Theory of Low-Energy Nucleon-Nucleon Scattering. IV. Numerical Results for High Partial Waves ($l \geqslant 2$)

D. AMATI AND E. LEADER*
CERN, Geneva, Switzerland

AND

B. VITALE†
Istituto di Fisica dell'Università, Roma, Italy
Istituto Nazionale di Fisica Nucleare, Sezione di Roma, Roma, Italy

(Received 1 October 1962)

The numerical results of a dispersive theory for nucleon-nucleon scattering developed in previous papers are presented for d and higher waves. Phase parameters as function of the energy are given up to 400 MeV. The ingredients of the theory—which evaluates the 2π exchanges—are amplitudes and parameters that are obtained from the theoretical analyses of simpler processes (π-N scattering, nuclear electromagnetic form factors, etc.). Of these ingredients, those which describe the low-wave (s and p) $N\bar{N} \rightarrow \pi\pi$ amplitudes are, at present, the least reliable. We have, therefore, presented the results in two parts: One—called "basic"—contains all contributions except for the s and p $N\bar{N} \rightarrow \pi\pi$ amplitudes so that they can be simply added when a better theoretical knowledge of them is available. The other part introduces the $N\bar{N} \rightarrow \pi\pi$ s and p waves through some phenomenological models based on the Bowcock, Cottingham, and Lurié analysis of π-N scattering. The effect of the ω is also tentatively introduced. These last results are compared with the phase parameters obtained from experimental data: The agreement is encouraging for one of the models used up to about 250 MeV both for $T=0$ and $T=1$ phase parameters.

1. INTRODUCTION

We are presenting here the first numerical results of our attempt[1] to treat the theory of low-energy nucleon-nucleon scattering within the framework of the Mandelstam representation.

In the earlier papers mentioned above, we discussed the division of the problem into the so called "high waves" and "low waves", corresponding roughly to the range of interaction involved, and we presented explicit formulas for the evaluation of the $T_{ij}(l)$ [2] (the singlet-triplet representation scattering amplitudes) for the high waves, i.e., $l \geqslant 2$. We also discussed the setting up of integral equations to determine the $T_{ij}(l)$ for the low waves. In this paper we shall restrict ourselves to the results of the high-wave case.

The earlier success of the one-pion exchange contribution[3] (OPEC) in accounting for the "very high" angular momentum phases ($l > 6$ up to 400 MeV) had led us to hope that the inclusion of the two-pion-exchange contribution (TPEC) might account quantitatively for the "high" angular momentum phases $l \geqslant 2$, and might help considerably in our understanding of the behavior of the low waves. The possibly important effects of a *resonant* three-pion-exchange contribution (e.g., the ω meson) were also considered.

The theory, as originally conceived, contained no free parameters in the sense that all the data needed could be obtained or related to various other branches of pion physics. In fact, the essential ingredient of the theory was the $N\bar{N} \rightarrow 2\pi$ amplitude, which it was hoped could be obtained from the theoretical analysis of low-energy pion-nucleon scattering, and electromagnetic form factor, etc. Unfortunately, our present knowledge of this amplitude is still ambiguous. The weakest points are:

(i) The low partial waves of the process $N\bar{N} \rightarrow 2\pi$ (say s and p waves)—which are influenced strongly by the $\pi\pi$ s- and p-wave interaction—are not well determined.

(ii) The high-energy behavior of the amplitude $N\bar{N} \rightarrow 2\pi$ is not known.

As far as the latter is concerned, our lack of knowledge of the $N\bar{N} \rightarrow 2\pi$ amplitude occurs in a region in which in any case multipion and other contributions can take place, and of course we have no idea how to handle these. We hope, however, that the same arguments which make us believe that the multipion effects will not contribute much to the high waves of NN elastic scattering, will also apply to the high-energy part (i.e., large t) of the 2π contribution. Nevertheless, the point at which we cease to add in the 2π contribution plays the part of a cutoff parameter. However, we find that for the "high" partial waves of NN elastic scattering, i.e., with $l \geqslant 2$, the sensitivity to the cutoff is small and this is an *a posteriori* justification of our neglect of the more distant parts of the t cut.

On the other hand, as regards point (i), we are forced to introduce a model dependence in our knowledge of the s- and p-wave $N\bar{N} \rightarrow 2\pi$ amplitude, as will be discussed in Sec. 3. Thus, the results presented here

* Present address: Lawrence Radiation Laboratory, University of California, Berkeley, California.

† Present address: Physics Department, University of Pennsylvania, Philadelphia, Pennsylvania.

[1] D. Amati, E. Leader, and B. Vitale, Nuovo Cimento **17**, 68 (1960) and **18**, 409 (1960), referred to, respectively, as I and II in the following; *ibid.* **18**, 458 (1960).

[2] See, for instance, M. L. Goldberger, Y. Nambu, and R. Oehme, Ann. Phys. (N.Y.) **2**, 226 (1957).

[3] P. Cziffra, M. H. MacGregor, M. J. Moravcsik, and H. Stapp, Phys. Rev. **114**, 880 (1959).

correspond to various attempts to describe the s- and p-wave $N\bar{N} \rightarrow 2\pi$ amplitude in an approximate fashion.

It is important to note, however, that the various contributions to the $T_{ij}(l)$ amplitudes are additive, so that if one can determine the $N\bar{N} \rightarrow 2\pi$ s- and p-wave amplitudes more accurately, then they can be incorporated very simply. We wish to stress, therefore, that we are presenting the numerical results in such a way that any future improvement in our knowledge of the large, resonant waves (p, perhaps s as well) in the $N\bar{N} \rightarrow 2\pi$ process can be trivially combined with these results without having to go through all the rigors of recalculating the whole two-pion-exchange contribution.

In Sec. 2 we briefly recapitulate the outline of the theoretical approach that was used and we indicate how and why the above-mentioned difficulties arise.

The various model descriptions of the $N\bar{N} \rightarrow 2\pi$ s and p waves are introduced in Sec. 3.

In Sec. 4 we present the numerical results for the phase parameters and compare them with those obtained from the analysis of experimental data.

The Appendix contains for easy reference the complete set of formulas used in the high-wave calculation as well as tables of the values of the $T_{ij}(l)$ arising from the "basic" 2π contribution.

2. SUMMARY OF THE THEORY

The general theory of low-energy nucleon-nucleon scattering in the double dispersion relation framework using the simplification of the Cini-Fubini approach[4] was considered in I. It was shown there that one can find a set of five linearly independent spin operator C_i matrices in the combined spin space of the two nucleons (actually linear combinations of the usual Fermi invariants) such that the scattering operator could be written

$$M=\sum c_i(w,t,\bar{t})C_i, \tag{2.1}$$

and such that the scalar functions $c_i(w,t,\bar{t})$ satisfy a Mandelstam representation in perturbation theory. Here w is the square of the c.m. energy, t is minus the square of the momentum transfer, and

$$w+t+\bar{t}=4m^2. \tag{2.2}$$

In addition, the $c_i(w,t,\bar{t})$ had very simple crossing properties under the exchange of $t\leftrightarrow\bar{t}$. In practice, however, it turned out more convenient to use the set of five scalar functions $p_i(w,t,\bar{t})$, the coefficients of the so-called perturbation invariants P_i which arise naturally in the calculation of the 2π exchange contribution. We shall, therefore, in this paper, discuss everything in terms of the $p_i(w,t,\bar{t})$, it being possible to calculate the $c_i^T(w,t,\bar{t})$ (where $T=0$, 1 refers to the total isotopic spin) from the $p_i(w,t,\bar{t})$ by

$$c_i^T(w,t,\bar{t})=\{\textstyle\sum_k U_{ik}(w,t)p_k^T(w,t,\bar{t})\} +(-1)^{i+T}\{t\leftrightarrow\bar{t}\}, \tag{2.3}$$

where, as usual, $\{t\leftrightarrow\bar{t}\}$ implies the expression previously contained in the bracket with the indicated exchange $t\leftrightarrow\bar{t}$. The matrix $U_{ik}(w,t)$ is given in Sec. 3.4 of II.

