# High-Energy Behavior of Scattering Amplitudes in Quantum Electrodynamics*

MAURICE LÉVY†
*Institute for Advanced Study, Princeton, New Jersey*
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The perturbation expansion of the elastic electron-electron scattering amplitude is examined in the limit of very high energy and finite momentum transfer (nearly forward scattering). Several classes of graphs involving the exchange of an arbitrary number of virtual photons are summed into an exponential form, but no evidence is found that the exponent contains terms depending logarithmically upon the energy with coefficients varying with the momentum transfer. This leads to the conclusion that the photon is probably not a "Regge pole." Its zero mass and its vector character are seen to be together responsible for this result. A symmetry property of the elastic scattering amplitude shows that terms involving the exchange of an even number of virtual photons do not, in any case, contribute at very high energy in the neighborhood of the forward direction.

## I. INTRODUCTION

SOME interest has been expressed recently[1] on the question as to whether the photon is an elementary particle, or rather the $J=1$ state of a composite system represented by a pole in the angular momentum plane moving with energy (Regge pole). In the list of the known "elementary" particles, the photon plays a special role because of its zero mass and because, being the agent of a universal interaction, it can be exchanged with equal probability by any pair of charged particles. Since the photon does not have a mass, there is no intrinsic scale which could make the photon pole move with energy. On the other end, it is hard to understand why this scale should depend particularly upon the mass of one among all the charged particles.[2]

A possible way to investigate this problem[3] is to calculate as a power series in $\alpha$, the fine structure constant, the scattering amplitude of two charged particles produced by the exchange of many photons in the limit of very high (squared) energy $s$ and finite (squared) momentum transfer $t$, and to isolate to every order of perturbation theory terms of the form

$$M_{n+1}=\left(\frac{\alpha^n}{n!}\sum_{p=0}^{n} F_{np}(t)\ln^p s\right)M_1,$$

where $M_1$ is the lowest order matrix element corresponding to the exchange of one photon. The sum of all these terms might then perhaps be recast into an exponential form:

$$M\equiv\sum_{n=0}^{\infty} M_{n+1}=G(t)\exp[\gamma(t)\ln s]M_1, \qquad (1)$$

where $G$ and $\gamma$ are expressed as power series in $\alpha$. Equation (1) would represent the characteristic behavior at high energy of a scattering amplitude dominated by a composite state formed in the crossed (particle-antiparticle) channel associated with a photon; $\gamma(t)$ would be identified with the photon trajectory. Actually, because of invariance under charge conjugation, one must distinguish between the exchange of an even or an odd number of photons. Only the terms corresponding to the exchange of an odd number of photons should be summed to give Eq. (1); the exchange of an even number of photons might or might not lead to the appearance of a second term on the right-hand side of Eq. (1), which would then correspond with the even counterpart of the photon trajectory, the lowest state of which presumably has a much higher mass.

The purpose of the present paper is to report the results of an investigation of elastic electron-electron scattering at high energy, which is summarized as follows:

(1) We first investigate to all orders the contribution to the elastic scattering amplitude coming from infrared photons, using the technique of Yennie, Frautschi, and Suura.[4] These terms are known to give rise to an exponential factor like in Eq. (1), but we show that the exponent vanishes to all orders in the limit of high $s$. (The infrared terms are calculated with a finite photon mass $\lambda$ which disappears at the end.)

(2) We calculate then exactly the contribution to the elastic scattering amplitude of the exchange of *two* photons and we show that the terms of order $(\ln s)^2M_1$ and $(\ln s)M_1$ cancel completely at high energy. By comparing with the preceding calculation, we find that these terms contain, in addition to the standard infrared

* Work supported in part by the National Science Foundation.
† Permanent address: Laboratoire de Physique Théorique et Hautes Energies, B. P. 12, Orsay (Seine-et-Oise), France.

[1] R. Blankenbecler, L. F. Cook, and M. L. Goldberger, Phys. Rev. Letters **8**, 463 (1962).

[2] A mechanism which can provide such a scale through a set of fixed masses, independently of the specific scattering process, is the photon-photon interaction. Simple arguments show that the "slope" of the photon trajectory should then be at least of order $\alpha^5m^{-2}$, where the electron mass $m$ is the lowest which can contribute. See, however, the discussion of Sec. IV.3.

[3] M. Lévy, Phys. Rev. Letters **9**, 235 (1962); M. Gell-Mann and M. L. Goldberger, *ibid.* **9**, 275 (1962); J. C. Polkinghorne (to be published); P. G. Federbush and M. T. Grisaru (to be published).

[4] D. R. Yennie, S. C. Frautschi, and H. Suura, Ann. Phys. (N. Y.) **13**, 379 (1961), abbreviated in the following as YFS. See also: K. E. Eriksson, Nuovo Cimento **21**, 383 (1961); K. T. Mahanthappa, Phys. Rev. **126**, 329 (1962).

contribution, terms which can still be considered as generated by the exchange of "soft photons," in the sense that their energy is small compared with the electron energy, but nevertheless cannot be neglected compared to the momentum transfer, which is kept finite.

(3) We calculate accordingly the scattering amplitude to all orders, assuming that the virtual photons momenta $k_i$ are small compared to the initial or final electrons momenta, but taking rigorously into account the energy-momentum conservation of all the exchanged photons: $\sum k_i=q$, where $q$ is the four-momentum transfer ($t=q^2$). The corresponding soft photon contribution can then be summed exactly in an exponential form, and shown to contain all the terms contributing to the exchange of two photons at high energy. We give reasons to believe that it contains all the $t$-dependent terms at high energy, to every order of perturbation theory, and we find that the $\ln(s)$ dependence of the exponent vanishes.

(4) We attempt to calculate corrections due to the ultraviolet contributions to the scattering matrix elements, and we show that they come from terms in which two of the exchanged photons are "hard," with large momenta of opposite sign, and are emitted or absorbed in immediate succession. This contribution is calculated and found to cancel again to every order of perturbation theory, in the limit of high energy.

The subsequent corrections are discussed, if not calculated, and arguments are given to indicate that, in the high-$s$ limit, they either vanish or give rise only to $\ln(s)$ terms with constant coefficients which would not induce a movement of the photon pole.

(5) We discuss next the influence of the modified photon propagators and of photon-photon interaction, and give an explicit formula for the $(n+1)$th order $t$-dependent (soft photon) part of the matrix element. We find that it is likely that the $\ln(s)$ dependence vanishes also in this case in the high-energy limit.

(6) Finally, we show that the elastic scattering amplitude at high energy has a symmetry property under the exchange of $s$ and $u=4m^2-(t+s)$ which implies that the contribution of graphs corresponding to the exchange of an *even* number of photons vanishes in any case for large values of $s$.

The general conclusion is that there is so far no evidence from perturbation theory that the photon is a Regge pole. Our calculations indicate rather strongly that it is a "fixed" pole, although they cannot be considered as representing a complete proof of this statement.

