quad (7.4b)$$

For $\mathrm{Re}s>0$ integral representations of the relevant dielectric-function Laplace transforms are

$$\mathbf{Y}_+(\lambda;s)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}y_+(\lambda;\omega)\frac{s}{s^2+\omega^2}, \quad (7.5a)$$

$$\mathbf{Y}_-(\lambda;s)=-i\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}y_-(\lambda;\omega)\frac{\omega}{s^2+\omega^2}. \quad (7.5b)$$

Using the assumption (7.4) with (7.2), we may immediately solve the simplified eigenvector problem to obtain

$$\mathbf{Z}(\lambda;s)=\frac{s^2+\omega_\lambda^2}{s^2+\omega_\lambda^2-i\omega_\lambda^2\mathbf{Y}_-(\lambda;s)}. \quad (7.6)$$

Used with (7.3), this gives

$$\mathbf{P}_-(\lambda;s)=-i\frac{\omega_\lambda^2}{s^2+\omega_\lambda^2-i\omega_\lambda^2\mathbf{Y}_-(\lambda;s)}. \quad (7.7)$$

The s-plane singularities of (7.7) determine the time behavior of the function $\mathcal{P}_-(\tau)$. Of particular importance in maser systems are two complex-conjugate poles $s_\pm$ which we represent in terms of two positive parameters Γ_λ, $\bar{\omega}_\lambda$:

$$s_\pm=-\tfrac{1}{2}\Gamma_\lambda\pm i\bar{\omega}_\lambda. \quad (7.8)$$

These poles lie in the left half-plane close to the imaginary axis and dominate the important coherence properties of the maser. They correspond to the poles $s=\pm i\omega_\lambda$ in the uncoupled function $\mathbf{P}_-(s)^0$. When the spectral functions $y_-(\lambda;\omega)$ vary slowly with ω in the neighborhood of $\bar{\omega}_\lambda$, the parameters Γ_λ, $\bar{\omega}_\lambda$ may readily be computed by a successive-approximation solution of the equations

$$\bar{\omega}_\lambda^2=\omega_\lambda^2+\tfrac{1}{4}\Gamma_\lambda^2-\omega_\lambda^2\left\{\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}y_-(\lambda;\omega)\frac{\omega-\bar{\omega}_\lambda}{(\omega-\bar{\omega}_\lambda)^2+\frac{1}{4}\Gamma_\lambda^2} +\mathrm{Im}\,y_-(\lambda;\bar{\omega}_\lambda+\tfrac{1}{2}i\Gamma_\lambda)\right\}\xrightarrow[\Gamma_\lambda\to 0]{}\omega_\lambda^2\left\{1-\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}y_-(\lambda;\omega)\frac{P}{\omega-\bar{\omega}_\lambda}\right\}, \quad (7.9a)$$

[16] Although derived by rather different arguments, equations similar to (6.7) and (6.9) have been considered by J. Schwinger, J. Math. Phys. **2**, 407 (1961); I. R. Senitzky, Phys. Rev. **119**, 670 (1960); **124**, 642 (1961).
[17] W. R. Bennett, Jr., Phys. Rev. **126**, 580 (1962).
[18] D. V. Widder, *The Laplace Transform* (Princeton University Press, Princeton, New Jersey, 1946).

$$\Gamma_\lambda=-\frac{\omega_\lambda^2}{\bar\omega_\lambda}\left\{\frac{1}{2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}y_-(\lambda;\omega)\frac{\Gamma_\lambda}{(\omega-\bar\omega_\lambda)^2+\frac{1}{4}\Gamma_\lambda^2}-\mathrm{Re}y_-(\lambda;\bar\omega_\lambda+\tfrac{1}{2}i\Gamma_\lambda)\right\}\xrightarrow[\Gamma_\lambda\to 0]{}\frac{\omega_\lambda^2}{2\bar\omega_\lambda}y_-(\lambda;\bar\omega_\lambda). \tag{7.9b}$$

The residues of the poles (7.8) in $\mathbf{P}_-(s)$ are

$$a_\pm=\frac{\omega_\lambda^2}{2is_\pm}\left\{1+\frac{\omega_\lambda^2}{2s_\pm}\left[\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\partial y_-}{\partial\omega}(\lambda;\omega)\frac{\frac{1}{2}\Gamma_\lambda\pm i(\omega-\bar\omega_\lambda)}{(\omega-\bar\omega_\lambda)^2+\frac{1}{4}\Gamma_\lambda^2}-\frac{\partial y_-}{\partial\bar\omega_\lambda}(\lambda;\bar\omega_\lambda\pm\tfrac{1}{2}i\Gamma_\lambda)\right]\right\}^{-1}$$

$$\xrightarrow[\Gamma_\lambda\to 0]{}\frac{\omega_\lambda^2}{2is_\pm}\left\{1+\frac{\omega_\lambda^2}{2s_\pm}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\partial y_-}{\partial\omega}(\lambda;\omega)\frac{P}{\omega-\bar\omega_\lambda}\mp i\frac{\omega_\lambda^2}{4\bar\omega_\lambda}\frac{\partial y_-}{\partial\bar\omega_\lambda}(\lambda;\bar\omega_\lambda)\right\}^{-1}. \tag{7.9c}$$

These equations are of fundamental importance for the analysis of steady-state maser operation, since they determine the position $\bar\omega_\lambda$, the width Γ_λ, and the strength $a_\pm$ of the narrow Lorentz peak which dominates the spectral functions of masers.

In addition to the poles (7.8), the function $\mathbf{P}_-(s)$ will display other singularities which reflect the detailed structure of the function $\mathbf{Y}_-(s)$. Although we shall not treat these components in detail, we can easily verify their existence by observing that the exact sum rule or moment relation [derived by using Eqs. (4.3) and (4.10) with the definitions (5.1), (5.2), and (7.1)],

$$\omega_\lambda^2=i\frac{\partial}{\partial\tau}\mathcal{P}_-(\tau)\Big|_{\tau=0}=i\lim_{s\to 0}s^2\mathbf{P}_-(\lambda;s)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\omega p_-(\lambda;\omega), \tag{7.10}$$

is not completely saturated by the pole residues (7.9c). Briefly, the importance of the various additional singularities in $\mathbf{P}_-(s)$ depends upon the strength of the channel-photon coupling and upon the extent to which the coupled cavity responds to driving frequencies different from the characteristic photon frequencies ω_λ. This is perhaps more clearly shown in the following representation of the spectral function $p_-(\lambda;\omega)$. Combining the definition (5.2) with relations of the present section, we have that with $\epsilon=0^+$

$$\mathbf{P}_\pm(\lambda;-i\omega+\epsilon)=\int_{-\infty}^{\infty}\frac{d\bar\omega}{2\pi}\frac{1}{\epsilon-i(\omega-\bar\omega)}p_\pm(\lambda;\bar\omega)=\tfrac{1}{2}p_\pm(\lambda;\omega)+i\int_{-\infty}^{\infty}\frac{d\bar\omega}{2\pi}\frac{P}{\omega-\bar\omega}p_\pm(\lambda;\bar\omega). \tag{7.11}$$