We also found that

$$\begin{aligned} p_i^{T=0}(w,t,\bar{t})&=3p_i^+-6p_i^-,\\ p_i^{T=1}(w,t,\bar{t})&=3p_i^++2p_i^-, \end{aligned} \tag{2.4}$$

where, as usual, the $\pm$ refer to isotopic spin 0 or 1 in the nucleon-antinucleon channel.

For the rest of this paper we shall deal exclusively with the high l waves, $l\geqslant 2$, for which the $p_i(w,t,\bar{t})$ had the extremely simple representation

$$p_i^\pm(w,t,\bar{t})=\begin{bmatrix} 0\\ -\dfrac{1}{2}\dfrac{g^2\delta_{i5}}{\mu^2-t}\end{bmatrix}+\left\{\frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{\rho_i^\pm(w,t')}{t'-t-i\epsilon}dt'\right\} \mp(-1)^i\{w\rightarrow\bar{t}'\}, \tag{2.5}$$

where

$$\bar{t}'=4m^2-t'-w. \tag{2.6}$$

Note that, as the deuteron pole is omitted in (2.5), that equation is not valid for the study of the ${}^3D_1$ wave.

The first term represents one-pion exchange. The second term reflects the effect of two-pion exchange and it was the central aim of our work to calculate this. Essentially the weight functions $\rho_i(w,t)$ are given by the $N\bar{N} \rightarrow 2\pi$ amplitude[5]:

$$\textstyle\sum_i[\rho_i^\pm(w,t)\mp(-1)^i\rho_i^\pm(\bar{t},t)]P_i \sim\sum_{\pi\pi}\langle\bar{p}_1p_2|\tau^\dagger|\pi\pi\rangle\langle\pi\pi|\tau|n_1\bar{n}_2\rangle, \tag{2.7}$$

so that a knowledge of the amplitude $\tau_{N\bar{N}\rightarrow2\pi}$ allows one to evaluate the $\rho_i(w,t)$.

We are now in a position to understand the difficulty (i) mentioned in the introduction. What are we to use for the $\tau_{N\bar{N}\rightarrow2\pi}$? We know that $\tau_{N\bar{N}\rightarrow2\pi}$ is given in terms of the well-known functions[5] $A^\pm(s,\bar{s},t)$ and $B^\pm(s,\bar{s},t)$ of pion-nucleon scattering, which have representations[6] of the form

$$B^-(s,\bar{s},t)=\left\{\frac{g^2}{m^2-s}+\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}\frac{\sigma_B^-(s',t)}{s'-s-i\epsilon}ds'\right\} +\{s\rightarrow\bar{s}\}+\frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{\nu_B^-(t',s-\bar{s})}{t'-t-i\epsilon}dt'. \tag{2.8}$$

Here t plays the role of the square of the c.m. energy of the $N\bar{N}$ system while s and $\bar{s}$ are the squares of momentum transfers. The weight functions $\sigma_A(s,t)$ and $\sigma_B(s,t)$ are dominated by the 33 resonance of πN scattering and are explicitly known.[5] The last term of (2.8) arises from the unitarity cut for the $N\bar{N} \rightarrow 2\pi$

---

[4] M. Cini and S. Fubini, Ann. Phys. (N. Y.) **3**, 352 (1960).

[5] G. F. Chew, M. L. Goldberger, F. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957), referred to as CGLN in the following.

[6] J. Bowcock, N. Cottingham, and D. Lurié, Nuovo Cimento **16**, 918 (1960) and **19**, 142 (1961), referred to as BCL in the following.

process. The functions $\nu_{A,B}(t', s-\bar{s})$ have distant cuts in s and $\bar{s}$ and are, therefore, expected to have a weak dependence on $s-\bar{s}$. This means that only the low partial waves of $N\bar{N}\to 2\pi$ will contribute to $\nu_{A,B}$. Now from unitarity, for $t'\leqslant 16\mu^2$, we have, roughly speaking,

$$\begin{aligned}\nu &\propto \mathrm{Im}\langle N\bar{N}|\pi\pi\rangle\\ &\propto \langle N\bar{N}|\pi\pi\rangle\langle\pi\pi|\pi\pi\rangle.\end{aligned}\tag{2.9}$$

Thus, the last term transforms the representation (2.8) into an integral equation for A, B with the low partial waves of low-energy $\pi\pi$ scattering as input data. In other words, it is only for the s and p waves of $N\bar{N}\to 2\pi$ that it should be necessary to retain the last term of (2.8). Thus for the higher waves of $N\bar{N}\to 2\pi$ we have an *explicit* representation for A and B, and, therefore, for $\tau_{N\bar{N}\to 2\pi}$, involving just the pole terms and 33 resonance integrals. This approximate representation of $\tau_{N\bar{N}\to 2\pi}$ we have referred to as the CGLN[5] representation. It is expected to be almost exact for the high l waves of $N\bar{N}\to 2\pi$ and may be badly wrong for the s and p waves.

We are left, therefore, with the problem of evaluating the s- and p-wave parts of $\tau_{N\bar{N}\to 2\pi}$. More exactly, we have to solve the partial-wave integral equations or in have to solve the partial-wave integral equations or in some other way evaluate the helicity amplitudes $f_+^{(\pm)J=0}$ and $f_\pm^{(-)J=1}$. In the next section we describe various attempts to estimate these amplitudes.

3. THE LOW PARTIAL WAVES OF $\pi\pi\to N\bar{N}$

From the analytic properties of the functions A and B one can write integral equations for the helicity amplitudes $f_\pm^J(t)$ as follows[7]:

$$f_\pm^J(t)=\frac{1}{\pi}\int_{-\infty}^{a}\frac{\mathrm{Im}f_\pm^J(t')}{t'-t-i\epsilon}dt'+\frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{\mathrm{Im}f_\pm^J(t')}{t'-t-i\epsilon}dt',\tag{3.1}$$

where $a=4\mu^2(1-\mu^2/4m^2)$ and where $\mathrm{Im}f_\pm^J(t')$ for $4\mu^2\leqslant t'\leqslant 16\mu^2$ is given by unitarity as

$$\mathrm{Im}f_\pm^J(t)=q^{2J+1}f_\pm^{J*}(t)f_{\pi\pi}^J(t).$$

Here

$$f_{\pi\pi}^J(t)=\exp(i\delta_{\pi\pi}^J)\sin\delta_{\pi\pi}^J/q^{2J+1}$$

is the $\pi\pi$ scattering amplitude in the $l=J$ state. The first integral in (3.1) is the CGLN term. In the case of $J=1$, i.e., the p wave, the second integral of (3.1) will be dominated by the $\pi\pi$ resonance, as first suggested by Frazer and Fulco.

Unfortunately, we do not have a really satisfactory method of solving Eq. (3.1). On the other hand, Bowcock, Cottingham, and Lurié[6] (BCL) attempted to replace the right-hand-cut integral by a Breit-Wigner resonance centered at the mass of the ρ meson and determined the parameters of this term by a study of low-energy πN scattering and the electromagnetic form factor. The functions $f_\pm^1(t)$ determined in this way are, of course, only approximate solutions of (3.1). In fact, the unitarity condition is reasonably satisfied in the region very close to the resonance and also presumably along the left-hand cut. However, in the region of interest to us, i.e., for values of t close to $4\mu^2$ the value of function $f_\pm^1(t)$ will depend on the matching of the left-hand cut integral and the far tail of the Breit-Wigner resonance.

In the absence of a satisfactory determination of $f_\pm^1(t)$ in this region, we have tried three models to take into account the p-wave $\pi\pi\to N\bar{N}$ amplitude in the region near $4\mu^2$.

(a) In this model we take for $f_\pm^1(t)$ the CGLN term $\mathfrak{f}_\pm^1(t)$ and we include the exchange of a ρ meson between the two nucleons (model A).

(b) We take

$$f_\pm^1(t)=\mathfrak{f}_\pm^1(t)$$

and no ρ-meson contribution (model B).

(c)

$$f_\pm^1(t)=f_{\rho\pm}(t);$$

i.e., the whole p wave given by the ρ resonance and its tail (model C).