## II. INFRARED CONTRIBUTION TO ELASTIC ELECTRON-ELECTRON SCATTERING

In this section, the "infrared" part of the elastic scattering amplitude is defined exactly in the same way as in Yennie *et al.*[4] (abbreviated in the following as YFS).

We consider the scattering of two electrons of initial four-momenta $p_1$, $p_2$ and final four-momenta $p_1'$, $p_2'$. We write as usual: $q=p_1-p_1'=p_2'-p_2$, $t=q^2$, $s=(p_1+p_2)^2$, and $u=(p_1-p_2')^2$. The latter quantities satisfy the relation:

$$s+t+u=4m^2, \tag{2}$$

where $m$ is the electron mass. The lowest order matrix element, corresponding to the exchange of one photon, can be expressed as

$$M_1(t)=\frac{i\alpha}{\pi}\frac{m^2}{(E_1E_2E_1'E_2')^{1/2}}\frac{[\gamma_\mu\times\gamma_\mu]}{t}, \tag{3}$$

where, following Tsai's notations,[5] we write in general: $[A\times B]=\bar{u}(p_1')Au(p_1)\bar{u}(p_2')Bu(p_2)$ (the $u$'s are the usual free Dirac spinors). Using the rules given by YFS, we find that, in the present case, the infrared part of the scattering amplitude can be separated from the rest in the form:

$$M=\exp(B+B')\bar{M}, \tag{4}$$

where $\bar{M}$ does not contain infrared contributions and

$$B(p_1,p_2)=-\frac{i\alpha}{8\pi^3}\int\frac{d^4k}{k^2-\lambda^2}\left[\frac{2p_1-k}{k^2-2p_1k}+\frac{2p_2+k}{k^2+2p_2k}\right]^2; \tag{5a}$$

$B'$ is obtained from $B$ through the relation:

$$B'=-B(p_1,-p_2'). \tag{5b}$$

In principle, one should add, to be complete, another similar factor involving the pair of momenta $(p_1,p_1')$ or $(p_2,p_2')$, but it corresponds to Feynman diagrams where the photon of momentum $k$ is emitted and absorbed by the same electron. In any case, it depends only on $t$ and does not contain, therefore, any of the $\ln(s)$ factors which are of interest to us [it can be absorbed in the function $G(t)$ of Eq. (1)].

We place ourselves in a physical situation of nearly forward scattering, where $s$ is large and positive and $t$ finite and negative. Because of Eq. (2), $u$ is consequently large and negative. It is then easy to see that $B'$ is real and can readily be computed in the limit of large $|u|$:

$$B'=-(\alpha/\pi)[\phi_1(u)+\phi_2(u;\lambda^2)], \tag{6}$$

where we have set

$$\phi_1(z)=z\int_0^1\frac{dx}{4m^2-z(1-x^2)-i\epsilon}\simeq-\frac{1}{2}\ln\left(\frac{-z}{m^2}\right) \tag{7}$$

and

$$\phi_2(z;\lambda^2)=-z\int_0^1\int_0^1\frac{ydydx}{y^2[m^2-(z/4)(1-x^2)-i\epsilon]+\lambda^2(1-y)}, \tag{8}$$

[5] Y. S. Tsai, Phys. Rev. **120**, 269 (1960). There is a small misprint in Eq. (9) of Tsai's paper: the (+) sign in front of the second line should be changed into (−). This does not affect his subsequent equations.

which, when $z$ is large and $<0$, can be evaluated as

$$\phi_2(z;\lambda^2)\cong\tfrac{1}{2}\ln^2(-z/m^2)+\ln(-z/m^2)\ln(m^2/\lambda^2);\quad (9)$$

($\phi_1$ and $\phi_2$ are, respectively, equal to $\frac{1}{4}z\mu_1$ and $-\frac{1}{2}z\mu_2$ in Tsai's notations).

$B$ can now be calculated from $B'$ through Eq. (5b). However, some care must be exercised in doing so, since $B$ contains, for $s>0$, an imaginary part which corresponds, in the relativistic theory, to the long-range contribution to the familiar Coulomb phase shift. Using integral expressions (7) and (8), one finds for large $s$:

$$B=(\alpha/\pi)[\phi_1(-s)+\phi_2(-s;\lambda^2)-i\pi\ln(s/\lambda^2)].\quad (10)$$

If $t$ is kept finite, we have $s\simeq -u$ in the limit $s\rightarrow\infty$, and, consequently,

$$B+B'\simeq -i\alpha\ln(s/\lambda^2).\quad (11)$$

The $\ln(s)$ and $(\ln s)^2$ terms have cancelled out, except for the unimportant purely imaginary term (11), the coefficient of which is independent of $t$ (even this term is cancelled by the remaining contribution calculated in Sec. III).

It should be noted at this point that the $\phi_1$ parts of $B$ and $B'$ are actually ultraviolet contributions, coming from the $k^2$ term in the numerator of $B$ (or $B'$) on the right-hand side of Eq. (5). They were added in YFS in order to maintain gauge invariance throughout the calculation.

### III. THE TWO-PHOTONS EXCHANGE DIAGRAMS

In order to understand the origin of the remaining contribution to the high-energy scattering amplitude coming from noninfrared terms, we turn now to an examination of the complete matrix element produced by the exchange of two photons. They have been calculated in detail by Redhead[6] and Tsai.[5] Let us call $M_2$ and $M_2'$ the respective contributions of diagrams (a) and (b) of Fig. 1. $M_2'$ is real whereas $M_2$, like $B$, contains an imaginary part. We have exhibited in Table I the various contributions to $M_2'$ and $\mathrm{Re}M_2$ in the limit of high $s\simeq -u$, and finite $t$. The first two columns contain the contribution of $B$ and $B'$ calculated in Sec. II, to the second order in $\alpha$. To make things clearer, we have separated them into a truly "infrared" part, and an ultraviolet part (independent of $t$). The last two columns contain the terms which were not extracted by the YFS rules. As can be seen immediately, the terms contained in each column cancel separately in the limit $s\simeq -u$. In addition, the imaginary part of $M_2$ can be written:

$$\mathrm{Im}M_2=[\mathrm{Im}B+\alpha\ln(s/-t)]M_1 = -\alpha\ln(-t/\lambda^2)M_1,\quad (12)$$

so that the $(\ln s)$ dependence cancels even there.

