the case where $C_2 \simeq C_2' \gg mR$ and obtain:

$$\mathrm{Re}(U+U') \simeq -\frac{\alpha}{2\pi} \ln\frac{4C_2C_2'}{m^2R^2} \ln\frac{C_2}{C_2'} \qquad \text{(A20)}$$

which cannot increase like $\ln(s)$ in a given x direction. Also, this is the worst possible situation that can happen. In all the other cases, $\mathrm{Re}(U+U') \to 0$ much faster.

The technique of Fourier transforms of matrix elements, which has not been much used in the past, can be applied to many problems in high-energy scattering. For instance, if one has to calculate an expression of the form

$$M_{\sigma\mu\cdots} = \int \frac{(k_\sigma k_\mu \cdots) d^4k}{\Delta(k)}, \qquad \text{(A21)}$$

in the high-energy limit, one can introduce a factor $\exp\{ikx\}$ and express all the k_σ, k_μ, etc., as derivatives with respect to x. One can the compute the Fourier transform of Δ^{-1} in the high-energy limit and let $x \to 0$ at the end. This eliminates many "components" of the matrix element which do not contribute at high energy.

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Energy and Momentum Density in Field Theory*

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It is shown that the energy density commutator condition in its simplest form is valid for interacting spin 0, $\frac{1}{2}$, 1 field systems, but not for higher spin fields. The action principle is extended, for this purpose, to arbitrary coordinate frames. There is a discussion of four categories of fields and some explicit consideration of spin $\frac{3}{2}$ as the simplest example that gives additional terms in the energy density commutator. As the fundamental equation of relativistic quantum field theory, the commutator condition makes explicit the greater physical complexity of higher spin fields.

INTRODUCTION

AFTER it had been noticed[1] that the energy and momentum density of a particular field system obeyed the equal time commutation relation

$$-i[T^{00}(x),T^{00}(x')] = -(T^{0k}(x)+T^{0k}(x'))\partial_k\delta(\mathbf{x}-\mathbf{x}'),$$

a general proof was constructed[2] by considering the response to an external gravitational field. The commutator condition applies to all systems for which $(-g^{00})T^{00}$ and $(-g_{00})^{1/2}T^{0k}$ are independent of the gravitational field, when it is of the special type

$$g_{kl}=\delta_{kl}, \quad g_{0k}=0, \quad -g_{00}(x)\neq 1.$$

How extensive is this distinguished class of physical systems? We shall find that fields with spin 0, $\frac{1}{2}$, 1 are included, but not fields of larger spin. Thus, higher spin fields can now be characterized, not merely as mathematically more complicated structures, but by their greater physical complexity.

The technical problem encountered here is the extension of the action principle to arbitrary coordinate frames, subject to the requirement of coordinate invariance. Even more is involved for, as Weyl[3] was the first to recognize, the description of spin entails the introduction, at each point, of an independent Lorentz coordinate frame, combined with the demand of invariance under local Lorentz transformations. This is the ultimate expression of the local field concept.

The relation between the local and the global coordinate systems is conveyed by a family of vector fields $e_a{}^\mu(x)$, which respond to general coordinate transformations and local Lorentz transformations as

$$\bar{e}_a{}^\mu(\bar{x}) = (\partial\bar{x}^\mu/\partial x^\nu)e_a{}^\nu(x),$$

and

$$\bar{e}_a{}^\mu(x) = l_a{}^b(x)e_b{}^\mu(x),$$

respectively. In the latter, the matrix l obeys the Lorentz invariance condition

$$l^T g l = g,$$

where g^{ab} is the Minkowski metric tensor, which we take to have the value -1 for its temporal component. The inverse vector set $e_\mu{}^a(x)$,

$$e_\mu{}^a e_b{}^\mu = \delta_b{}^a, \quad e_\mu{}^a e_a{}^\nu = \delta_\mu{}^\nu,$$

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[1] J. Schwinger, Phys. Rev. **127**, 324 (1962).

[2] J. Schwinger, Phys. Rev. **130**, 406 (1963).