It is perfectly clear that none of these models represents the true situation, since none of them in fact satisfy the integral equation for $f_\pm^1(t)$. Nevertheless, we believe that the correct solution lies somewhere in the region spanned by the models. It turns out actually that model (c) leads to results somewhat in agreement with experiment, whereas (a) and (b) seem to provide too great an attraction between the nucleons, resulting in too large phase shifts. In the Appendix we shall write down the formulas necessary to improve this work if and when a reasonable solution for $f_\pm^1(t)$ is known.

Let us turn now to the s-wave part of $\tau_{N\bar{N}\to 2\pi}$. Here again one should, in principle, solve the partial-wave integral equation for $f_+^{J=0}(t)$, [$f_-^{J=0}(t)$ is identically zero]. However, we have preferred to rely on a more phenomenological estimate of $f_+^0(t)$ which comes from the works of BCL. In the work of BCL the s-wave part of $\tau_{N\bar{N}\to 2\pi}$ arises from the addition of a constant C_A^+ to the representation for $A^+(s,t)$, which represents the effect of the distant cuts and which is adjusted so that the theory predicts correctly the πN s-wave scattering lengths. Thus, they use

$$A^+(s,t)=\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}\sigma_A^+(s',t)\left[\frac{1}{s'-s}+\frac{1}{s'-\bar{s}}\right]ds' +\frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{\nu_A^+(t')}{t'-t-i\epsilon}dt'+C_A^+,\tag{3.2}$$

with $\nu_A^+=0$ and $C_A^+/4\pi=-0.9$. There is, therefore, no contribution to the $N\bar{N}\to 2\pi$ s wave arising from *direct* low-energy s-wave $\pi\pi$ interactions. It appears

[7] W. Frazer and J. Fulco, Phys. Rev. **117**, 1603, 1609 (1960) and **119**, 1420 (1960).

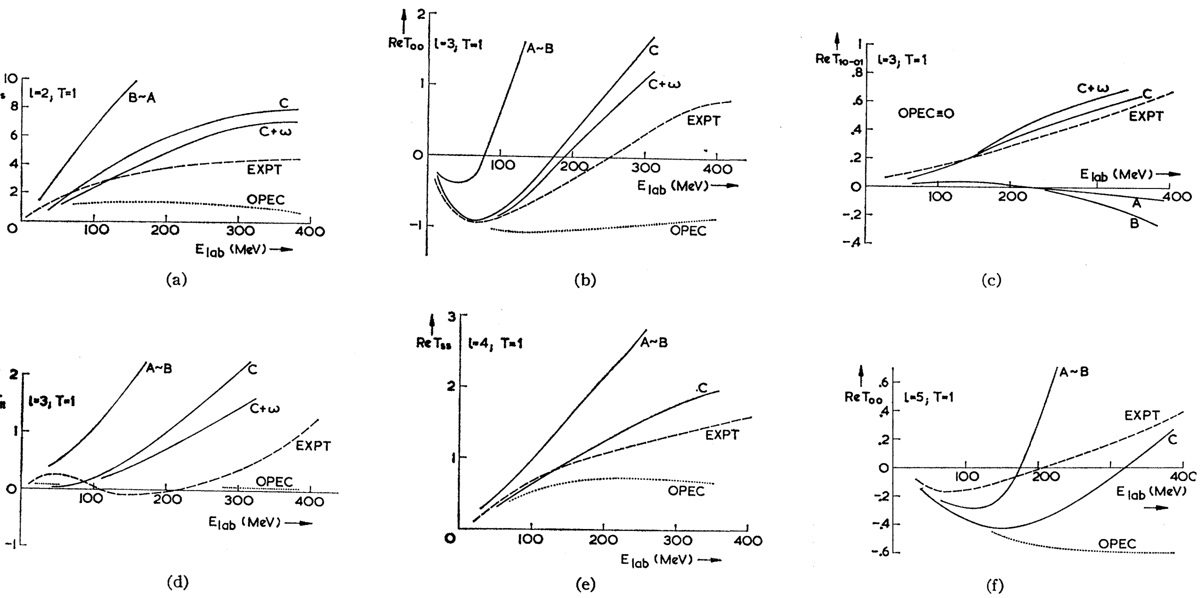

FIG. 1. Real parts of some of the $T_{ij}(l)$ for isotopic spin one. Shown for comparison are OPEC, the various theoretical models, and some experimental curves (SMMN).

now, however, that the $\pi\pi$ s-wave interaction may be quite strong so that a more accurate treatment of the term involving $\nu_A{}^+(t')$ is desirable. This would mean exactly to solve the integral equation for $f_+{}^0(t)$ and this will be possible as soon as there is more reliable information available on the actual s-wave $\pi\pi$ phase shift. The formulas of the next section allow also for the inclusion, in a simple matter, of any new information about $f_+{}^0(t)$.

Finally, the discovery of a sharp resonance in the 3π continuum, i.e., the ω meson, which seems to be coupled to nucleons at least as strongly as the ρ suggests that even if we have no idea how to handle a 3π exchange we should at least include the ω exchange as a Breit-Wigner-type contribution centered at t_ω the mass squared of the ω. The coupling constant of ω to the nucleon is not known, though there are indications that its charge coupling $g_{1\omega}$ is of the order of—or bigger than —$g_{1\rho}$, whereas its magnetic moment coupling $g_{2\omega}$ is almost zero on the assumption that the ω is largely responsible for the isoscalar magnetic moment form factor of the nucleon. We have, therefore, been content to take $g_{2\omega}=0$ and to try various values $g_{1\omega}$ of the order of a few times $g_{1\rho}$. To allow for possible changes when these coupling constants are better known, we have explicitly written down the effect of the contribution of the ω.

In the same way, further contributions arising from the newly discovered unstable particles, e.g., the η could be taken into account when more details of their properties are known.

4. RESULTS AND CONCLUSIONS

We come finally to the presentation of the numerical results of the theory. This will be done at two levels. Firstly, we compare graphically the $T_{ij}(l)$ and the phase parameters, calculated using the rough estimates for the $N\bar{N}\to 2\pi$ s and p waves and the ω contributions as discussed in Sec. 3, with the experimental values. Secondly, in the Appendix, we shall list tables of values of the $T_{ij}{}^T(l)$ at various energies, for the "basic" 2π part, i.e., $T_{ij}{}^T(l)$ with no s- or p-wave $N\bar{N}\to 2\pi$ amplitudes and no ω contribution, and which is completely model independent in the framework of this calculation. The meaning and use of these $T_{ij(\mathrm{basic})}(l)$ will be discussed in the Appendix.

Comparison with Experiment

Figures 1 to 3 show the comparison between the theoretical and phenomenological values of phases and scattering amplitudes. The experimental situation has recently become more satisfactory[8–10] but there are still large uncertainties in the phases. The best determined sets of phase shifts seem to be those of Breit *et al.*[8] (YLAM), and those of MacGregor, Moravcsik, Stapp, and Noyes[9] (SMMN) who have now obtained five possible sets of phases (at all energies) all of a similar character, and not differing very much in their χ^2 values. The main difference arises from the choice of the data used in the phase-shift analysis and in the number of free parameters used in the search. In general, there are more parameters searched in SMMN than in YLAM. Thus in YLAM the H waves are forced

[8] G. Breit, M. H. Hull, K. E. Lassila, K. D. Pyatt, and H. M. Ruppel, Phys. Rev. **120**, 2227 (1960). M. H. Hull, K. E. Lassila, H. M. Ruppel, F. A. McDonald, and G. Breit, *ibid.* **122**, 1606 (1961) and preceding papers quoted.

[9] Cf. M. H. MacGregor, M. J. Moravcsik, and H. P. Stapp, Ann. Rev. Nucl. Sci. **10**, 291 (1960) and preceding papers with H. P. Noyes therein quoted.