Using the dielectric-approximation expression (7.7) for $\mathbf{P}_-(s)$ and the spectral representation (7.5b) for $\mathbf{Y}_-(s)$, we find from this result that

$$p_-(\lambda;\omega)=2\,\mathrm{Re}\mathbf{P}_-(\lambda;-i\omega+\epsilon)=\omega_\lambda^4[\omega\epsilon/\omega_\lambda^2+y_-(\lambda;\omega)]\Big/\left[\left(\omega^2-\omega_\lambda^2-\omega_\lambda^2\int_{-\infty}^{\infty}\frac{d\bar\omega}{2\pi}\bar\omega y_-(\lambda;\bar\omega)\frac{P}{\omega^2-\bar\omega^2}\right)^2+\frac{\omega_\lambda^4}{4}\left(\frac{\omega\epsilon}{\omega_\lambda^2}+y(\lambda;\omega)\right)^2\right]. \tag{7.12}$$

The corresponding expression in the absence of photon/channel coupling is

$$p_-(\lambda;\omega)^0=\frac{\omega_\lambda^2\omega\epsilon}{(\omega^2-\omega_\lambda^2)^2+(\omega\epsilon)^2/4}=\tfrac{1}{2}\omega_\lambda[\delta(\omega-\omega_\lambda)-\delta(\omega+\omega_\lambda)]. \tag{7.13}$$

In general, $p_-(\lambda;\omega)$ will display several maxima whose locations along the real ω axis and whose magnitudes will govern the properties of $\mathcal{P}_-(\tau)$. If we write ($\epsilon=0^+$)

$$p_-(\lambda;\omega)=\omega^2\frac{[\omega\epsilon+\omega_\lambda^2 8\pi\chi''(\lambda;\omega)]}{\{\omega^2-\omega_\lambda^2[1+4\pi\chi'(\lambda;\omega)]\}^2+\frac{1}{4}[\omega\epsilon+\omega_\lambda^2 8\pi\chi''(\lambda;\omega)]^2}, \tag{7.14}$$

we may distinguish two important general classes of maxima: (1) those associated with minima (or zeros) of $[\omega^2-\omega_\lambda^2(1+4\pi\chi')]$ in regions where $\omega_\lambda^2\chi''(\lambda;\omega)$ is small and slowly varying and (2) those associated with maxima of $\omega_\lambda^2\chi''(\lambda;\omega)$ in regions where $[\omega^2-\omega_\lambda^2\times(1+4\pi\chi')]$ is slowly varying and greater in magnitude than $\omega_\lambda^2\chi''(\lambda;\omega)$. The δ-function maxima of (7.13) are of type (1), as are their interacting analogs, the Lorentzian maxima characterized by $\bar\omega_\lambda$ and Γ_λ, when the iteration solution of (7.9) is most suitable—that is, when the function $\omega_\lambda^2\chi''(\lambda;\omega)$ is slowly varying in the neighborhood of $\omega=\bar\omega_\lambda$.

Clearly the two classes of maxima we have cited do not cover all contingencies. In particular, if both denominator terms of (7.14) vary comparably in the neighborhood of a particular maximum, that maximum does not belong to one of these simple classes. Physically this corresponds to a situation in which two oscillators (for example, the λ-mode photon field and one of the coupling channels) are so strongly coupled that the effect of one on the other cannot simply be viewed as a small perturbation of the original oscillator frequencies and damping coefficients, the physical meaning of the iterative solution of Eq. (7.9). Instead, both oscillators must be treated as a coupled unit and the resulting quadratic secular equation solved exactly. A second exceptional case which deserves special mention is that for which $\chi''(\lambda;\omega)$ displays *rapid* variations at one or more points in the neighborhood of a (relatively broad) minimum of $[\omega^2-\omega_\lambda^2(1+4\pi\chi')]$. This would occur, for example, if narrow holes were burned into the spectral function $y_-(\lambda;\omega)$ as a result of strong maser action.[17] As is clear from (7.14), these modulations will be reflected into the function $p_-(\lambda;\omega)\sim 1/\chi''(\lambda;\omega)$ in the region of interest. This second special case is of the general type (1) above except that the maximum does not have the Lorentz shape characteristic of an ω-independent $\chi''(\lambda;\omega)$.

Finally, it is important to understand that these remarks are, in fact, not peculiar to masers. They are well-known aspects of dielectric (or linear-response) theory, as our deliberate susceptibility ($\chi=\chi'+i\chi''$) notation (7.14) was meant to emphasize.

VIII. $\mathcal{P}_+(\tau)$ IN THE DIELECTRIC APPROXIMATION; A FLUCTUATION/DISSIPATION THEOREM

It is easy to verify—an expansion of $Z(\tau)$ in powers of the susceptibility function $Y_-(\tau)$ is sufficient—that the expression (6.8) appropriate to the function $\mathcal{P}_-(\lambda;\tau)$ is independent of t, consistent with the stationary property of the maser ensemble. The t independence of the anticommutator function $\mathcal{P}_+(\lambda;\tau)$ is less easily established, because the unknown inhomogeneous terms of (6.8) reflect the character of the ensemble and enter fundamentally into the invariance proof. However, if we assume that the right-hand side of Eq. (6.8) is independent of t, we may use that invariance to determine certain of the unknown expectation values by passing to the limit[19] $t\to\infty$ with τ fixed.

Using Eqs. (4.2) and (6.10), we may readily establish that the t dependence of term (1) of (6.8) is governed by factors of the form

$$\int_0^t dt''\, Z(\lambda;t-t'')\times\begin{Bmatrix}\cos\omega_\lambda t''\\ \sin\omega_\lambda t''\end{Bmatrix}\sim\mathcal{P}_-(\lambda;t)\underset{t\to\infty}{\longrightarrow}0. \tag{8.1}$$

Similar remarks apply to the t' components of term (1) and to the factors derived from $P_{\lambda''}(t'')^0$ and $P_{\lambda'''}(t''')^0$ in term (2). It follows that in the large-t limit only the terms (3) remain.

$$\mathcal{P}_\pm(\lambda;\tau)=\lim_{t\to\infty}-\int_0^{t+\tau}dt'\int_0^t dt''\,\mathcal{P}_-(\lambda;t')\mathcal{P}_-(\lambda;t'')Y_\pm(\lambda;t+\tau-t'-t+t''),$$

$$=-\int_0^\infty dt'dt''\,\mathcal{P}_-(\lambda;t')\mathcal{P}_-(\lambda;t'')Y_\pm(\lambda;\tau-t'+t''). \tag{8.2}$$