For our purpose, which is to extend the calculation of high-energy contributions to all orders, the terms contained in the third column of Table I are of special interest. We interpret them as coming from photon energies which are still small compared to electron energies, but are nevertheless of the same order as the momentum transfer, which we have kept finite. We choose to call them "soft," to distinguish them from the "infrared" terms[7] where $k^2\ll q^2$. It must be realized, however, that the soft photons occupy, in the present kinematical situation, almost the complete range of the spectrum. To prove our interpretation, we compare the "truly infrared" part of $B'M_1$, calculated in the preceding section:

$$B'M_1\simeq -\frac{i\alpha}{4\pi^3}(M_1t)\frac{1}{q^2}\int\frac{4p_1\cdot p_2'd^4k}{(k^2-\lambda^2)(k^2-2p_1k)(k^2-2p_2k)} \simeq -\frac{\alpha}{\pi}\phi_2(u;\lambda^2)M_1,\quad (13)$$

with a similar expression in which the energy-momentum conservation of the two exchanged photons has been restored (in other words, $k\ll p_1$ or $p_2'$, but $k\simeq q$):

$$U'M_1\simeq -\frac{i\alpha}{4\pi^3}(M_1t)\times\int\frac{4p_1p_2'd^4k}{(q-k)^2(k^2-\lambda^2)(k^2-2p_1k)(k^2-2p_2'k)} \simeq -\frac{\alpha}{\pi}M_1[\phi_2(u;\lambda^2)-\phi_2(u;-t)].\quad (14)$$

TABLE I. High-energy contributions to the two-photons exchange diagrams.

| | YFS terms | | Other terms | |
|---|---|---|---|---|
| | Infrared | Ultraviolet | "Soft" | Ultraviolet |
| $\pi\,\mathrm{Re}M_2/\alpha M_1$ | $\phi_2(-s;\lambda^2)$ | $\phi_1(-s)$ | $-\phi_2(-s;-t)$ | $\phi_1(-s)$ |
| $\pi M_2'/\alpha M_1$ | $-\phi_2(u;\lambda^2)$ | $-\phi_1(u)$ | $-\phi_2(u;-t)$ | $-\phi_1(u)$ |

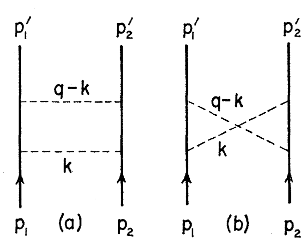


FIG. 1. The two basic diagrams for the exchange of two photons.

[6] M. L. G. Redhead, Proc. Roy. Soc. (London) A220, 219 (1953).

[7] Starting with Sec. IV, we shall actually call "soft" the *sum* of the infrared part of YFS and the soft part defined here. It consists of all the contributions coming from photon momenta which are small compared to $p_1$, $p_2$ or $p_2'$, but have any magnitude compared to $q$.

As can be seen, this new expression contains both contributions coming from the first and third column of Table I (a similar verification can be made for $M_2$). It can also be verified that the value of the integral (14) in the high-energy limit is not changed if one neglects $k^2$ compared to $2p_1k$ or $2p_2'k$ in the denominator of the integrand. This remark is of importance for the discussion of the $n+1$ photons exchange diagrams in the next section.

The results of this section can be summarized as follows:

(1) In the 2-photons diagrams, the double logarithms (contained in $\phi_2$) are produced either from "infrared" or "soft" photon energies, but they cancel separately for each of the graphs (a) and (b) of Fig. 1.

(2) The remaining single logarithms cancel by adding the contributions of crossed and uncrossed diagrams. One last word about the ultraviolet contribution: it is easy to see that it should, in general, *only give rise to* $\ln(s)$ *terms with a constant coefficient* (like $\phi_1$ here). The reason is that, with our separation of the "soft" photons terms, only values of $k$ comparable to $p_1$, $p_2$ or $p_2'$ are included in the ultraviolet part. We can, therefore, neglect $q$ compared to $k$ in these terms, which then depend only on $s$ or $u$, and no longer on $t$. This point is of great importance, since we are mostly interested in the movement of the photon pole when $t$ varies.

## IV. GENERAL TREATMENT OF THE HIGH-ENERGY SCATTERING AMPLITUDE

In this section, we use the information obtained from our discussion of the two-photons diagrams to give an approximate treatment of the high-energy scattering amplitude which includes, to all orders, the various kinds of contributions which we have isolated in the last section.

In the following, we shall often call "line (1)," for instance, the continuous set of electron lines which link the external electron line of momentum $p_1$ to the external electron line of momentum $p_1'$ [and similarly for "line (2)"]. Now, if we isolate in line (1) the emission of the first photon of momentum $k_1$, and polarization $\nu_1$, we can write the corresponding contribution to the matrix element:

$$\bar{u}(p_1')\Gamma(p_1', p_1-k_1)S_F(p_1-k_1)\gamma_{\nu_1}u(p_1) = \bar{u}(p_1')\Gamma(p_1', p_1-k_1)\frac{[N_{\nu_1}^{\text{el.}}+N_{\nu_1}^{\text{mag.}}]}{k_1^2-2p_1\cdot k_1}u(p_1), \quad (15)$$

where we have set:

$$N_\nu^{\text{el.}}=2p_\nu-k_{1\nu},$$
$$N_\nu^{\text{mag.}}=\tfrac{1}{2}[\mathbf{k}_1,\gamma_\nu]. \quad (16)$$

By separating explicitly more photons from the vertex function $\Gamma$ of Eq. (15), we can then define the "electric" and the "magnetic" parts of the emission or absorption operators for each photon successively.

### 1. The "Soft Photon" Contribution

We consider first the part of the matrix elements contributed by exchanged photons of momenta $k_i$ which are small compared to $p_1$, $p_2$ and $p_2'$, but can, nevertheless, be of the same order as $q$, the momentum transfer. We saw in the previous section that they contribute the main $t$-dependent part of the 2-photons exchange diagrams at high energy. As long as the integrals continue to converge in the ultraviolet, the separation of the soft photon contribution can be achieved by replacing $N_\nu^{\text{el.}}$ of Eq. (16) by $2p_\nu$, by disregarding $N_\nu^{\text{mag.}}$, and by neglecting, for instance, $k_1^2$ compared to $-2p_1\cdot k_1$ in the denominator of the right-hand side[8] of Eq. (15). The summation of the diagrams to every order can then be achieved in a relatively simple way.

We shall consider here $M_{n+1}$, the matrix element corresponding to the exchange of $(n+1)$ photons. Only $n$ of these will have independent moment $k_1\cdots k_n$ (and polarizations $\nu_1\cdots\nu_n$). We shall assume that they are all emitted from line (1) *in that order* [if one of the photons $(i)$ is actually absorbed by line (1), we label its momentum $-k_i$, and the result is unchanged]. Then, the $(n+1)$th photon emitted from line (1) will have a momentum $k_0=q-\sum k_i$ and polarization $\mu$. There are $(n+1)!$ different graphs contributing to $M_{n+1}$. Each of them can be characterized completely by a permutation $(\alpha_1\cdots\alpha_{n+1})$ of the $n+1$ first integers, which defines the order in which the $n+1$ photons are absorbed by line (2). We shall call such a graph $\{\alpha_1\cdots\alpha_{n+1}\}$ or, more simply, $\{\alpha\}$.