[10] Iu.M. Kazarinov and I. N. Silin (to be published).

to be exactly the OPEC phases as is also the $^1G_4$ wave. In those graphs where YLAM is not shown it is implied that YLAM for that phase was taken as the OPEC phase. There is also some recent work of Kazarinov and Silin[10] (KS) in which the phases are very similar to the SMMN sets. In the figures will be found occasional points taken from these authors with the errors on their phases which are estimated in their papers. It is important, in drawing any conclusions from the figures, to remember that the errors in the experimental phases are still quite big. Some errors evaluated by the Yale group are also plotted. In Figs. 1 to 3, error bars taken from the "pp" results of reference 8 are indicated by I. Those taken from reference 10 are shown as Φ.

In order to avoid unintelligible figures, we have shown on the isotopic spin 1 graphs just the YLAM phases and set 3 of the SMMN group.

The isotopic spin $T=1$ phase shifts, i.e., coming from pp scattering are, of course, much better known than the $T=0$ phases. Nevertheless, we have attempted some comparison with the $T=0$ data of references 8 and 10.

In some graphs are also shown the OPEC values so as to give an idea of the size of contribution coming from the 2π exchange.

As was mentioned in Sec. 3, we have tried three models, A, B, and C, to represent that part of the 2π exchange which is in a p state. Although we have no idea why it should be so from a theoretical point of view, it is found that only model C, i.e., in which the Breit-Wigner ρ resonance represents the whole p wave, yields any sort of agreement with the experimental data. Both models A and B appear to produce far too much attraction so that the phase shifts soon grow much too large. We have, therefore, not drawn in the

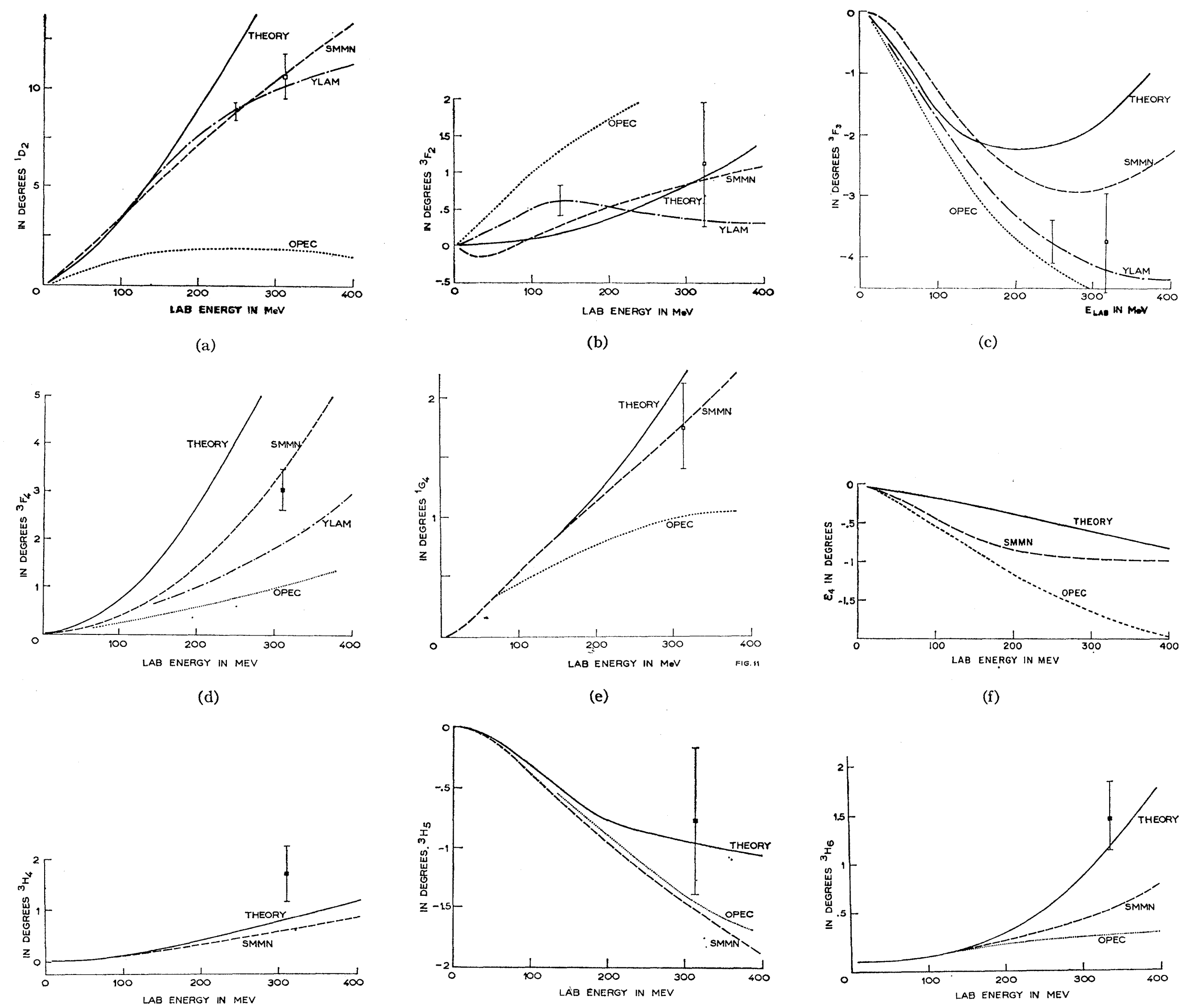

FIG. 2. Comparison of theoretical phase shifts (model $C+\omega$) with experiment, for isotopic spin one. The different experimental curves are explained in Sec. 4.

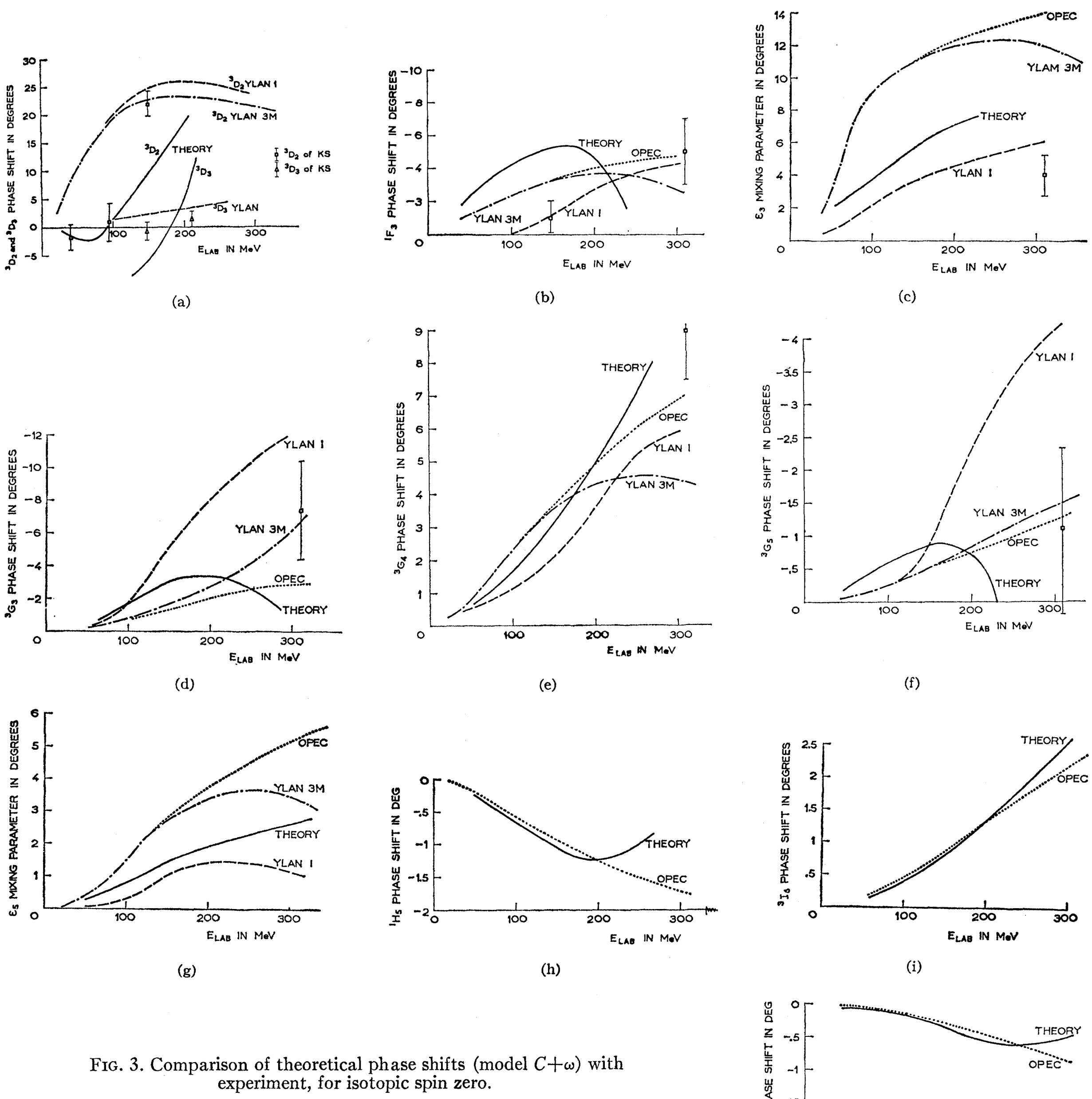

FIG. 3. Comparison of theoretical phase shifts (model $C+\omega$) with experiment, for isotopic spin zero.

curves of A and B in all the figures, but merely indicated in some of them what type of behavior these models cause.