Using the spectral functions $p_\pm(\lambda;\omega)$ defined by (5.2) and the line-shape functions $y_\pm(\lambda;\omega)$ introduced in (6.5), we find directly from (8.2) that

$$p_\pm(\lambda;\omega)=y_\pm(\lambda;\omega)\left\{[\tfrac{1}{2}p_-(\lambda;\omega)]^2+\left[\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\frac{P}{\omega-\omega'}p_-(\lambda;\omega')\right]^2\right\}. \tag{8.3}$$

It is not practicable to determine the function $p_-(\lambda;\omega)$ by solving the nonlinear equation (8.3).[20] The importance of Eq. (8.3) derives instead from the fact that it implies

$$\frac{p_+(\lambda;\omega)}{p_-(\lambda;\omega)}=\frac{y_+(\lambda;\omega)}{y_-(\lambda;\omega)}, \tag{8.4}$$

an expression which may be used to establish $p_+(\lambda;\omega)$ once $p_-(\lambda;\omega)$ has been computed from (7.12). [The expression (7.12) for $p_-(\lambda;\omega)$ is consistent with (8.3).]

It is interesting to note in passing the physical significance of the fact that all terms of Eq. (6.8) vanish as $t\to\infty$ except the term (3) retained in (8.2). Briefly, factors of the type (8.1) vanish because they refer to a system component (the λ radiation mode) having a finite number (two) of degrees of freedom. The nonvanishing term (3) contains in its $Y_+(\tau)$ factor a reference to the pump-coupled channel systems which in our spectral treatment (7.5) implicitly have an infinite

[19] Cf., the remarks of Sec. III.

[20] Since Eqs. (8.3) do not display an explicit ω_λ dependence, their solution is clearly not unique without additional restrictions such as that provided by the sum rule (7.10).

(continuum) number of degrees of freedom, a familiar aspect of relaxation theory. Our particular example also shows that the steady-state expectation values are independent of the initial distribution of energy among any finite number of degrees of freedom [terms (1) and (2) in (6.8)] and depend only upon the energy characteristics of the continuum degrees of freedom [term (3)]. Our procedure treats the latter degrees of freedom as if they were an infinite reservoir with which the finite systems are in contact, a realistic assumption since the "pumps" presumably do contain very large reservoirs.

The equality (8.4) of the two anticommutator-commutator ratios p_+/p_- and y_+/y_- constitutes a rudimentary fluctuation-dissipation theorem.[6] If the stationary ensemble used to define our correlation functions was that appropriate to a thermal equilibrium of temperature T, then the Fourier transforms $p_\pm(\omega)$ and $y_\pm(\omega)$ would have the ratios

$$\frac{p_+(\lambda\lambda';\omega)}{p_-(\lambda\lambda';\omega)}=\frac{y_+(\lambda\lambda';\omega)}{y_-(\lambda\lambda';\omega)}=\coth\frac{\omega}{2kT}=2[(e^{\omega/kT}-1)^{-1}+\tfrac{1}{2}]. \quad (8.5)$$

The last line is a general thermal-equilibrium result and obtains for all anticommutator-commutator spectral-function ratios not otherwise modified by chemical potentials. For the driven (nonequilibrium) stationary maser ensemble no such general statement obtains. Although we do have the limited relation (8.4), the further identification of those ratios with a universal thermal weight function is not generally possible.

Using (8.4) with (7.12) in (5.3a), one may easily establish that the maser oscillator spectrum in the dielectric approximation is

$$\mathcal{K}_\lambda(\omega)=\{\omega_\lambda^2[(\omega^2+\omega_\lambda^2)y_+(\lambda;\omega)-2\omega\omega_\lambda y_-(\lambda;\omega)]/2\}\Bigg/\left\{\left(\omega^2-\omega_\lambda^2-\omega_\lambda^2\int_{-\infty}^{\infty}\frac{d\bar{\omega}}{2\pi}\bar{\omega}y_-(\lambda;\bar{\omega})\frac{P}{\omega^2-\bar{\omega}^2}\right)^2+\frac{\omega_\lambda^4}{4}[y_-(\lambda;\omega)]^2\right\}. \quad (8.6)$$

The corresponding expression for the response function (5.3b) is

$$r(\lambda;\omega)=[\omega_\lambda y_-(\lambda;\omega)(\omega+\omega_\lambda)^2/2i]\Bigg/\left[\left(\omega^2-\omega_\lambda^2-\omega_\lambda^2\int_{-\infty}^{\infty}\frac{d\bar{\omega}}{2\pi}\bar{\omega}y_-(\lambda;\bar{\omega})\frac{P}{\omega^2-\bar{\omega}^2}\right)^2+\frac{\omega_\lambda^2}{4}[y_-(\lambda;\omega)]^2\right]. \quad (8.7)$$

If for $\omega>0$ all significant components of the spectrum $\mathcal{K}_\lambda(\omega)$ lie in a narrow interval about $\omega=\omega_\lambda$, it follows from (8.6) and (8.7) that

$$\mathcal{K}_\lambda(\omega)\approx\frac{y_+(\lambda;\omega)-y_-(\lambda;\omega)}{y_-(\lambda;\omega)}\frac{i\omega_\lambda}{2}r(\lambda;\omega). \quad (8.8)$$

IX. ENERGY TRANSFER BETWEEN PHOTON AND CHANNEL SYSTEMS

The rate of energy transfer between the cavity electromagnetic field and the various channel systems is also a feature of importance in the analysis of maser systems. If we assume that the various channel systems are uncoupled except through their interaction with the photons in the maser cavity, the operator γ_λ in the photon-channel interaction (4.9) will contain a sum ($\sum_c$) over the independent channels,

$$H_I=\sum_c\sum_\lambda\gamma_{c\lambda}P_\lambda. \quad (9.1)$$

The dielectric spectral functions $y_\pm(\lambda;\omega)$ will display a similar sum structure,

$$y_\pm(\lambda;\omega)=\sum_c y_\pm(\lambda;\omega)_c, \quad (9.2)$$

if to eliminate trivial complications we make the reasonable assumption that $\langle\gamma_{c\lambda}\rangle=0$ for each channel c. (That is, we assume that the channels do not display a net static current or moment.)