[3] H. Weyl, Z. Physik. **56**, 330 (1929). Our approach derives most directly from this source rather than the later developments of Schrödinger, Bargmann, Belinfante, and others.

has analogous transformation properties,

$$\bar{e}_\mu{}^a(\bar{x})=(\partial x^\nu/\partial\bar{x}^\mu)e_\nu{}^a(x),$$
$$\bar{e}_\mu{}^a(x)=l^a{}_b(x)e_\mu{}^b(x).$$

In particular,

$$(dx)\ \det e_\mu{}^a(x),\quad (dx)=dx^0dx^1dx^2dx^3,$$

is an invariant for both types of transformations (with $\det l=+1$).

The change of the invariant action operator W for an infinitesimal alteration of the $e_\mu{}^a(x)$ can be written as

$$\delta_e W=\int (dx)(\det e_\nu{}^b(x))\delta e_\mu{}^a(x)T^\mu{}_a(x),$$

which defines the set of Hermitian operators $T^\mu{}_a$. If $\delta e_\mu{}^a$ refers to an infinitesimal local Lorentz transformation,

$$\delta e_\mu{}^a(x)=\delta\omega^a{}_b(x)e_\mu{}^b(x),$$
$$\delta\omega_{ab}=-\delta\omega_{ba},$$

local Lorentz invariance requires that

$$T^{ab}(x)=T^{ba}(x),$$

where

$$T^{ab}=e_\mu{}^aT^{\mu b}.$$

The same symmetry property applies to the tensor

$$T^{\mu\nu}=e_a{}^\mu e_b{}^\nu T^{ab}=T^{\mu b}e_b{}^\nu=T^{\nu\mu}.$$

That symmetry can be exploited by writing

$$\delta e_\mu{}^aT^\mu{}_a=\delta e_\mu{}^ae_{\nu a}T^{\mu\nu}$$
$$=\tfrac{1}{2}\delta(e_\mu{}^ae_{\nu a})T^{\mu\nu},$$

which introduces the symmetrical tensor

$$g_{\mu\nu}=e_\mu{}^ag_{ab}e_\nu{}^b=e_\mu{}^ae_{\nu a}.$$

Since

$$g=\det g_{\mu\nu}=-(\det e_\mu{}^a)^2,$$

we can conclude that

$$\delta_eW=\delta_gW$$
$$=\int (dx)(-g)^{1/2}\tfrac{1}{2}\delta g_{\mu\nu}T^{\mu\nu},$$

which makes contact with the more familiar definition of the stress tensor.

ACTION PRINCIPLE

We take the action operator to have the following standard form in a Minkowski space:

$$W=\int (dx)[\tfrac{1}{4}(\chi A^\mu\partial_\mu\chi-\partial_\mu\chi A^\mu\chi)-\mathcal{H}(\chi)],$$

where the A^μ are four constant numerical matrices and the $\chi(x)$ are a set of Hermitian fields. To generalize this to arbitrary coordinate frames, we write

$$W=\int (dx)\ \det e_\mu{}^a(x)\mathcal{L}(x),$$

where $\mathcal{L}$ is tentatively constructed as

$$\mathcal{L}(x)=\tfrac{1}{4}e_a{}^\mu(x)[\chi(x)A^a\partial_\mu\chi(x)-\partial_\mu\chi(x)A^a\chi(x)]-\mathcal{H}(\chi(x)).$$

Here the A^a are the same set of four constant numerical matrices now defined relative to the local coordinate system. The requirement of invariance under general coordinate transformations is trivially satisfied if the $\chi(x)$ behave as scalars,

$$\bar{\chi}(\bar{x})=\chi(x).$$

But what of invariance under local Lorentz transformations?