In all the theoretical curves shown, the $N\bar{N}\rightarrow 2\pi$ s wave was given by the BCL constant $C_A{}^+/4\pi=-0.9$ as explained in Sec. 3, and the coupling was set at $g_{1\omega}=2g_{1\rho}$; $g_{2\omega}=0$.

The effect of increasing the ω coupling was usually to move the theoretical curves towards the experimental values. Thus, the repulsion created by a strongly coupled ω seems to be exactly what is needed to improve the theoretical curves, which show in general too much attraction at higher energies. We did not feel justified in taking $g_{1\omega}$ much larger, but if it should turn out that $g_{1\omega}\sim 3$ or 4 times $g_{1\rho}$, then some of the high-energy discrepancy between the theoretical and experimental curves would disappear.

In Figs. 1(a) to 1(f) are shown the real parts of some of the $T_{ij}(l)$ for isotopic spin 1 compared with the experimental results of SMMN. It is seen that only model C gives results something like the experimental curves. Particularly interesting is the curve of $(T_{10}\text{-}T_{01})$, shown for $l=3$, Fig. 1(c), since there is no OPEC contribution at all to this combination of T_{ij}. It is seen that model C or $(C+\omega)$ are of the right order of magnitude and agree quite well with the experimental curve.

In Figs. 2 and 3 we have plotted a comparison between the theoretical phase shifts (represented by model C and including the ω) and the phenomenological phase shifts SMMN, YLAM for isotopic spin 1, and YLAM1, YLAM3M for isotopic spin 0.

It should be remembered that the $T=0$ case is more sensitive to the assumptions about the $\pi\pi$ p wave than is the $T=1$ case but that the $T=0$ experimental results are much less accurately known. We shall, therefore, lean most heavily on the $T=1$ data for our comparison.

It can be seen that, in general, theory and experiment are in good agreement up to about 200 MeV. More specifically, for the "very high" waves, i.e., $l\geqslant 6$ where OPEC is assumed to give perfectly adequate phases all the way up to 400 MeV, we find that our phases are almost indistinguishable from OPEC and differ from it by about 5–10% at 400 MeV.

For the H waves ($l=5$), Figs. 2(g) to 2(i), it is difficult to draw any conclusions since it is not clear how sensitive the experimental analysis is to variations in such very small phase shifts. However, both $^3H_5$ and $^3H_4$ are reasonably like experiment and not far off the OPEC values. $^3H_6$, on the other hand, seems to rise rather alarmingly after 250 MeV, but even this behavior may turn out to be quite compatible with experiment. (Actually, the other four of the five SMMN solutions are almost identical with the curve and there is also a point of KS at 310 MeV close to this curve.)

For $l\leqslant 4$, the experimental situation is better determined and it may be meaningful to draw conclusions from the comparison.

In Fig. 2(e) it can be seen that for $^1G_4$ the OPEC diverges from experiment already at about 80 MeV, whereas the theory curve is in excellent agreement and differs by about 6% at 250 MeV.

Of the $l=3$ phases [Figs. 2(b) to 2(d)], the $^3F_1$ is in good agreement with experiment being small all the way up to 400 MeV. OPEC gives too large a phase. For $^3F_3$ the theory curve seems to have the same characteristic shape as SMMN and agrees with SMMN up to about 200 MeV. On the other hand, YLAM favors the OPEC curve.

For $^3F_4$ we have a situation very similar to $^3H_6$ in which experiment lies roughly midway between the theory and the pure OPEC curves. Agreement is poor $\sim$50% up to 200 MeV.

For $l=2$, i.e., the $^1D_2$ wave [Fig. 2(a)] we are at the limiting region of the "high-wave" treatment. It is, therefore, very encouraging that the agreement between theory and experiment is excellent (<20% up to 200 MeV). OPEC, on the other hand, fails badly already at 50 MeV.

The mixing parameter ϵ_4 is shown in Fig. 2(f). The fit is not good, though better than pure OPEC.

Let us turn now to the $T=0$ results. As mentioned earlier, the experimental situation is less well defined. Nevertheless, it can be seen that, broadly speaking, up to about 250 MeV the same relationship exists between theory and experiment as in the $T=1$ case.

For $l=5$ and 6 the theory agrees with OPEC up to 300 MeV [see Figs. 3(h) to 3(j)]. These phases were not searched for in YLAN1 and YLAN3M.

For $l=4$, i.e., $^3G_3$, $^3G_4$, $^3G_5$ [Figs. 3(d) to 3(f)], theory seems to lie midway between YLAN1 and YLAN3M up to 200 MeV and then turns down unreasonably sharply. The $^1F_3$ phase Fig. 3(b) shows the same behavior. The $^3D_2$ and $^3D_3$, Fig. 3(a), show a peculiar rapidly increasing behavior. The $^3D_2$ phase, while differing considerably from the YLAN results, lies close to two of the low-energy KS points; the $^3D_3$ shows a very poor agreement with both experimental fits.

All in all, we do not wish to labor the comparison for the isotopic spin zero case since, as mentioned, the experimental results are still extremely uncertain and the $T=0$ phases are rather sensitive to the assumptions about the $N\bar{N}\rightarrow 2\pi$ p wave.

In summary, then, we feel that the inclusion of the 2π effects improves somewhat our understanding of the behavior of the nucleon-nucleon phase shifts at moderate energies. It will be extremely interesting to see whether a more accurate treatment of the s- and p-wave $N\bar{N}\rightarrow 2\pi$ amplitude can further improve the situation.

In the Appendix can be found the complete set of formulas we have used in deriving these results as well as all the formulas necessary to modify these results in terms of better determined $N\bar{N}\rightarrow 2\pi$ s- and p-wave amplitudes.

We are deeply indebted to D. Lake for his invaluable aid in the programming of the numerical work, to A. Rambaldi for programming the calculation of the phase parameters from the $T_{ij}(l)$, and to W. Klein for various calculations.

We are very grateful also to Professor G. Breit and Professor H. P. Noyes for regularly informing us of the results of their phenomenological analyses.

APPENDIX

We present here for convenience of the reader a completely self-contained set of formulas[11] for calculating the nuclear-bar phase shifts from the theory and for adjusting the numerical results of the present paper when a better knowledge of the s- and p-wave $\pi\pi\rightarrow N\bar{N}$ amplitudes and the ω coupling constants exists.

This first section explains the path leading from the functions $\bar{p}_\alpha(w,y)$ to the phases. The second section specifies the $\bar{p}_\alpha(w,y)$ as obtained from our theory.

[11] Our units are $\hbar=c=1$.