The energy intrinsic to the cavity photon field is described by the Hamiltonian H_0^p of (4.2). The rate at which energy flows into the λ mode of the photon cavity from the c channel as a result of the interaction (9.1) is described by the operator

$$\mathcal{R}_{c\lambda}=i[\gamma_{c\lambda}P_\lambda,H_0^p]=\omega_\lambda^2\gamma_{c\lambda}Q_\lambda. \quad (9.3)$$

In the stationary maser ensemble the average λ-mode energy must be a constant of the motion. The net rate of energy transfer to any photon mode must be zero:

$$0=\sum_c\langle\mathcal{R}_{c\lambda}\rangle=\omega_\lambda^2\sum_c\langle\gamma_{c\lambda}Q_\lambda\rangle=\omega_\lambda^2\langle\gamma_\lambda Q_\lambda\rangle. \quad (9.4)$$

To evaluate the stationary expectation value of the rate operator (9.3), we can utilize a time-independence technique similar to that employed in the preceding section. Doing that and making the now familiar dielectric approximation, we find that

$$\begin{aligned}\langle\mathcal{R}_{c\lambda}\rangle&=\lim_{t\to\infty}\omega_\lambda^2\langle\gamma_{c\lambda}(t)Q_\lambda(t)\rangle\\&=\frac{i}{2}\int_0^\infty dt\left\{Y_+(\lambda;t)_c\frac{\partial}{\partial t}\mathcal{P}_-(\lambda;t)+Y_-(\lambda;t)_c\frac{\partial}{\partial t}\mathcal{P}_+(\lambda;t)\right\}\\&=\frac{1}{4}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\omega\{p_-(\lambda;\omega)y_+(\lambda;\omega)_c-p_+(\lambda;\omega)y_-(\lambda;\omega)_c\}.\end{aligned} \quad (9.5)$$

Summing the right-hand side of (9.5b) over the different channels c and using (8.4), we verify immediately that the time-invariance restriction (9.4) obtains. In the stationary thermal equilibrium ensemble for which the ratio (8.5) applies to the separate functions $y_{\pm}(\lambda;\omega)_c$, it follows that $\langle \mathcal{R}_{c\lambda}\rangle=0$ for each c, λ.

Equations (9.4) and (9.5) are of immense practical importance since they determine through the implicit saturation properties of the channel systems the detailed steady-state properties of the functions $Y_{\pm}(\lambda;\tau)$.

X. PUMP MODULATION

The central feature of the dielectric approximation is the replacement of the operator combinations (6.3) by numerical functions $Y_{\pm}(\tau)$ related in Eq. (6.4) to steady-state expectation values. As we noted, this approximation is valid if the rates of energy transfer between photon and channel systems is not too large and if the channel correlation properties are nominally time independent. In maser systems of interest the power-transfer restriction is not usually serious. However, the time-independence assumption does significantly limit the usefulness of our results. Postponing to subsequent works any rigorous generalization of the present analysis, we, nevertheless, indicate one relatively simple extension of physical importance.

Let us first consider a system in which the pumps are adiabatically modulated over time intervals much greater than any correlation time in $\mathcal{P}_{\pm}(\lambda;\tau)$ or $Y_{\pm}(\lambda;\tau)_c$. For this case the ensemble of Sec. III is not completely time independent but changes slowly. We are led, therefore, to append a macroscopic-time index t to the ensemble expectation value $\langle\cdots\rangle_t$ and consequently to the correlation functions $\mathcal{P}_{\pm}(\lambda;\tau)_t$, $Y_{\pm}(\lambda;\tau)_t$. All of the results of Secs. VII–IX will obtain as before, provided only that we include the new t index.

Of particular interest is the rate equation (9.5) which governs the macroscopic energy dynamics of the maser system. That equation becomes

$$\begin{aligned}\langle \mathcal{R}_{c\lambda}\rangle_t&=\frac{1}{4}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\omega\{p_-(\lambda;\omega)_t y_+(\lambda;\omega)_{c,t}\\&\qquad -p_+(\lambda;\omega)_t y_-(\lambda;\omega)_{c,t}\},\\&=\int_0^{\infty}\frac{d\omega}{2\pi}\omega\left\{\left[\tfrac{1}{2}y_+(\lambda;\omega)_{c,t}-\frac{\omega\omega_\lambda}{\omega^2+\omega_\lambda^2}y_-(\lambda;\omega)_{c,t}\right]\right.\\&\qquad\left.\times p_-(\lambda;\omega)_t-\frac{\omega_\lambda^2}{\omega^2+\omega_\lambda^2}y_-(\lambda;\omega)_{c,t}\mathcal{H}_\lambda(\omega)_t\right\},\end{aligned}\qquad(10.1)$$

where we have used (5.3) to replace $p_+(\lambda;\omega)_t$ by the energy spectral function $\mathcal{H}_\lambda(\omega)_t$. If $y_{\pm}(\lambda;\omega)_t$ are smooth, slowly varying functions of ω and if $p_{\pm}(\lambda;\omega)_t$ are each dominated for $\omega\geq 0$ by a single sharp maximum centered at $\omega=\bar{\omega}_\lambda$, it follows from (10.1) and the normalization conditions (2.4) and (7.10) that

$$\begin{aligned}\frac{\partial}{\partial t}\bar{n}_\lambda(t)&\equiv\frac{\partial}{\partial t}\langle n_\lambda\rangle_t=\frac{1}{\omega_\lambda}\sum_c\langle \mathcal{R}_{c\lambda}\rangle_t,\\&=\tfrac{1}{4}\omega_\lambda[y_+(\lambda;\bar{\omega}_\lambda)_t-y_-(\lambda;\bar{\omega}_\lambda)_t]\\&\qquad-\tfrac{1}{2}\omega_\lambda y_-(\lambda;\bar{\omega})_t\bar{n}_\lambda(t).\end{aligned}\qquad(10.2)$$

The $\omega_\lambda(y_+-y_-)/4$ term in (10.2) represents spontaneous emission, the $\omega_\lambda\bar{n}_\lambda y_-/2$ term the net rate of cavity photon dissipation (absorption less stimulated emission).

The adiabatic procedure underlying (10.2) is strictly legitimate (but relatively uninteresting) only if the pump variations are very slow, slow relative even to the long maser coherence time $(\Gamma_\lambda)^{-1}$ calculable from (7.9). However, when in the steady-state situation the spectral functions $p_{\pm}(\lambda;\omega)$ would be sharply peaked about a single maximum at $\omega=\omega_\lambda$, the specific result (10.2) is more generally valid. It is valid whenever the pump variations are slow relative to the much shorter time $(T_{2c})_{\max}$. This new limiting time is the maximum over the different channels c of the coherence time T_{2c} of the susceptibility functions $Y_{\pm}(\lambda;\tau)_c$. The generalization derives from the plausible assumption that the pump modulations will primarily distort the sharp $p_{\pm}(\lambda;\omega)$ maxima at $\omega=\bar{\omega}_\lambda$ by spreading them (through sideband generation) over a frequency interval on the order of the modulation frequency. Since the sum rules (2.4) and (7.10) fix the area under the spectral curves, Eqs. (10.2) will not be sensitive to these modulation distortions as long as $y_{\pm}(\lambda;\omega)_t$ do not change appreciably as a function of ω in the sideband neighborhood of $\bar{\omega}_\lambda$.

These conclusions are important since Eq. (10.2) underlies most thermodynamic analyses of starting transients and of "spiking.[3]" These phenomena involve rapid system changes which clearly cannot be described as slow in our original restricted sense (adiabatic).