The general form of $M_{n+1}$ is as follows:

$$M_{n+1}=-\frac{4\pi^2m^2}{(E_1E_2E_1'E_2')^{1/2}}\left(\frac{-i\alpha}{4\pi^3}\right)^{n+1} \times\int\prod_{i=1}^{n}\frac{d^4k_i}{k_i^2-\lambda^2}\frac{1}{[q-\sum k_i]^2}\times[\mathcal{A}\times\sum_{\{\alpha\}}\mathcal{B}_\alpha]. \quad (17)$$

$\mathcal{A}$ is the contribution from line (1) and can be written relatively simply in the soft photon approximation. Using the notation:

$$K_{r,s}=\sum_{i=r}^{s}k_i, \quad (18)$$

we have:

$$\mathcal{A}=\gamma_\mu\prod_{i=1}^{n}\frac{2p_{1\nu_i}}{[-2p_1\cdot K_{1,i}]}. \quad (19)$$

Since the photons will be absorbed by line (2) in all possible orders, we can replace $\mathcal{A}$ by its symmetrized form with respect to all $k_i$ (and divide it by $n!$).

The following identity, relative to a set of numbers

[8] See the remark after Eq. (14). It is not possible to neglect $k^2$ in the denominator if the corresponding integral diverges in the ultraviolet like in the YFS formalism. Even there, however, this would not be terribly serious, since the logarithmically divergent terms cancel when crossed and uncrossed parts are added.

$a_1 \cdots a_n$, is useful:

$$\sum_{\text{perm.}\{a_i,a_j\}} \frac{1}{a_1(a_1+a_2)\cdots(a_1+a_2+\cdots+a_n)} = \prod_{i=1}^{n}\left(\frac{1}{a_i}\right). \quad (20)$$

The symmetrized form of $\mathcal{A}$ can then be written, using this identity:

$$\mathcal{A}_{\text{sym.}}=\frac{1}{n!}\sum_{\text{perm.}\{k_i,k_j\}}\mathcal{A}=\frac{\gamma_\mu}{n!}\prod_{i=1}^{n}\frac{2p_{1\nu_i}}{(-2p_1\cdot k_i)} =\frac{\gamma_\mu}{n!}\prod_{i=1}^{n}\frac{2p_{1\nu'_i}}{(-2p_1\cdot k_i')}, \quad (21)$$

where $\{k_1'\cdots k_n'\}$, $\{\nu_1'\cdots\nu_n'\}$ are arbitrary permutations of all the $\{k_1\cdots k_n\}$, $\{\nu_1\cdots\nu_n\}$, resepectively.

We now consider a graph $\{\alpha\}$, which can be constructed as follows: We give ourselves a permutation $k_1'\cdots k_n'$ of the $k_i$'s and assume that the photon $k_0$ is absorbed by line (2) at the position $(m+1)$. In other words, the order of absorption of the $n+1$ photons will be: $k_1'\cdots k_m'$; $k_0$; $k_{m+1}'\cdots k_n'$. The corresponding $\mathcal{B}_\alpha$ can then be written:

$$\mathcal{B}_\alpha=\gamma_\mu\prod_{i=1}^{n}\frac{2p_{2\nu_i'}}{[+2p_2\cdot K_{i,i'}]}\prod_{j=m+1}^{n}\frac{2p_{2\nu_j'}}{[-2p_2'\cdot K_{j,n+1'}]}. \quad (22)$$

In order to get the sum of all $\mathcal{B}_\alpha$: (a) we sum over all permutations of the first group $\{k_1'\cdots k_m'\}$ and of the second group $\{k_{m+1}'\cdots k_n'\}$ separately; (b) we multiply the number of ways in which a group of $m$ photons can be chosen from a larger set of $n$, that is $n!(m!)^{-1}[(n-m)!]^{-1}$; (c) we vary $m$ from 0 to $n$ to allow $k_0$ to take all the possible positions on line (2).

The result is

$$\mathcal{B}=\sum_{\{\alpha\}}\mathcal{B}_\alpha=\gamma_\mu\sum_{m=0}^{n}\frac{n!}{m!(n-m)!}\prod_{i=1}^{m}\frac{2p_{2\nu_i'}}{[2p_2\cdot k_i']} \times\prod_{j=m+1}^{n}\frac{2p_{2\nu_j'}{}'}{[-2p_2'\cdot k_j']}, \quad (23)$$

so that, finally, we can write (returning to the variables $k_i$) $M_{n+1}{}^0$, the soft photon part of $M_{n+1}$:

$$M_{n+1}{}^0 =-\frac{4\pi^2m^2}{(E_1E_2E_1'E_2')^{1/2}}\left(\frac{-i\alpha}{4\pi^3}\right)^{n+1}\sum_{m=0}^{n}\frac{1}{m!(n-m)!} \times\int\frac{[\gamma_\mu\times\gamma_\mu]}{(q-\sum k_i)^2}\prod_{i=1}^{m}\frac{4p_1\cdot p_2 d^4k_i}{(k_i^2-\lambda^2)[-2p_1\cdot k_i][2p_2\cdot k_i]} \times\prod_{j=m+1}^{n}\frac{4p_1\cdot p_2' d^4k_j}{(k_j^2-\lambda^2)[-2p_1\cdot k_j][-2p_2'\cdot k_j]}. \quad (24)$$

This expression can be simplified by introducing the Fourier transform of the lowest order matrix element:

$$M_1(x)=\frac{1}{(2\pi)^4}\int M_1(q^2)e^{iqx}d^4q, \quad (25)$$

where $M_1(q^2)$ is defined by Eq. (3), and the two functions:

$$U(p_1,p_2;x)=\frac{-i\alpha}{4\pi^3}\int\frac{(4p_1\cdot p_2)e^{ikx}d^4k}{(k^2-\lambda^2)(-2p_1\cdot k)(2p_2\cdot k)}, \quad (26)$$

and

$$U'(p_1,p_2';x)=-U(p_1,-p_2';x). \quad (27)$$

Then, $M_{n+1}{}^0$ becomes

$$M_{n+1}{}^0=\sum_{m=0}^{n}\frac{1}{m!(n-m)!}\int U^m(x)U'^{\,n-m}(x) \times M_1(x)e^{-iqx}d^4x, \quad (28)$$

and, after summation over $n$, we obtain

$$M^0=\sum_{n=0}^{\infty}M_{n+1}{}^0=\int\exp[U(x)+U'(x)] \times M_1(x)e^{-iqx}d^4x. \quad (29)$$

Equation (29) for $M^0$ has a form similar to the one obtained by YFS, except that we now have, at the end, a four-dimensional integration over $x$ which reflects the fact that we have rigorously taken into account the energy-momentum conservation of the photon lines [the YFS expression can be obtained from Eq. (24) by neglecting $\sum k_i$ compared to $q$ in the first factor of the integrand]. Our discussion of the previous section showed that it is essential to maintain the photon momentum conservation if we want to obtain *all* the $t$-dependence terms at high energy.