A. General

Firstly, the nuclear-bar phase shifts are obtained from the "scattering parameters[12] α" by the following equations (the notation is δ_{lj} for the phases and ϵ_l for the mixing parameters):

$$\delta_l=\tfrac{1}{2}\arcsin \mathrm{Re}\alpha_l,\qquad \delta_{l,l}=\tfrac{1}{2}\arcsin \mathrm{Re}\alpha_{l,l}, \tag{A1}$$

and the numerical inversion of the equations

$$\begin{aligned}\mathrm{Re}\alpha_{l+2,l+1}&=\cos 2\epsilon_{l+1}\sin 2\delta_{l+2,l+1},\\ \mathrm{Re}\alpha_{l,l+1}&=\cos 2\epsilon_{l+1}\sin 2\delta_{l,l+1},\\ \mathrm{Re}\alpha^{l+1}&=\sin 2\epsilon_{l+1}\cos(\delta_{l,l+1}+\delta_{l+2,l+1}).\end{aligned} \tag{A2}$$

The parameters α are given in terms of the partial-wave singlet-triplet representation[2] of the scattering amplitude, $T_{ij}(l)$, as follows:

$$\begin{aligned}\alpha_l&=[k/(2l+1)]T_{ss}(l),\\ \alpha_{l,l}&=[k/(2l+1)][T_{11}(l)-(l+2)(l-1)T_{1-1}(l)+\sqrt{2}T_{01}(l)],\\ \alpha_{l,l+1}&=\frac{k}{2l+1}\frac{1}{2l+3}[(l+2)T_{11}(l)+\sqrt{2}l(l+1)T_{10}(l)\\ &\quad+l(l+2)(l-1)T_{1-1}(l)-\sqrt{2}l(l+2)T_{01}(l)+(l+1)T_{00}(l)],\\ \alpha_{l,l-1}&=\frac{k}{2l+1}\frac{1}{2l-1}[(l-1)T_{11}(l)-\sqrt{2}l(l+1)T_{10}(l)+(l+2)(l+1)(l-1)T_{1-1}(l)\\ &\quad+\sqrt{2}(l-1)(l+1)T_{01}(l)+lT_{00}(l)],\\ \alpha^{l+1}&=-\frac{k}{2l+1}\frac{[(l+1)(l+2)]^{1/2}}{2l+3}[T_{11}(l)-\sqrt{2}lT_{10}(l)\\ &\quad+l(l-1)T_{1-1}(l)-\sqrt{2}lT_{01}(l)-T_{00}(l)],\end{aligned} \tag{A3}$$

where

$$k^2=\text{lab kinetic energy in MeV}/1876=\tfrac{1}{4}w-1,$$

and the partial wave $T_{ij}(l)$ are defined by

$$T_{ij}(l)=\frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}\times\int_{-1}^{+1}T_{ij}(w,\cos\theta)P_l^m(\cos\theta)d\cos\theta, \tag{A4}$$

with

$$m=|i-j|.$$

The general equation for the $T_{ij}(l)$ for isotopic spin $T=0$ or 1 in terms of the functions $\bar{p}_\alpha(w,y)$ is the following:

$$T_{ij}{}^T(l)=\nu(Tmls)(2l+1)\times\frac{(l-m)!}{(l+m)!}\sum_\alpha\int_1^{1+t_{\max}/2k^2}W_{ij}{}^\alpha(w,y)\times\bar{p}_\alpha{}^T(w,y)(y^2-1)^{m/2}Q_l{}^m(y)dy. \tag{A5}$$

Here

$$m=|i-j|,\quad \nu(Tmls)=1-(-1)^{l+m+T+s},$$

where s is the total spin of the N-N system, i.e., 0 or 1 for singlet or triplet states, respectively. The limit of integration is decided by the largest t value, $t_{\max}$, up to which the discontinuity on the t cut is known. The $Q_l{}^m$ are the second type associated Legendre functions as defined in Morse and Feshbach.[13]

Finally the matrix W is given by

$$W_{ij}{}^\alpha=\textstyle\sum_\beta V_{ij}{}^\beta U_{\beta\alpha}, \tag{A6}$$

where the matrices $V_{ij}{}^\beta$ and $U_{\beta\alpha}$ are given in section 3.4 of II.[14]

It is important to notice that the phase shifts are defined in terms of the *real* parts of the T_{ij} amplitudes. This is because the amplitudes as given by the theory are not perfectly unitary. In fact their *imaginary* parts would correspond to the imaginary parts of amplitudes formed from the OPEC phase shifts. Of course, the OPEC *amplitudes* are pure real, so the inclusion of 2π effects, as carried out in this theory,[15] helps very much to satisfy the inner consistency required by unitarity.

B. Contributions to the $\bar{p}_\alpha$ Functions

We write

$$T_{ij}{}^T(l)=T_{ij}{}^T(l)_{[\mathrm{OPEC}]}+T_{ij}{}^T(l)_{[2\pi\ \mathrm{BASIC}]}+T_{ij}{}^T(l)_{[2\pi,p]}+T_{ij}{}^T(l)_{[2\pi,s]}+T_{ij}{}^T(l)_{[\omega]} \tag{A7}$$

to indicate the contributions arising from one-pion exchange, the exchange of 2 pions in all other states besides s and p waves, the 2π p wave, 2π s wave, and the ω meson (3π contribution).

We stress that the well-known [OPEC] term and the [2π basic] term are essentially model independent so that only the latter 3 terms need be recalculated in utilizing improved information on the $N\bar{N}\rightarrow 2\pi$ s- and p-wave and ω contributions. We have, therefore, included tables of numerical values of the [2π basic] parts of the $T_{ij}(l)$.

[12] H. P. Stapp, T. J. Ypsilantis, and N. Metropolis, Phys. Rev. **105**, 302 (1957).

[13] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw Hill Book Company, Inc., New York, 1953), p. 1327.

[14] The row for $V_{10}{}^\beta$, which was not separately given in II, is

	$\beta=1$	$\beta=2$	$\beta=3$	$\beta=4$	$\beta=5$
$(4\pi\lambda/m)V_{10}{}^\beta$	$\sqrt{2}(\lambda^2-1)$	$\sqrt{2}(\lambda-1)y$	0	$-\sqrt{2}\lambda(\lambda-1)y$	0.

[15] This is in contrast to certain models which try to lump all the 2π effects into the exchange of a composite particle.

TABLE I. "Basic" 2π contribution to the singlet-triplet representation scattering matrices.

E_{lab} (MeV)	Isotopic spin =1, $l=3$: T_{11}	T_{1-1}	T_{10}	T_{01}	T_{00}	$l=2$: T_{ss}	Isotopic spin =0, $l=2$: T_{11}	T_{1-1}	T_{10}	T_{01}	T_{00}	$l=3$: T_{ss}
40	−0.21	−0.003	0.007	−0.009	−0.19	−1.32	−10	−0.57	0.20	−0.35	−9.8	−1.22
68	−0.49	−0.006	0.017	−0.031	−0.47	−2.11	−15	−0.82	0.99	−1.6	−15	−2.7
105	−0.79	−0.009	0.039	−0.068	−0.77	−2.57	−16	−0.88	1.6	−1.9	−16	−3.6
147	−0.99	−0.014	0.062	−0.081	−0.98	−2.6	−12	−1.0	1.8	−0.21	−13	−3.3
210	−1.1	−0.018	0.087	−0.11	−1.1	−2.29	−3.9	−0.98	1.7	−1.8	−5.5	−0.92
250	−1.1	−0.019	0.097	−0.11	−1.1	−1.98	+2	−0.91	1.4	−1.3	0.23	1.3
310	−0.95	−0.019	0.11	−0.12	−0.97	−1.13	10	−0.82	1.0	−0.9	8.0	4.9
380	−0.77	−0.20	0.11	−0.13	−0.80	−1	19	−0.70	0.44	−0.18	17	9.4

Recalling that for $T=0$, 1 we have

$$\bar{p}_\alpha{}^0=3\bar{p}_\alpha{}^+-6\bar{p}_\alpha{}^-,\qquad \bar{p}_\alpha{}^1=3\bar{p}_\alpha{}^++2\bar{p}_\alpha{}^-, \tag{A8}$$

and defining

$$t=2k^2(y-1), \tag{A9}$$

we have the following

(a) *The one-pion exchange.* This well-known contribution is given by

$$_{[\mathrm{OPEC}]}\bar{p}_\alpha{}^-(w,y)=-\frac{g^2}{4k^2}\delta_{\alpha 5}\delta(y-1-\mu^2/2k^2),\qquad _{[\mathrm{OPEC}]}\bar{p}_\alpha{}^+(w,y)=0, \tag{A10}$$

where g^2 is the renormalized, rationalized pion-nucleon coupling constant, i.e., $g^2/4\pi\approx 14.4$.