XI. AN ILLUSTRATIVE TWO-CHANNEL MASER

To illustrate the application of the preceding formalism to a concrete system, we consider in this section a simple maser having two channels: an activating a channel which will excite cavity photons and a dissipative d channel which will absorb photons. We shall not attempt to calculate the functions $y_{\pm}(\lambda;\omega)_c$ from physical models but shall assume simply that these functions have known elementary forms.

We assume that the pump (sink)-coupled d channel is nominally a two-level system characterized by an equilibrium ensemble of temperature T_d and by a broad Lorentz line shape. Specifically, we assume consistent with (6.5) and (8.5) that

$$y_+(\lambda;\omega)_d=\Omega_{d\lambda}[g(\omega)_d+g(-\omega)_d],\qquad(11.1a)$$

$$y_-(\lambda;\omega)_d=\Omega_{d\lambda}\left[g(\omega)_d\tanh\frac{\omega}{2kT_d}-g(-\omega)_d\tanh\frac{(-\omega)}{2kT_d}\right],$$

$$=y_+(\lambda;\omega)_d\tanh\frac{\omega}{2kT_d},\qquad(11.1\text{b})$$

with $\Omega_{d\lambda}\geq 0$ and

$$g(\omega)_d=\frac{\Delta\nu_d}{(\omega-\epsilon_d)^2+\frac{1}{4}(\Delta\nu_d)^2}.\qquad(11.1\text{c})$$

The parameter $\Omega_{d\lambda}$ is a photon-channel coupling constant having the dimensions of energy (or angular frequency since $\hbar\equiv 1$).

Similarly, we assume that the pump (source)-coupled a channel is nominally a two-level system characterized by a (somewhat narrower, $\Delta\nu_a\ll\Delta\nu_d$) Lorentz line shape and by suitable statistical parameters. To express the fact that the a channel will be excited by the action of the pump, we supplement the positive channel temperature T_a by a chemical potential $\mu_{a\lambda}$.[21] The appropriate analogs of Eqs. (11.1) are

$$y_+(\lambda;\omega)_a=\Omega_{a\lambda}[g(\omega)_a+g(-\omega)_a],\qquad(11.2\text{a})$$

$$y_-(\lambda;\omega)_a=\Omega_{a\lambda}\left[g(\omega)_a\tanh\frac{\omega-\mu_{a\lambda}}{2kT_a}-g(-\omega)_a\tanh\frac{-\omega-\mu_{a\lambda}}{2kT_a}\right],\qquad(11.2\text{b})$$

with $\Omega_{a\lambda}\geq 0$ and

$$g(\omega)_a=\frac{\Delta\nu_a}{(\omega-\epsilon_a)^2+\frac{1}{4}(\Delta\nu_a)^2}.\qquad(11.2\text{c})$$

Although the y_+/y_- connection in the presence of the chemical potential $\mu_{a\lambda}$ is different from that in (8.5), the analogy between (11.2b) and the first of Eqs. (11.1b) is obvious. The plausibility of (11.2) can be verified by a statistical analysis[8] of $Y_\pm(\tau)$ in which we put $\gamma=\rho_{12}+\rho_{21}$ [with $\rho_{\mu\mu'}$ the $(\mu\mu')$ element of the channel-system density operator] and in which the chemical potential $\mu_{a\lambda}$ acts through the population difference $(\rho_{22}-\rho_{11})$.

If a particular channel system c, $c=a$ or d, is composed of many independent identical subsystems, the number of such subsystems will appear as a factor in the coupling constant $\Omega_{c\lambda}$. Both $g(\pm\omega)_c$ components of the function $y_+(\lambda;\omega)_c$ are proportional to the total number of subsystems in the c channel, whereas the important resonant component of $y_-(\lambda;\omega)_c$ is proportional to the number of subsystems in the lower state less the number in the upper state. For $\omega>0$ this latter population difference is governed by the factor $\tanh(\omega-\mu_{c\lambda})/2kT_c$ and is, therefore, mainly a function of the energy difference ϵ_c, of the chemical potential $\mu_{c\lambda}$, and of the temperature T_c. In discussing Eq. (10.2), we remarked that the term $\omega_\lambda(y_+-y_-)/4$ describes stimulated emission. This may be verified here by observing that the resonant contribution of that term—the resonant component is the only one of importance in maser systems—is proportional to the population of the upper channel state:

$$\Gamma_{c\lambda}{}^s(\omega)\equiv[\tfrac{1}{4}\omega_\lambda(y_+-y_-)]_{\text{res}}(\lambda;\omega)_c \underset{\omega>0}{=}\frac{1}{2}\Omega_{c\lambda}\omega_\lambda\frac{e^{-(\omega-\mu_{c\lambda})/kT_c}}{1+e^{-(\omega-\mu_{c\lambda})/kT_c}}g(\omega)_c.\qquad(11.3)$$

Note that from the present viewpoint spontaneous-emission noise and thermal noise are equivalent. Thermal noise is noise which steams from the spontaneous emission of systems which are thermally excited.[22]

Taking $y_\pm(\lambda;\omega)$ to be the sum (9.2) of the separate functions (11.1) and (11.2), we can immediately evaluate the expressions considered in previous sections. We restrict ourselves to a single photon mode λ such that ω_λ lies near the line centers ϵ_a, ϵ_d. (The results are particularly relevant to microwave masers where cavity mode isolation is most effective.) Using the $\Gamma_\lambda\to 0$ forms of Eqs. (7.9) and neglecting the nonresonant components of $y_\pm(\lambda;\omega)$, we find that the dominant coherent component of the $p_\pm(\lambda;\omega)$ spectra has the linewidth

$$\Gamma_\lambda=\frac{\omega_\lambda{}^2}{2\bar{\omega}_\lambda}\left\{\Omega_{d\lambda}g(\bar{\omega}_\lambda)_d\tanh\frac{\bar{\omega}_\lambda}{2kT_d}+\Omega_{a\lambda}g(\bar{\omega}_\lambda)_a\tanh\frac{\bar{\omega}_\lambda-\mu_{a\lambda}}{2kT_a}\right\},\qquad(11.4\text{a})$$

[21] Certain authors prefer to utilize an artificial "negative temperature" in place of a chemical potential. While this technique is satisfactory in systems having a few sharp energy levels, its introduction is, in fact, not necessary and, moreover, inadmissible in more general circumstances. W. A. Barker, Phys. Rev. **124**, 124 (1961), discusses three-level masers in terms of chemical potentials and positive temperatures.