The cancellation of all $\ln(s)$ terms for $M^0$ can now be understood intuitively from Eqs. (29) and (27). If $U(x)$, for example, contains a term of the form $F(t,x)\ln(s)$, the change $p_2\rightleftharpoons -p_2'$ leaves $F(t,x)$ unchanged, but transforms $s$ into $u$. In our kinematical situation where $s\simeq -u$, Eq. (27) amounts, therefore, in this case, to

$$U'(x)=-\mathrm{Re}U(x). \quad (30)$$

Actually, $U(x)$ is somewhat more complicated. $U$ and $U'$ are calculated explicitly in the Appendix, and it is shown that when $x$ is finite, no term of order $\ln(s)$ remains in the exponential of Eq. (29). (For $x\to 0$, the cancellation has already been proved on the expressions for $B$ and $B'$ of YFS.)

The physical interpretation of the approximation which leads to Eq. (29) can be given as follows: In an $n$th order diagram, the maximum contribution to the scattering is obtained if each electron undergoes small changes of momentum at each of the $n$ vertices, the sum of which amounts to the (relatively small) total momentum transfer $q$. When there occurs one large change of momentum, the corresponding contribution is strongly decreased by the phase factor which appears in Eqs. (26) and (27). In this sense, the variable $x$ which we have introduced is closely related to the scattering in configuration space. Our approximation is very reminiscent of the "stationary phase" approximation which has been applied to the problem of high-energy scattering by a potential by Schiff[9] and others, and our Eq. (29) resembles closely the result obtained in those calculations [compare, for example, Eq. (13) in Schiff's paper].

### 2. Ultraviolet Corrections

One of our main arguments, in our contention that the photon pole does not move with energy, is that, once the "infrared" and "soft" photon parts have been separated out, the remaining ultraviolet contributions, which correspond to $k_i \gtrsim p_1$, $p_2$ or $p_2'$, do not depend on $t$. It is, therefore, quite important to examine this point further, and, in particular, to ask ourselves how we can obtain the next order correction to our expression for $M^0$.

From the discussion above, it is clear that this correction will come from terms *where each of the electrons undergoes two large changes of momentum of opposite sign*. Now, whenever one internal electron line has a momentum which is very different from a free momentum, the contribution to the matrix element is reduced by the denominator of the corresponding $S_F$ function. Therefore, the two photons of large and opposite momenta must be emitted and absorbed in succession in order to minimize this effect. The order of absorption of these two "hard" photons can either be the same as or opposite to their order of emission. Consequently, each of the graphs which we must consider for our correction will contain a "core" consisting of the two basic fourth-order diagrams, the remaining part being filled by soft photons emitted or absorbed in any order before or after the "core" (Fig. 2).

We can now apply to this class of graphs the same technique as was used in the previous paragraph. The calculation is straightforward, and leads to the following expression for $M^{(1)}$, the correction to $M^0$ of Eq. (29):

$$M^{(1)}=\int \exp[U(x)+U'(x)] \times[M_2(x)-M_2{}^0(x)]e^{-iqx}d^4x, \quad (31)$$

where $M_2(x)$ is the Fourier transform of the complete fourth-order matrix element and $M_2{}^0$ the part of it which is obtained by expanding the exponential in Eq. (29):

$$M_2{}^0(x)=(U+U')M_1(x). \quad (32)$$

[9] L. I. Schiff, Phys. Rev. **103**, 443 (1956).

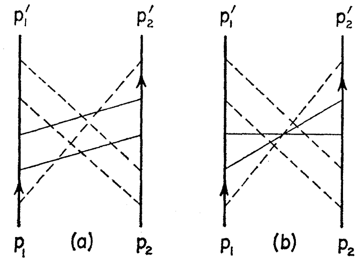


FIG. 2. General diagrams for the first ultraviolet corrections to the matrix element. Full and dashed lines indicate hard and soft photon, respectively.

Now, we have seen in Sec. III that $M_2-M_2{}^0$ does not depend on $q^2$ in the high-energy limit. Consequently, its Fourier transform will be proportional to $\delta_4(x)$ so that $M^{(1)}$ can be simplified as follows:

$$M^{(1)}=\exp(B+B')[M_2(q^2)-M_2{}^0(q^2)]; \quad (33)$$

[we have used the fact that $U(0)\simeq B$ and $U'(0)\simeq B'$, as was already pointed out in Sec. III after Eq. (14); there is a logarithmic divergence which cancels when $B$ and $B'$ are added].

The results of Secs. II and III show then that $M^{(1)}$ vanishes in the high-energy limit. But, what is more important is that, even before cancellation, *each term contributing to* $M^{(1)}$ *does not depend on the momentum transfer*.

It is easy to understand Eq. (33): if two of the exchanged photons have large momenta of opposite signs, the remaining $(n-1)$ soft photons contributing to $M_{n+1}{}^{(1)}$ are no longer sensitive to the over-all energy momentum conservation of the photon lines, as long as $q$ retains a moderate value. Then, the YFS method applies, and Eq. (33) is essentially their result, obtained in a different way, with no reference to the infrared divergence (which would, in any case, disappear from $B+B'$).

The next approximation $M^{(2)}$ can be calculated by assuming: (a) either that *three* successive virtual photons have large momenta which add up in such a way that their sum is small; (b) or that *two* virtual photons only have large momenta of opposite sign, but that a soft photon is emitted in between so that two large energy denominators are present in the matrix element. It is likely that the last effect will be dominant. In any case, following the same method as previously, $M^{(2)}$ must be calculated from a "core" of sixth-order matrix elements in which at least two of the three virtual photons are hard. Although the calculation is understandably more difficult, we do not see any reason which would make the general features of the result any different from those of $M^{(1)}$.

### 3. Influence of Photon-Photon Interaction

Two elements have been missing from our analysis so far: the effect of a modified photon propagator (closed loops with only two photon external lines) and the influence of photon-photon interaction.

It is relatively easy, in principle, to include the effect of electron loops in our formalism. In the soft photon contribution, for instance, all one has to do is to change $U(x)$ and $U'(x)$ into

$$\bar{U}(p_1,p_2;x)=\frac{-i\alpha}{4\pi^3}\int\frac{e^{ikx}(4p_1p_2)D_F'(k^2-\lambda^2)d^4k}{(-2p_1k)\cdot(+2p_2k)}, \quad (26a)$$

and

$$\bar{U}'(x)=-\bar{U}(p_1,-p_2';x), \quad (27a)$$

the matrix element being still given by Eq. (29). Their corresponding evaluation is, of course, considerably more difficult. However, Eq. (27a) which expresses the relation between crossed and uncrossed graphs makes it likely that the cancellation of the leading $\ln(s)$ or $\ln|u|$ terms still occurs in this case.