For such a transformation,

$$\bar{e}_a{}^\mu(x)=l_a{}^b(x)e_b{}^\mu(x),$$

we have

$$\bar{\chi}(x)=L(x)\chi(x),$$

where

$$\mathcal{H}(\bar{\chi})=\mathcal{H}(\chi)$$

and

$$L^TA^aL=l^a{}_bA^b$$

ensure invariance if the Lorentz transformation does not depend upon position. But our tentative Lagrange function $\mathcal{L}$ is not invariant under arbitrary local Lorentz transformations:

$$\bar{\mathcal{L}}-\mathcal{L}=\tfrac{1}{4}e_a{}^\mu[\chi A^a(L^{-1}\partial_\mu L)\chi-\chi(L^{-1}\partial_\mu L)^TA^a\chi].$$

To identify the structure of $L^{-1}\partial_\mu L$, we remark that two infinitesimally neighboring Lorentz transformations are connected by an infinitesimal transformation,

$$l_{ab}+\delta l_{ab}=l_{ac}(\delta_b{}^c+\delta\omega^c{}_b),$$

where

$$\delta\omega_{ab}=l^c{}_a\delta l_{cb}=-\delta\omega_{ba}.$$

The corresponding group composition property of L, expressed in terms of the spin matrices, is

$$L+\delta L=L(1+\tfrac{1}{2}i\delta\omega_{ab}S^{ab}),$$

and therefore

$$L^{-1}\partial_\mu L=\tfrac{1}{2}iS^{ab}l^c{}_a\partial_\mu l_{cb}.$$

This result indicates that the structure of the Lagrange function must be modified into

$$\mathcal{L}=\tfrac{1}{4}e_a{}^\mu[\chi A^a(\partial_\mu-\tfrac{1}{2}i\omega_{\mu bc}S^{bc})\chi-(\partial_\mu-\tfrac{1}{2}i\omega_{\mu bc}S^{bc})\chi A^a\chi]-\mathcal{H}(\chi),$$

where the vector fields $\omega_{\mu ab}=-\omega_{\mu ba}$ are given such local Lorentz transformation properties that

$$L^{-1}(\partial_\mu-\tfrac{1}{2}i\bar{\omega}_{\mu ab}S^{ab})L=\partial_\mu-\tfrac{1}{2}i\omega_{\mu ab}S^{ab}.$$

The required transformation law is

$$\bar{\omega}_{\mu ab} = l_a{}^{a'} l_b{}^{b'} (\omega_{\mu a' b'} + l_{ca'}\partial_\mu l^c{}_{b'}) = l_a{}^{a'} l_b{}^{b'} \omega_{\mu a' b'} + l_b{}^{b'}\partial_\mu l_{ab'}.$$

An alternative statement, in terms of the functions

$$\omega_{abc}(x) = e_a{}^\mu(x)\omega_{\mu bc}(x),$$

is

$$\bar{\omega}_{abc} = l_a{}^{a'} l_b{}^{b'} l_c{}^{c'} \omega_{a'b'c'} + l_a{}^{a'} l_c{}^{c'} e_{a'}{}^\mu \partial_\mu l_{bc'}.$$

Now let us observe the following local Lorentz transformation property:

$$(\bar{e}_a{}^\mu \partial_\mu \bar{e}_b{}^\nu)\bar{e}_{\nu c} = l_a{}^{a'} l_b{}^{b'} l_c{}^{c'} (e_{a'}{}^\mu \partial_\mu e_{b'}{}^\nu) e_{\nu c'} + l_a{}^{a'} l_c{}^{c'} e_{a'}{}^\mu \partial_\mu l_{bc'}.$$

Accordingly,

$$\omega_{abc} - (e_a{}^\mu \partial_\mu e_b{}^\nu) e_{\nu c} = \Gamma_{abc}$$

has the simple behavior

$$\bar{\Gamma}_{abc} = l_a{}^{a'} l_b{}^{b'} l_c{}^{c'} \Gamma_{a'b'c'}.$$

One should not be mislead by the notation to conclude that $\Gamma_{abc}(x)$, like $\omega_{abc}(x)$, is a scalar under general coordinate transformations. Rather

$$\bar{\Gamma}_{abc}(\bar{x}) - \Gamma_{abc}(x) = -e_a{}^\mu e_b{}^\lambda (\partial_\mu \partial_\lambda \bar{x}^\nu) e_{\kappa c} (\partial x^\kappa / \partial \bar{x}^\nu),$$