[22] In the preceding analysis, we implicitly assumed that all relevant photons are entirely contained within the maser cavity, which also contains all active channel systems (uniformly distributed, cf., Secs. IV and VII). If the maser cavity is coupled through imperfect walls to another cavity nearly in thermal equilibrium, spontaneous emission from the quantized coupling-channel subsystems will be supplemented by "shot noise" resulting from the random penetration into the maser cavity of real (not "zero-point") external photons. The probability for such penetration clearly depends upon the number of such photons present in the external cavity—that is, to $1/[\exp(\hbar\bar{\omega}_\lambda/kT)-1]\to kT/\hbar\bar{\omega}_\lambda$ as $T\to\infty$—and accounts for the kT dependence of the linewidth computed in reference 1. If the external temperature is low ($kT<\hbar\bar{\omega}_\lambda$, as in most optical masers) or if the external-cavity coupling is small, spontaneous emission will dominate, as we have assumed above. Cf., the discussion of noise by J. Weber, Rev. Mod. Phys. **31**, 681 (1959), and the references contained therein.

and the position

$$\bar{\omega}_\lambda=\omega_\lambda\left\{1+\Omega_{d\lambda}\frac{\bar{\omega}_\lambda-\epsilon_d}{(\bar{\omega}_\lambda-\epsilon_d)^2+\frac{1}{4}(\Delta\nu_d)^2}\tanh\frac{\bar{\omega}_\lambda}{2kT_d}\right.$$
$$\left.+\Omega_{a\lambda}\frac{\bar{\omega}_\lambda-\epsilon_a}{(\bar{\omega}_\lambda-\epsilon_a)^2+\frac{1}{4}(\Delta\nu_a)^2}\tanh\frac{\bar{\omega}_\lambda-\mu_{a\lambda}}{2kT_a}\right\}^{1/2}. \quad (11.4\text{b})$$

The case of most interest is that for which $\mu_{a\lambda}>\bar{\omega}_\lambda$. (In our two-level system pumping implies $\mu_{a\lambda}>0$; population inversion requires the stronger condition $\mu_{a\lambda}>\bar{\omega}_\lambda$.) In that case $\tanh[(\bar{\omega}_\lambda-\mu_{a\lambda})/2kT_a]<0$ and Γ_λ is reduced below its empty-cavity value.

When the linewidth of the damping channel is large ($\Delta\nu_d\gg\bar{\omega}_\lambda-\epsilon_d$) so that the damping rate is insensitive to small changes in $\bar{\omega}_\lambda$ and when Γ_λ and $(\bar{\omega}_\lambda-\omega_\lambda)$ are very small relative to ω_λ, Eqs. (11.4) simplify to

$$\Gamma_\lambda=\Gamma_{d\lambda}+\Gamma_{a\lambda}(\bar{\omega}_\lambda), \quad (11.5\text{a})$$

$$\bar{\omega}_\lambda=\omega_\lambda+\frac{\Gamma_{d\lambda}}{\Delta\nu_d}(\bar{\omega}_\lambda-\epsilon_d)+\frac{\Gamma_{a\lambda}(\bar{\omega}_\lambda)}{\Delta\nu_a}(\bar{\omega}_\lambda-\epsilon_a)$$
$$=\left(\omega-\epsilon_d\frac{\Gamma_{d\lambda}}{\Delta\nu_d}-\epsilon_a\frac{\Gamma_{a\lambda}(\bar{\omega}_\lambda)}{\Delta\nu_a}\right)\Big/\left(1-\frac{\Gamma_{d\lambda}}{\Delta\nu_d}-\frac{\Gamma_{a\lambda}(\bar{\omega}_\lambda)}{\Delta\nu_a}\right), \quad (11.5\text{b})$$

where

$$\Gamma_{d\lambda}=\frac{2\omega_\lambda}{Q_{d\lambda}}=\frac{2\omega_\lambda\Omega_{d\lambda}}{\Delta\nu_d}\tanh\frac{\omega_\lambda}{2kT_d} \quad (11.6)$$

measures the intrinsic damping rate of the d-coupled cavity and where

$$\Gamma_{a\lambda}(\bar{\omega}_\lambda)=\frac{1}{2}\omega_\lambda\Omega_{a\lambda}g(\bar{\omega}_\lambda)_a\tanh\frac{\bar{\omega}_\lambda-\mu_{a\lambda}}{2kT_a} \quad (11.7\text{a})$$

measures the strength of the exciting-channel-photon coupling. Equations (11.5) are already familiar in the literature of maser theory.[1,2] The factor $\tanh[(\omega-\mu_{a\lambda})/2kT_a]$ usually varies negligibly over the line $g(\omega)_a$. In that case it is convenient to introduce the nominal maximum.

$$\Gamma_{a\lambda}(\max)\equiv\left|\tfrac{1}{2}\omega_\lambda\Omega_{a\lambda}g(\epsilon_a)_a\tanh\frac{\epsilon_a-\mu_{a\lambda}}{2kT_a}\right|$$
$$=\left|\frac{2\omega_\lambda\Omega_{a\lambda}}{\Delta\nu_a}\tanh\frac{\epsilon_a-\mu_{a\lambda}}{2kT_a}\right| \quad (11.7\text{b})$$

to facilitate the presentation (at the end of this section) of numerical results.

If for $\omega>0$ the functions $p_\pm(\lambda;\omega)$ are dominated by a single sharp peak having the characteristics (11.5), it follows from (10.2) and (11.3) that in the steady state

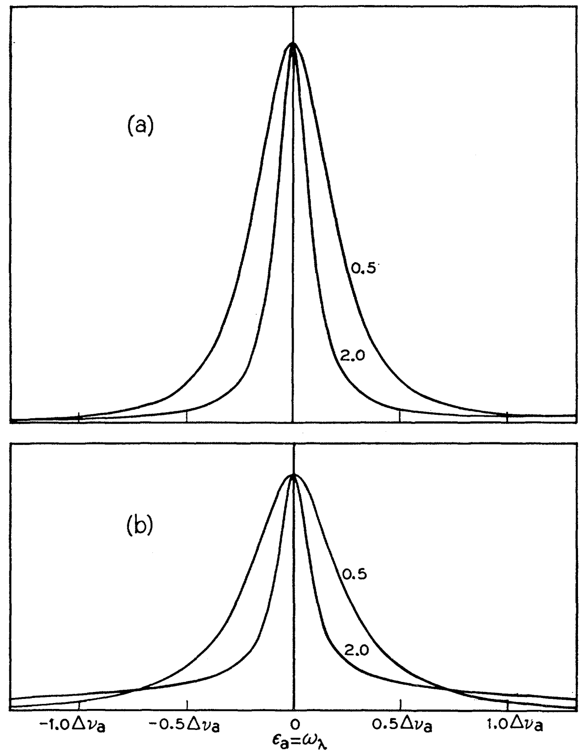

FIG. 3. Oscillator output spectra (a) and amplifier response characteristic (b) for $\Gamma_d=0.5\ \Delta\nu_a$ and $2.0\ \Delta\nu_a$ with $\epsilon_a=\omega_\lambda$ and $\Gamma_a(\max)=\Gamma_d-0.5\ \Delta\nu_a$, $\mu_{a\lambda}>\epsilon_a$.