As to the photon-photon interaction, it is sufficient to consider a graph like Fig. 3, where $n+1$ photons emerge from a black box in which the electron "line (1)" enters and goes out. As in Sec. 1 of this section, the photon momenta are labeled $k_1\cdots k_n$ and $k_0=q-\sum k_i$. Then, in the soft photon approximation, the general form of the matrix element is still given by Eq. (17), except that now we replace $\mathcal{A}$ by an unknown but *symmetric* function of the photon momenta and polarizations

$$\mathcal{A}\equiv(1/n!)F_\mu^{(n)}(k_1\cdots k_n;\nu_1\cdots\nu_n); \quad (34)$$

($F_\mu^{(n)}$ depends also on $p_1$ and $q$, but we can leave this dependence implicit). The calculation of $\mathcal{B}$ is unchanged but, because of the symmetric character of the photons, we can, by suitable changes of variables, express it differently from Eq. (23):

$$\mathcal{B}=\gamma_\mu\prod_{i=1}^{n}\left[\frac{2p_{2\nu_i}}{(2p_2\cdot k_i)}+\frac{2p_{2\nu_i}'}{(-2p_2'k_i)}\right]; \quad (35)$$

$\bar{M}_{n+1}^{(0)}$, the soft photon contribution to $\bar{M}_{n+1}$ can now be expressed quite generally as:

$$\bar{M}_{n+1}^{0}=-\frac{4\pi^2m^2}{(E_1E_2E_1'E_2')^{1/2}}\left(\frac{-i\alpha}{4\pi^3}\right)^{n+1}\frac{1}{n!}\times\int\frac{[F_\mu^{(n)}(k_1\cdots k_n;\nu_1\cdots\nu_n)\times\gamma_\mu]}{(q-\sum k_i)^2}\times\prod_{i=1}^{n}D_F'(k_i^2-\lambda^2)\left[\frac{2p_{2\nu_i}}{(2p_2\cdot k_i)}-\frac{2p_{2\nu_i}'}{(2p_2'\cdot k_i)}\right]d^4k_i. \quad (36)$$

The same cancellation which we have already observed still seems to occur here because, in the limit of high $p_i$ and small $q$, $p_2\simeq p_2'$, so that each of the factors in the infinite product on the right-hand side goes to zero in that limit. On the other hand, it is well known that the photon-photon interaction introduces a strong convergence factor for large values of $k^2$; the influence of the ultraviolet contributions should then be even less important than we have estimated in Sec. IV.2.

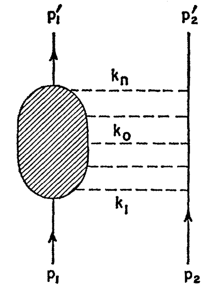


FIG. 3. The general graph for the exchange of $n+1$-photons with photon-photon interactions.

The total matrix element $\bar{M}^0$ can be summed into an exponential, as in Eq. (29), only if we assume that, at high energy, $F_\mu^{(n)}$ factorizes as a product involving separately the momenta and polarizations $(k_i,\nu_i)$. The corresponding generalization of the functions $U(x)$ and $U'(x)$ is rather obvious; therefore, we do not need to write the result here.

## 4. A Symmetry Property of the High-Energy Scattering Amplitude

We have not made any difference, so far, between matrix elements involving an even or an odd number of exchanged photons. Nevertheless, as was pointed out in the Introduction, there is an important physical difference between them. We want to show here that this difference reflects itself in a symmetry property of the scattering amplitude at high energy.

In the kinematical situation which we have considered in this paper where $s\simeq|u|$ is very large and $t$ is finite, the leading terms of the scattering amplitude are not modified if we exchange everywhere $p_2$ and $-p_2'$, leaving $p_1$ and $p_1'$ as they are, so that $t$ remains the same, but $s$ and $u$ are exchanged. To order $\alpha^n$ with respect to $M_1$, for example, this would not affect the real part of a term like $\alpha^nF_n(t)(\ln s)^nM_1$. We can then introduce a transformed matrix element $\tilde{M}$ through the formula:

$$M(p_1,p_2;p_1',p_2')\to\tilde{M}(p_1,p_2;p_1',p_2')=\tfrac{1}{2}[M(p_1,p_2;p_1',p_2')+M(p_1,-p_2';p_1',-p_2)]. \quad (37)$$

This transformation has the effect to suppress a large number of terms in the perturbation expansion of $M$. For instance, in the analysis of the previous sections, because of relations (5b), (27), and (27a), the exponents in Eqs. (4), (29), or (33) change their sign if one makes the exchange $p_2\rightleftharpoons-p_2'$. Consequently, the odd powers of $\alpha$ in the perturbation expansion of the corresponding matrix elements (which are due to the exchange of an *even* number of photons) disappear altogether in $\tilde{M}$. Similarly, the cancellation of the $\ln(s)$ terms in the complete two-photons exchange graphs becomes obvious, since:

$$M_2'(p_1,p_2;p_1',p_2')\cong -M_2(p_1,-p_2';p_1',-p_2) \quad (38)$$

at high energy.

This type of cancellation is still maintained if one takes into account photon-photon interactions, as is clear for Eq. (36) where the exchange $p_2 \rightleftarrows -p_2'$ changes $\bar{M}_{n+1}^0$ into $(-1)^n\bar{M}_{n+1}^0$. Actually, this symmetry property does not hold only for the "soft photon" part of the matrix elements. *It is true in general.* To see this, it is sufficient to look at the general diagram of Fig. 3, and to compare two graphs $\{\alpha_1\}$ and $\{\alpha_2\}$ where the photon $k_0$ occupies the positions $m+1$ and $n-m+1$, respectively. Then, because of the symmetry of $F_\mu^{(n)}$ with respect to the exchange of all $k_i$ and $\nu_i$, one can see that, for each factor $\gamma_{\nu_i}S_F(p_2+K_{1,i})$ in one graph, there is a factor $\gamma_{\nu_i}S_F(p_2'-K_{1,i})$ in the other one, and vice versa (one has, for the second graph, to relabel all the $k_i$, $\nu_i$ in the opposite order). Since $S_F(-p)=-S_F(p)$ for large $p$, it follows that, in general:

$$\tilde{M}_{n+1}(p_1,p_2;p_1',p_2') =\tfrac{1}{2}[1+(-1)^n]M_{n+1}(p_1,p_2;p_1',p_2'). \quad (39)$$

Since the change $M\rightarrow\tilde{M}$ does not affect the leading logarithmic terms at high energy,[10] the result is that the diagrams of even order cannot contribute when $s\rightarrow\infty$, $t$ remaining finite. For the case where $n+1$ is odd, the number of independent graphs is reduced by a factor 2. [Using the labeling $\{\alpha_1\cdots\alpha_{n+1}\}$ introduced in Sec. IV.1, one finds that the change $M\rightarrow\tilde{M}$ transforms $\{\alpha_1\cdots\alpha_{n+1}\}$ into $(-1)^n\{\alpha_{n+1}\cdots\alpha_1\}$.]

We conclude that there cannot be, in any case, an "even counterpart" (even under charge conjugation, that is) to the photon trajectory; actually, as we have already emphasized, we do not believe that there is a photon trajectory either!

## V. CONCLUDING REMARKS

The conclusion of our work is that there does not seem to be any $\ln(s)$ terms in the elastic electron-electron scattering amplitude for high values of the squared energy $s$. If there are terms of high order which, for one reason or another, we have missed, they can only come from ultraviolet contributions to the matrix element, and should not depend on the momentum transfer, if the latter remains finite. One can conclude, therefore, that the photon pole does not "move" with energy.