(b) *The 2π "basic" contribution.* We shall not write in detail the lengthy formulas for these contributions. They are given by

$$_{[2\pi\ \mathrm{BASIC}]}\bar{p}_\alpha{}^\pm(w,y)=\frac{\Theta(y-1-2\mu^2/k^2)}{\pi}\times[\rho_\alpha{}^\pm(w,y)\mp(-1)^\alpha\rho_\alpha{}^\pm(\bar{t}',y)], \tag{A11}$$

where the real parts of the $\rho^\pm$ are the sum of the expression given by Eqs. (3.13), (3.17), and (3.18) of II. We restrict ourselves only to correcting some misprints and errors in paper II and give the numerical results for the $T_{ij}(l)$ as function of energy arising from the 2π basic.

Due to the fact that our definition of the invariant B of πN scattering [Eq. (3.14) of I] has the opposite sign to the usual definition,[5] and due to the fact that [2π BASIC] are quadratic in g^2, G_A, and G_B, the values of G_A and G_B defined by Eq. (3.2) of II shall be given by

$$G_A/4\pi\approx -7.4-15t/m^2,\qquad G_B/4\pi\approx -17+6.5t/m^2. \tag{A12}$$

The arc-tangential functions in paper II are all defined in the range $-\pi/2$ to $\pi/2$; the third expression of Eq. (3.10) should read

$$I^0(s's'')=\frac{4\pi}{(\zeta\eta-\xi^2)^{1/2}}\left[\arctan\left(\frac{(\zeta\eta-\xi^2)^{1/2}}{\xi+\eta}\right)-\arctan\left(\frac{(\zeta\eta-\xi^2)^{1/2}}{\xi}\right)\right]\quad\text{for}\quad \xi^2-\eta\zeta<0. \tag{A13}$$

In the expression for γ_2 [Eq. (3.12)], G_A must be replaced by $-G_A$, while $\alpha_0{}^+$ (given wrongly in the footnote on the same page) should be

$$\alpha_0{}^+=-\frac{G_A}{q\kappa}\arctan\frac{1}{H}+\frac{m}{\kappa^2}\left[G_B\left(H\arctan\frac{1}{H}-1\right)+g^2\left(h\arctan\frac{1}{h}-1\right)\right]. \tag{A14}$$

In the expression for $\mathrm{Re}\rho_1{}^+$ of Eq. (3.17) the term

$$-NG_A{}^2\frac{\pi}{\kappa^2q^2}\left(\arctan\frac{1}{H}\right)^2$$

contained in the first bracket must be replaced by

$$-N\pi\alpha_0{}^{+2}.$$

In the expression for $\rho_1{}^+$ and $\rho_2{}^+$ of Eq. (3.18), $S(Mm)$ should read $S(mM)$.

In Tables I–III the 2π basic contribution to the

TABLE II. "Basic" 2π contribution to the singlet-triplet representation scattering matrices.

E_{lab} (MeV)	Isotopic spin =1, $l=5$: T_{11}	T_{1-1}	T_{10}	T_{01}	T_{00}	$l=4$: T_{ss}	Isotopic spin =0, $l=4$: T_{11}	T_{1-1}	T_{10}	T_{01}	T_{00}	$l=5$: T_{ss}
40	−0.0004	-1.4×10^{-5}	1×10^{-4}	-0.4×10^{-5}	−0.0037	−0.025	−0.19	−0.0013	0.004	−0.007	−0.18	−0.024
68	−0.020	-6×10^{-5}	3.8×10^{-4}	-6.2×10^{-4}	−0.019	−0.089	−0.61	−0.0042	0.017	−0.027	−0.60	−0.11
105	−0.058	-1.9×10^{-4}	0.0015	−0.0021	−0.057	−0.19	−1.1	−0.0091	0.047	−0.60	−1.1	−0.27
147	−0.105	-3.6×10^{-4}	0.0034	−0.0044	−0.104	−0.29	−1.3	−0.014	0.080	−0.90	−1.4	−0.36
210	−0.17	-6.1×10^{-4}	0.0069	−0.0081	−0.17	−0.37	−0.63	−0.019	0.11	−0.11	−0.81	−0.18
250	−0.19	-7.4×10^{-4}	0.0089	−0.0091	−0.19	−0.39	+0.24	−0.020	0.10	−0.11	−0.002	−0.15
310	−0.22	-9.3×10^{-4}	0.012	−0.014	−0.22	−0.39	1.9	−0.022	0.088	−0.076	1.6	0.88
380	−0.22	-11×10^{-4}	0.015	−0.016	−0.22	−0.35	4.3	−0.022	0.039	−0.015	3.9	2.0

TABLE III. "Basic" 2π contribution to the singlet-triplet representation scattering matrices.

E_{lab} (MeV)	Isotopic spin=1, $l=7$, T_{11}	T_{1-1}	T_{10}	T_{01}	T_{00}	$l=6$, T_{ss}	Isotopic spin=0, $l=6$, T_{11}	T_{1-1}	T_{10}	T_{01}	T_{00}	$l=7$, T_{ss}
40	-8.2×10^{-5}	-1.1×10^{-7}	1×10^{-6}	-1.1×10^{-6}	-7.7×10^{-5}	-5.2×10^{-4}	-0.0037	-8×10^{-6}	5×10^{-5}	-6×10^{-5}	-0.0034	-4.8×10^{-4}
68	-8.9×10^{-4}	-1.1×10^{-6}	1×10^{-5}	-1.8×10^{-5}	-8.6×10^{-4}	-0.0039	-0.026	-5.8×10^{-5}	4.2×10^{-4}	-5.8×10^{-4}	-0.025	-0.0049
105	-0.0043	-5.7×10^{-6}	7.6×10^{-5}	-1×10^{-4}	-0.0043	-0.014	-0.082	-2.2×10^{-4}	0.0021	-0.0026	-0.084	-0.021
147	-0.011	-1.5×10^{-5}	2.5×10^{-4}	-2.5×10^{-4}	-0.011	-0.031	-0.14	-4.9×10^{-4}	0.0050	-0.0056	-0.17	-0.040
210	-0.025	-3.7×10^{-5}	6.9×10^{-4}	-8.1×10^{-4}	-0.025	-0.058	-0.11	-9.2×10^{-4}	0.0094	-0.0096	-0.13	-0.036
250	-0.034	-4.5×10^{-5}	0.0010	-0.0012	-0.034	-0.073	$+0.054$	-0.0011	0.010	-0.011	0.087	$+0.012$
310	-0.047	-7.4×10^{-5}	0.0016	-0.0019	-0.047	-0.09	0.34	-0.0015	0.010	-0.010	0.28	0.15
380	-0.058	-9.9×10^{-5}	0.0023	-0.0026	-0.058	-0.1	0.92	-0.0017	0.0050	-0.002	0.84	0.43

partial-wave amplitudes $T_{ij}(l)$ for $2\leq l\leq 7$ are given as function of the lab kinetic energy both for $T=1$ and $T=0$.