($d\bar{n}_\lambda/dt=0$)

$$\bar{n}_\lambda=\frac{y_+(\lambda;\bar{\omega}_\lambda)-y_-(\lambda;\bar{\omega}_\lambda)}{y_-(\lambda;\bar{\omega}_\lambda)},$$
$$=(1/\Gamma_\lambda)\{\Gamma_{d\lambda}{}^s(\bar{\omega}_\lambda)+\Gamma_{a\lambda}{}^s(\bar{\omega}_\lambda)\}. \quad (11.8)$$

That is, the steady-state cavity population $\bar{n}_\lambda$ is determined by the net rate (Γ_λ) at which spontaneous emission [$\Gamma_{c\lambda}{}^s$, compare Eq. (11.3)] is dissipated.[22] Using this result with (9.5), we find by the methods used to derive (10.2) that the net steady-state rate of energy transfer from channel a into photon mode λ is

$$\langle\mathcal{R}_{a\lambda}\rangle=\tfrac{1}{4}\omega_\lambda{}^2[y_+(\lambda;\bar{\omega}_\lambda)_a-y_-(\lambda;\bar{\omega}_\lambda)_a]-\tfrac{1}{2}\omega_\lambda{}^2y_-(\lambda;\bar{\omega}_\lambda)_a\bar{n}_\lambda,$$
$$\approx\omega_\lambda\{\Gamma_{a\lambda}{}^s(\bar{\omega}_\lambda)-\Gamma_{a\lambda}(\bar{\omega}_\lambda)\bar{n}_\lambda\}. \quad (11.9)$$

(Recall that for a pumped channel $\Gamma_{a\lambda}<0$.) When maser action starts, Γ_λ decreases, $\bar{n}_\lambda$ increases, and the energy transfer rate (11.9) increases. The ultimate limit to this process—that is, steady-state saturation—is determined by the rate at which the a-channel pump supplies energy to the channel. Given that pumping rate (a function of T_a and $\mu_{a\lambda}$) and the rate at which absorbed photon energy can be removed from the d channel (a function of T_d), we can determine from Eqs. (11.8) and (11.9) the steady-state values of T_a,

T_d, $\mu_{a\lambda}$, Γ_λ, and $\bar{n}_\lambda$. Although more complicated in its details, the procedure is basically the same when the shape of the spectral functions $g(\omega)_c$ in (11.1) and (11.2) is also a function of the maser operating level.

If $\Gamma_{d\lambda} \ll \Delta\nu_a$, the functions $p_\pm(\lambda;\omega)$ are dominated by a Lorentz line component having the characteristic parameters (11.5). If $\Gamma_{d\lambda} \gtrsim \Delta\nu_a$, the a-channel coupling necessary to achieve maser action must be so strong that the two denominator terms in (7.14) will vary comparably in the neighborhood of the $\omega \approx \omega_\lambda$ maximum. As we have already noted in Sec. VII, this will cause the coherent line (7.9) or (11.5) to be accompanied in $p_-(\lambda;\omega)$ by less coherent components having a frequency spread $\gtrsim \Delta\nu_a$. These less coherent components will also appear in $p_+(\lambda;\omega)$ and in $\mathcal{H}_\lambda(\omega)$ but with considerably reduced amplitude, a reflection of the peaked character near $\omega = \bar{\omega}_\lambda$ of the ratio (8.4).

We may illustrate these and other features most easily by a few specific numerical examples for which we can compute the energy spectrum $\mathcal{H}_\lambda(\omega)$ of (8.6) and the linear-response function $r(\lambda;\omega)$ of (8.7). In each case we assume that

$$\mu_{a\lambda} > \epsilon_d = \epsilon_a \approx \omega_\lambda \gg \Delta\nu_d \gg \Delta\nu_a \qquad (11.10)$$

and consider only the frequency region near $\omega = \epsilon_a$. The results are shown in Figs. 3 to 6.

The horizontal scale in each figure is in units of $\Delta\nu_a$, the Lorentz width of the active-channel functions

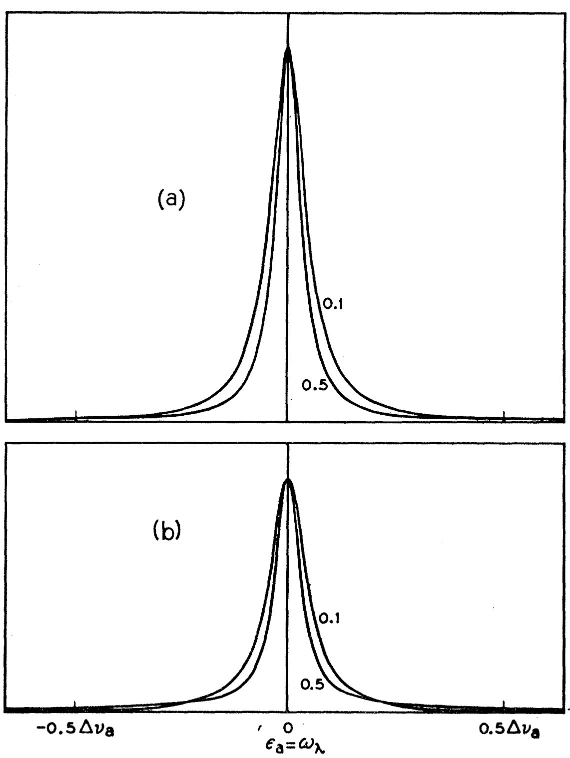

FIG. 4. Oscillator output spectra (a) and amplifier response characteristic (b) for $\Gamma_d = 0.1\ \Delta\nu_a$ and $0.5\ \Delta\nu_a$ with $\epsilon_a = \omega_\lambda$ and $\Gamma_a(\max) = \Gamma_d - 0.1\ \Delta\nu_a$, $\mu_{a\lambda} > \epsilon_a$.