Another conclusion which should be emphasized is that, when summing subsets of graphs in the high-energy limit, any kind of "ladder" approximation is rather dangerous.

[10] This would not be the case if the leading term of the $(n+1)$th order matrix element were of order $s^{-1}(\ln s)^nM_1$, as seems to be the case for the exchange of scalar particles (see M. Gell-Mann and M. L. Goldberger, and J. C. Polkinghorne, quoted in footnote 3). Then, the role of the even and odd graphs should be reversed.

Finally, we would like to speculate on the significance of possible $\ln(s)$ terms with constant coefficients, if they turn out to exist after all. These terms might be summed up into the form $\exp\{c(\alpha)\ln(s)\}M_1$, where $c(\alpha)$ should be an even function of $\alpha$ only. We assume now that the exchange of a particle of fixed angular momentum $J$ leads to a scattering amplitude behaving like $s^J$ at high energy. $M_1$ behaves like $s$ in that limit, and since we know that, if the photon has a fixed angular momentum, it must be $J=1$, we conclude that one should impose the consistency condition $c(\alpha)=0$, which can be used to determine the fine structure constant! It is more likely, however, that this condition will be satisfied identically for each power of $\alpha^2$. Even if this were not the case, the computation of $c(\alpha)$ to all orders should be a formidable problem indeed.

The author would like to thank Professor Robert Oppenheimer for extending once more to him his kind hospitality at the Institute for Advanced Study.

## APPENDIX

### Calculation of $U(x)$ and $U'(x)$

We start with the calculation of $U'(x)$:

$$U'(x)=-\frac{i\alpha|u|}{2\pi^3}\times\int\frac{e^{ikx}d^4k}{(k^2-\lambda^2)(-2p_1k+i\epsilon)(-2p_2'k+i\epsilon)}. \quad (A1)$$

We first combine the two terms linear in $k$ in the denominator through a Feynman parametric integral:

$$\frac{1}{(-2p_1k)(-2p_2'\cdot k)}=\frac{1}{2}\int_{-1}^{+1}\frac{dz}{(-2P_z\cdot k+i\epsilon)^2}, \quad (A2)$$

where $2P_z=(1-z)p_1+(1+z)p_2'$. We then express the denominators in an exponential form, through the formula:

$$\frac{1}{(D+i\epsilon)^{n+1}}=\frac{(-i)^{n+1}}{n!}\int_0^\infty e^{iaD}a^n da, \quad (A3)$$

so that

$$U'(x)=\frac{\alpha|u|}{4\pi^3}\int_{-1}^{+1}dz\int_0^\infty\int_0^\infty bdbda\times\int\exp\{i[kx+a(k^2-\lambda^2)-2bk\cdot P_z]\}d^4k. \quad (A4)$$

Making a displacement of the $k$ integration and using the formula:

$$\int e^{i\rho k^2}d^4k=-\frac{i\pi^2}{\rho^2}\epsilon(\rho), \quad (A5)$$

we get, after changing $a$ into $a^{-1}$:

$$U'(x)=-\frac{i\alpha|u|}{4\pi}\int_{-1}^{+1}dz \times\int_0^\infty\int_0^\infty \exp[-ia(bP_z-\tfrac{1}{2}x)^2-i\lambda^2/a]bdbda. \quad \text{(A6)}$$

Putting

$$\Delta=(bP_z-\tfrac{1}{2}x)^2=b^2P_z^2-b(P_z\cdot x)+\tfrac{1}{4}x^2, \quad \text{(A7)}$$

we find that we can do the $a$ integration[11]:

$$\int_0^\infty e^{-ia\Delta-i\lambda^2/a}da = -\frac{\pi\lambda}{\Delta^{1/2}}H_1^{(2)}(2\lambda\Delta^{1/2}), \quad \text{if } \Delta>0$$

$$=\frac{2i\lambda}{(-\Delta)^{1/2}}K_1[2\lambda(-\Delta)^{1/2}], \quad \text{if } \Delta<0, \quad \text{(A8)}$$

where $H_1^{(2)}$ and $K_1$ are the usual Hankel functions; $\lambda$ in these expressions acts like a regularizing factor in the limit of high $b$, since the Hankel functions force then the integral to converge. We shall actually take the limit $\lambda\rightarrow 0$ immediately, but put a cutoff $b_{\max}$ in the $b$ integration. The cutoff will disappear when $U$ and $U'$ are added. We then have very simply:

$$U'(x)=-\frac{\alpha|u|}{4\pi}\int_{-1}^{+1}dz\int_0^{b_{\max}}\frac{bdb}{\Delta}. \quad \text{(A9)}$$

We divide this expression into

$$U'^{(1)}=-\frac{\alpha|u|}{8\pi}\int_{-1}^{+1}\frac{dz}{P_z^2}\int_0^{b_{\max}}\frac{[2bP_z^2-(P_z\cdot x)]}{\Delta} \quad \text{(A10)}$$

and

$$U'^{(2)}=-\frac{\alpha|u|}{8\pi}\int_{-1}^{+1}dz\frac{(P_z\cdot x)}{P_z^2}\int_0^\infty\frac{db}{\Delta}. \quad \text{(A11)}$$

($U'^2$ converges when $b\rightarrow\infty$.) $U'^{(1)}$ is real when $x^2<0$ but has an imaginary part when $x^2>0$. Its real part is always given, in the limit of high $|u|$, by

$$\mathrm{Re}U'^{(1)}=\frac{2\alpha}{\pi}\phi_1(-u)\ln\frac{(2b_{\max})}{|x^2|}-\frac{\alpha}{2\pi}\phi_2\left(-u;\frac{1}{|x^2|}\right), \quad \text{(A12)}$$

where $\phi_1$ and $\phi_2$ are defined by Eqs. (7) and (8) of the text. $\mathrm{Re}U'^{(1)}$ *is cancelled by a similar term coming from* $\mathrm{Re}U^{(2)}$ exactly in the same way as in Secs. II and III [intuitively, one can understand $|x^2|^{-1}$ as a continuous variable which takes values between $\lambda^2$ and $(-t)$, the cancellation occurring for *each* of these intermediate values].