(c) *The effect of* $N\bar{N}\rightarrow\pi\pi$ *s and p waves.*

$$_{[2\pi,s,p]}\bar{p}_\alpha^{\pm}(w,y)=\Theta(y-1-2\mu^2/k^2)\chi_\alpha^{\pm}(w,y),\quad\text{(A15)}$$

where in terms of the helicity amplitudes

$$\chi_1^{+}(w,y)=\frac{8}{(t-4m^2)}\left(\frac{t-4\mu^2}{t}\right)^{1/2}|f_+^{0(+)}(t)|^2,$$

$$\chi_1^{-}(w,y)=\frac{3}{2}\left(\frac{t-4\mu^2}{t}\right)^{1/2}\frac{t-4\mu^2}{(t-4m^2)^2}(4m^2-2w-t)\times\left|\frac{1}{\sqrt{2}}f_-^{1(-)}(t)-f_+^{1(-)}(t)\right|^2,$$

$$\chi_2^{-}(w,y)=\frac{3}{2\sqrt{2}}\left(\frac{t-4\mu^2}{t}\right)^{1/2}\frac{t-4\mu^2}{t-4m^2}f_-^{1(-)}(t)\times\left[\frac{1}{\sqrt{2}}f_-^{1(-)}(t)-f_+^{1(-)}(t)\right],\quad\text{(A16)}$$

$$\chi_4^{-}(w,y)=-\frac{3}{16}\left(\frac{t-4\mu^2}{t}\right)^{1/2}(t-4\mu^2)|f_-^{1(-)}(t)|^2,$$

$$\chi_2^{+}=\chi_3^{\pm}=\chi_4^{+}=\chi_5^{\pm}=0.$$

When one treats the p-wave $\pi\pi$ resonance as the exchange of a ρ meson, the p-wave contributions of the formulas above (A16) reduce to the simple form

$$_{[\rho]}\chi_1^{-}(w,y)=\frac{1}{16k^2}(4m^2-m_\rho^2-2w)g_{2\rho}^2\delta\left(1-y+\frac{m_\rho^2}{2k^2}\right),$$

$$_{[\rho]}\chi_2^{-}(w,y)=-\frac{1}{4k^2}g_{2\rho}g_{1\rho}\delta\left(1-y+\frac{m_\rho^2}{2k^2}\right),\quad\text{(A17)}$$

$$_{[\rho]}\chi_4^{-}(w,y)=-\frac{1}{4k^2}g_{1\rho}^2\delta\left(1-y+\frac{m_\rho^2}{2k^2}\right),$$

where $g_{2\rho}$, $g_{1\rho}$ are the renormalized, rationalized coupling constants of the ρ to the nucleon in an interaction Hamiltonian of the form

$$\mathcal{H}_I=ig_{1\rho}\bar{\psi}(p')\boldsymbol{\tau}\gamma_\mu\psi(p)\boldsymbol{\phi}_{[\rho]}{}^{\mu}-g_{2\rho}\bar{\psi}(p')\boldsymbol{\tau}\psi(p)\boldsymbol{\phi}_{[\rho]}{}^{\mu}(p+p')_\mu/2.\quad\text{(A18)}$$

These coupling constants are related to the parameters introduced in BCL by

$$g_{1\rho}=\sqrt{6\pi}\left(\frac{(\frac{1}{4}m_\rho^2-\mu^2)^{3/2}}{m_\rho}\right)^{1/2}\frac{C_1+2mC_2}{\Gamma^{1/2}},$$

$$g_{2\rho}=2\sqrt{6\pi}\left(\frac{(\frac{1}{4}m_\rho^2-\mu^2)^{3/2}}{m_\rho}\right)^{1/2}C_2/\Gamma^{1/2}.\quad\text{(A19)}$$

The passage from (A16) to (A17) follows using the formulas of II, Sec. 4.2.

(d) *The effect of the ω meson.* Treating the ω as the limiting case of the exchange of a narrow Breit-Wigner resonant form, with mass m_ω, we get

$$
{}_{[\omega]}\bar{p}_\alpha{}^{+}(w,y)=\delta\left(y-1-\frac{m_\omega{}^2}{2k^2}\right)\lambda_\alpha(w,y),
\qquad
{}_{[\omega]}\bar{p}_\alpha{}^{-}(w,y)=0, \tag{A20}
$$

with

$$
\begin{aligned}
\lambda_1(w,y)&=(1/24k^2)(4m^2-m_\omega{}^2-2w)g_{2\omega}{}^2,\\
\lambda_2(w,y)&=(1/6k^2)g_{2\omega}g_{1\omega},\\
\lambda_4(w,y)&=-(1/6k^2)g_{1\omega}{}^2,\\
\lambda_3&=\lambda_5=0,
\end{aligned} \tag{A21}
$$

where $g_{2\omega}$ and $g_{1\omega}$ are the renormalized, rationalized coupling constants of the ω to nucleons using the same type of coupling as in (A16). As mentioned earlier, we have used $g_{1\omega}\sim 2g_{1\rho}$, $g_{2\omega}=0$ in this paper.

PHYSICAL REVIEW VOLUME 130, NUMBER 2 15 APRIL 1963

Use of Radiative π Decay to Limit the Neutrino Mass

ALFRED S. GOLDHABER*
Palmer Physical Laboratory, Princeton, New Jersey
(Received 13 December 1962)

A limit on the μ-neutrino mass may be obtained by observing radiative π decay. If one measures the γ-ray energy, then a value within $\frac{3}{4}$ MeV of the maximum possible γ energy is required to limit the neutrino mass to 1 MeV. If one also measures the coincident μ momentum, then the γ energy must be within, say, 3 MeV of the maximum, and the μ momentum within 0.1 MeV/c of its maximum. "Useful" events for both processes occur once in about 10^8 π decays.

THE present limit on the mass of the μ neutrino is 3.6 MeV.[1] Barkas, Birnbaum, and Smith obtained this limit by measuring the μ momentum in ordinary π decay,

$$\pi\rightarrow\mu+\nu.$$

The limit is so poor because the μ momentum is insensitive to the mass of the highly relativistic neutrino. An obvious way to improve this result is to find a reaction which gives little energy to the neutrino. Since there are no known two-body decays with this property, the next best thing is to look at three-body decays with low-energy neutrinos. One such process is radiative π decay,

$$\pi\rightarrow\mu+\nu+\gamma,$$

in the kinematic region with low neutrino momentum. If we assume that, even for finite neutrino mass, the decay coupling is pseudoscalar and $(1-i\gamma_5)$, then the branching ratio, R, of radiative π decay to ordinary π decay has the same formal dependence on momenta and masses as in the zero-mass case:

$$R=\frac{2\alpha}{\pi}\frac{1}{(1-\mu^2)^2}\int d\epsilon\int dE\left(\frac{p\cdot k}{l\cdot k}+B\frac{(\mathbf{l}\times\hat{k})^2}{(l\cdot k)^2}\right),$$

where $M_\pi=1=c$; $\mu=$mass of μ; $\nu=$mass of ν; $\alpha=$fine structure constant$=1/137$; $A=\frac{1}{2}(1-\mu^2)=29.80\pm0.04$ MeV/c (see reference 1); $B=\frac{1}{2}(1-\nu^2-\mu^2)$; $k=(k_0,\mathbf{k})$, $p=(p_0,\mathbf{p})$, and $l=(l_0,\mathbf{l})$ are the four-momenta of γ, ν, and μ, respectively; $\hat{k}=\mathbf{k}/|\mathbf{k}|$; $E=p_0$ is the energy of the ν; $\epsilon=A-k_0$ is the difference between the maximum possible γ energy and the given γ energy.[2]

The above expression lacks a term, contributing appreciably near $k=0$, which would cancel the logarithmic infrared divergence. This is permissible because we shall only need the formula in the neighborhood of $|\mathbf{k}|=A$.

A more serious defect is the neglect of all structure in the π meson, as well as possible intermediate boson effects. We may estimate the error from these sources by assuming of Neville's dimensionless form factors[3] that $|h_1|\leqslant 1$ and $|h_2|\leqslant 1$. This results in contributions to the rate for slow neutrinos of about the same size as that computed here from "inner bremmstrahlung" (I.B.) alone. The structure amplitude might interfere destructively with the I.B. amplitude. In that case, the branching ratio for slow neutrinos could be much smaller than my estimate. While such a cancellation seems quite unlikely, it is possible.

If we correct for the intermediate boson alone, assuming it has a mass greater than the K meson, we have $h_1<1/10$, $h_2=0$,[3] which leads to a negligible correction.

* National Science Foundation Predoctoral Fellow.

[1] W. H. Barkas, W. Birnbaum, and F. M. Smith, Phys. Rev. **101**, 778 (1956).

[2] This is consistent with Eqs. (2.7) and (2.8) of D. E. Neville, Phys. Rev. **124**, 2037 (1961).

[3] See reference in footnote 2.