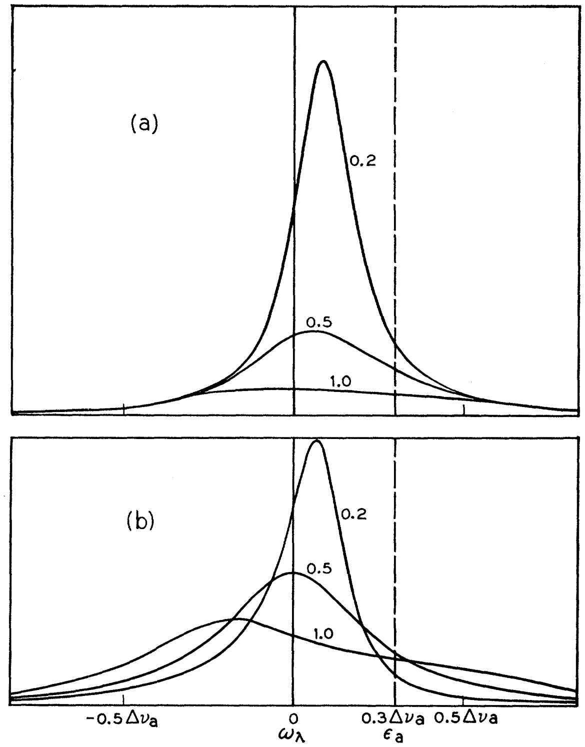

FIG. 5. Oscillator output spectra (a) and amplifier response characteristic (b) for $\Gamma_a(\max) = \Gamma_d - 0.2\ \Delta\nu_a$, $\Gamma_d - 0.5\ \Delta\nu_a$, and $\Gamma_d - 1.0\ \Delta\nu_a$ with $\epsilon_a = \omega_\lambda + 0.3\ \Delta\nu_a$ and $\Gamma_d = \Delta\nu_a$. In each case $\mu_{a\lambda} > \epsilon_a$.

(11.2). The vertical scales in the different figures are unrelated; each has been chosen such that for clarity the curves are as large as possible; but within each part of each figure all curves have been drawn to the same scale. In each figure the upper curves (a) represent the oscillator output spectrum $\mathcal{H}_\lambda(\omega)$; the lower curves (b) represent the amplifier response function $r(\lambda;\omega)$.

Figures 3 and 4 illustrate how for Γ_d comparable to $\Delta\nu_a$ changes in Γ_d modify the spectral functions. Observe in Fig. 3(a) with $\Gamma_d \gtrsim \Delta\nu_a$ that for the same peak intensity the width and total power (area under curve) of the oscillator output decreases as Γ_d increases. Since $\Gamma_a(\max)$, defined in (11.7b), must increase as Γ_d increases in order to keep the peak intensity fixed, the output efficiency of a maser oscillator clearly decreases as Γ_d increases. In Fig. 4(a) with $\Gamma_d \lesssim \Delta\nu_a$ the same general remarks apply but quantitatively the Γ_d changes are less significant. That is, for fixed peak intensity the output is less sensitive to Γ_d as Γ_d decreases below $\Delta\nu_a$.

An important feature to note in the response spectrum of Fig. 3(b) is the broad plateau extending to each side of the sharp central peak. This plateau increases in breadth as Γ_d increases. Physically it reflects the fact that a heavily damped (Γ_d large) cavity is not highly selective in the absence (e.g., off resonance) of active-channel effects. (Since $\Delta\nu_d \gg \Delta\nu_a$, the parameter Γ_d is not frequency sensitive for $\omega \approx \omega_\lambda$.) Figures 5 and 6,

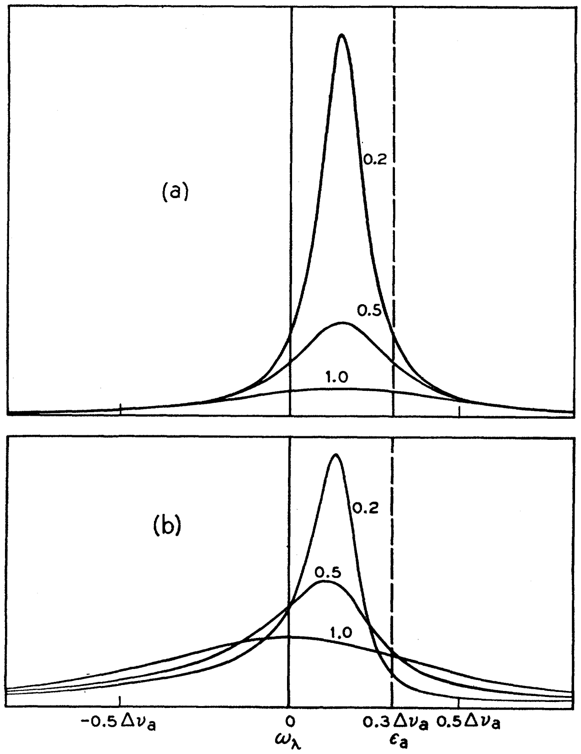

FIG. 6. Oscillator output spectra (a) and amplifier response characteristic (b) for $\Gamma_a(\max)=\Gamma_d-0.2\ \Delta\nu_a$, $\Gamma_d-0.5\ \Delta\nu_a$, and $\Gamma_d-1.0\ \Delta\nu_a$ with $\epsilon_a=\omega_\lambda+0.3\ \Delta\nu_a$ and $\Gamma_d=0.5\ \Delta\nu_a$. In each case $\mu_{a\lambda}>\epsilon_a$.

which illustrate frequency-pulling effects appropriate to $\epsilon_a\neq\omega_\lambda$, are in qualitative agreement with Eq. (11.5b). Figures 5 and 6 also illustrate how the response and output functions vary with $\Gamma_a(\max)$ when ϵ_a and Γ_d are held fixed.

In the limit $\Gamma_d/\Delta\nu_a\rightarrow 0$ the only aspect of these spectra of importance will be the sharp central Lorentz peak associated with the parameters (11.4) or (11.5). The Γ_d sensitivity apparent in Figs. 3 and 4, the response plateau apparent in Fig. 3, and the line-shape asymmetry apparent in Figs. 5 and 6, all become inconsequential by comparison.

XII. REMARKS

In the preceding sections we indicated by the analysis of rudimentary experiments that maser qualities of experimental interest can often be unambiguously expressed in terms of simple correlation functions. These functions are of such a type that they may be studied by general operator techniques of considerable power originally developed for the solution of relativistic and, subsequently, multiparticle nonrelativistic problems. Using methods of this type, we were led to a simple dielectric approximation whose general structure relative to the photon field of the maser was essentially independent of the specific maser realization, although much of our vocabulary was for definiteness oriented toward pump-photon coupling channels consisting of many identical ions or atoms with quantized energy levels.

In the dielectric approximation the properties of the coupling channels manifest themselves in a pair of channel correlation (susceptibility) functions. In order for the dielectric approximation to be valid it is necessary that the linear differential response of the channel systems be a macroscopic (average) characteristic of those systems independent in the short term of the strength of the photon field. In the long term the electromagnetic field affects the channel populations and indirectly the susceptibility functions. Nonlinear saturation, hole-burning, and channel cross-relaxation effects are already implicit in our analysis, since they affect the long-term (average) channel properties.

By formal mathematical expressions and by simple numerical examples we indicated how the response of a steady-state maser amplifier and the output of a steady-state maser oscillator depend upon the coupling-channel susceptibility functions. We also determined in terms of channel and photon spectral functions the rate at which energy is transferred between the photon and channel systems. As an important generalization of the steady-state theory we indicated that spectrally insensitive macroscopic rate equations[3] derived from the steady-state energy-transfer expressions are more generally valid.