$U'^{(2)}$ is somewhat more difficult to evaluate, and, to avoid lengthy developments, we shall calculate it only in the case $x^2<0$ (the case $x^2>0$ gives a similar result). Writing $-x^2=R^2$ and $[(P_z\cdot x)^2-x^2P_z^2]^{1/2}=Q_z$, we can do the $b$ integration, and obtain

$$U'^{(2)}=-\frac{\alpha|u|}{8\pi}\int_{-1}^{+1}\frac{dz}{P_z^2}\frac{(P_z\cdot x)}{Q_z}\ln\frac{R^2P_z^2}{[Q_z+(P_z\cdot x)]^2}. \quad \text{(A13)}$$

Now, since $P_z^2=m^2+\frac{1}{4}|u|(1-z^2)$, it is clear that the main contribution to the integral will come from values where $P_z^2$ is small, i.e., near $z=\pm 1$. We define then a number $\epsilon$ such that

$$m^2/|u|\ll\epsilon\ll 1, \quad \text{(A14)}$$

write $C_1=(p_1\cdot x)$, $C_2=(p_2\cdot x)$, $C_2'=(p_2'\cdot x)$, and express $U'^{(2)}$ as

$$U'^{(2)}=(\alpha/2\pi)[G(R,C_2')+G(R,C_1)] \quad \text{(A15)}$$

with

$$G(R,C)=-\frac{|u|}{4}\int_{1-\epsilon}^{1}\frac{dz}{P_z^2}\frac{C}{[R^2P_z^2+C^2]^{1/2}} \times\ln\frac{R^2P_z^2}{[(R^2P_z^2+C^2)^{1/2}+C]^2}. \quad \text{(A16)}$$

We can then write approximately $P_z^2\simeq m^2+\frac{1}{2}|u|(1-z)$, change to the variable $\sinh\chi=C(RP_z)^{-1}$, $\sinh\chi_0=C(mR)^{-1}$, $\sinh\chi_1=2C(R\epsilon|u|)$, and obtain, in the limit defined by (A14):

$$G(R,C)\simeq\int_{\chi_1}^{\chi_0}2\chi d\chi\simeq\ln^2\left[\frac{C+(m^2R^2+C^2)^{1/2}}{mR}\right]. \quad \text{(A17)}$$

The calculation of $\mathrm{Re}U^{(2)}$ goes along similar lines, except that care must be taken because several expressions change sign in the integration domain. It is clear that the only cases which are of interest to us are those in which the $C_i$'s are large compared to $mR$ (otherwise $G\rightarrow 0$ automatically). If we suppose then that $2C_i^2(sR^2)^{-1}\gg\epsilon$, the calculation of $\mathrm{Re}U^{(2)}$ becomes relatively easy, with the result:

$$\mathrm{Re}U^{(2)}\simeq-(\alpha/2\pi)[G(R,C_2)+G(R,C_1)], \quad \text{(A18)}$$

so that, finally, the only remaining parts of $U+U'$ are

$$\mathrm{Re}(U+U')\simeq-(\alpha/2\pi)[G(R,C_2)-G(R,C_2')]. \quad \text{(A19)}$$

(One verifies that $U+U'=0$ if $p_2=p_2'$, as is apparent from their definition.) Since, in our kinematical situation, $p_2\simeq p_2'$, we can approximate Eq. (A19) in

[11] W. Magnus and F. Oberhettinger, *Formeln und Sätze für die Speziellen Functionen der Mathematischen Physik* (Springer-Verlag, Berlin, 1948), 2nd ed., p. 174.

the case where $C_2 \simeq C_2' \gg mR$ and obtain:

$$\mathrm{Re}(U+U') \simeq -\frac{\alpha}{2\pi} \ln\frac{4C_2C_2'}{m^2R^2} \ln\frac{C_2}{C_2'} \qquad \text{(A20)}$$

which cannot increase like $\ln(s)$ in a given $x$ direction. Also, this is the worst possible situation that can happen. In all the other cases, $\mathrm{Re}(U+U') \to 0$ much faster.

The technique of Fourier transforms of matrix elements, which has not been much used in the past, can be applied to many problems in high-energy scattering. For instance, if one has to calculate an expression of the form

$$M_{\sigma\mu\cdots} = \int \frac{(k_\sigma k_\mu \cdots) d^4k}{\Delta(k)}, \qquad \text{(A21)}$$

in the high-energy limit, one can introduce a factor $\exp\{ikx\}$ and express all the $k_\sigma$, $k_\mu$, etc., as derivatives with respect to $x$. One can the compute the Fourier transform of $\Delta^{-1}$ in the high-energy limit and let $x \to 0$ at the end. This eliminates many "components" of the matrix element which do not contribute at high energy.

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# Energy and Momentum Density in Field Theory*

JULIAN SCHWINGER
*Harvard University, Cambridge, Massachusetts*
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It is shown that the energy density commutator condition in its simplest form is valid for interacting spin 0, $\frac{1}{2}$, 1 field systems, but not for higher spin fields. The action principle is extended, for this purpose, to arbitrary coordinate frames. There is a discussion of four categories of fields and some explicit consideration of spin $\frac{3}{2}$ as the simplest example that gives additional terms in the energy density commutator. As the fundamental equation of relativistic quantum field theory, the commutator condition makes explicit the greater physical complexity of higher spin fields.

## INTRODUCTION

AFTER it had been noticed[1] that the energy and momentum density of a particular field system obeyed the equal time commutation relation

$$-i[T^{00}(x),T^{00}(x')] = -(T^{0k}(x)+T^{0k}(x'))\partial_k\delta(\mathbf{x}-\mathbf{x}'),$$

a general proof was constructed[2] by considering the response to an external gravitational field. The commutator condition applies to all systems for which $(-g^{00})T^{00}$ and $(-g_{00})^{1/2}T^{0k}$ are independent of the gravitational field, when it is of the special type

$$g_{kl}=\delta_{kl}, \quad g_{0k}=0, \quad -g_{00}(x)\neq 1.$$

How extensive is this distinguished class of physical systems? We shall find that fields with spin 0, $\frac{1}{2}$, 1 are included, but not fields of larger spin. Thus, higher spin fields can now be characterized, not merely as mathematically more complicated structures, but by their greater physical complexity.

The technical problem encountered here is the extension of the action principle to arbitrary coordinate frames, subject to the requirement of coordinate invariance. Even more is involved for, as Weyl[3] was the first to recognize, the description of spin entails the introduction, at each point, of an independent Lorentz coordinate frame, combined with the demand of invariance under local Lorentz transformations. This is the ultimate expression of the local field concept.

The relation between the local and the global coordinate systems is conveyed by a family of vector fields $e_a{}^\mu(x)$, which respond to general coordinate transformations and local Lorentz transformations as

$$\bar{e}_a{}^\mu(\bar{x}) = (\partial\bar{x}^\mu/\partial x^\nu)e_a{}^\nu(x),$$

and

$$\bar{e}_a{}^\mu(x) = l_a{}^b(x)e_b{}^\mu(x),$$

respectively. In the latter, the matrix $l$ obeys the Lorentz invariance condition

$$l^T g l = g,$$

where $g^{ab}$ is the Minkowski metric tensor, which we take to have the value $-1$ for its temporal component. The inverse vector set $e_\mu{}^a(x)$,

$$e_\mu{}^a e_b{}^\mu = \delta_b{}^a, \quad e_\mu{}^a e_a{}^\nu = \delta_\mu{}^\nu,$$

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[1] J. Schwinger, Phys. Rev. **127**, 324 (1962).

[2] J. Schwinger, Phys. Rev. **130**, 406 (1963).

[3] H. Weyl, Z. Physik. **56**, 330 (1929). Our approach derives most directly from this source rather than the later developments of Schrödinger, Bargmann, Belinfante, and others.