which also shows that the combination $\Gamma_{abc} - \Gamma_{bac}$ is a scalar. The antisymmetry of ω_{abc} in b and c implies that

$$\Gamma_{abc} + \Gamma_{acb} = -e_a{}^\mu [(\partial_\mu e_b{}^\nu) e_{\nu c} - (\partial_\mu e_{\nu b}) e_c{}^\nu] = e_a{}^\mu e_b{}^\lambda e_c{}^\nu \partial_\mu g_{\nu\lambda}$$

or, with an evident definition,

$$\Gamma_{\mu\nu\lambda} + \Gamma_{\mu\lambda\nu} = \partial_\mu g_{\nu\lambda}.$$

If we adopt the invariant symmetry restriction

$$\Gamma_{\mu\nu\lambda} - \Gamma_{\nu\mu\lambda} = 0,$$

we can construct $\Gamma_{\mu\nu\lambda}$ explicitly. It is the Christoffel symbol

$$\Gamma_{\mu\nu\lambda} = \tfrac{1}{2}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}).$$

The same symmetry restriction in the form

$$\Gamma_{abc} - \Gamma_{bac} = 0$$

gives

$$\omega_{abc} - \omega_{bac} = \Omega_{cab},$$

where

$$\Omega_{cab} = e_{\nu c}[e_a{}^\mu \partial_\mu e_b{}^\nu - e_b{}^\mu \partial_\mu e_a{}^\nu] = -\Omega_{cba}.$$

From this we derive

$$\omega_{abc} = \tfrac{1}{2}[\Omega_{bca} + \Omega_{cab} - \Omega_{abc}].$$

ENERGY AND MOMENTUM DENSITY

It suffices, for present purposes, to choose the field $e_a{}^\mu(x)$ as

$$e_{(0)}{}^0(x) \neq 1, \quad e_{(k)}{}^0 = 0,$$
$$e_{(0)}{}^k(x) \simeq 0, \quad e_{(l)}{}^k = \delta_l{}^k,$$

where (0) and (k) are local coordinate indices and $\simeq 0$ implies that $e_{(0)}{}^k$ differs only infinitesimally from zero. The inverse vector system is

$$e_0{}^{(0)} = (e_{(0)}{}^0)^{-1}, \quad e_k{}^{(0)} = 0,$$
$$e_0{}^{(k)} = -e_0{}^{(0)} e_{(0)}{}^k \simeq 0, \quad e_l{}^{(k)} = \delta_l{}^k,$$

and

$$-g_{00} = (e_0{}^{(0)})^2, \quad g_0{}^k = e_0{}^{(k)}, \quad g_{kl} = \delta_{kl}.$$

The dependence of the action operator upon the variable functions is given by

$$\delta_e W = \int (dx)[-T^{(0)(0)}\delta e_0{}^{(0)} + T^{(0)}{}_{(k)}\delta e_0{}^{(k)}],$$

where, with $g_{0k} = 0$,

$$T^{(0)(0)} = (-g_{00})T^{00}, \quad T^{(0)}{}_{(k)} = (-g_{00})^{1/2} T^0{}_k$$

are just the quantities of interest.

The elements of Ω_{abc} that differ from zero are

$$\Omega_{(0)(k)(0)} = -\Omega_{(0)(0)(k)} = e_{(0)}{}^0 \partial_k e_0{}^{(0)},$$

and

$$\Omega_{(l)(k)(0)} = -\Omega_{(l)(0)(k)} = -e_{(0)}{}^0 \partial_k e_0{}^{(l)}.$$

The action operator is now given by

$$W = \int (dx)[\tfrac{1}{4}(\chi A^{(0)}\partial_0\chi - \partial_0\chi A^{(0)}\chi) - e_0{}^{(0)} T^{(0)(0)} + e_0{}^{(k)} T^{(0)}{}_{(k)}],$$

where

$$T^{(0)(0)} = \mathcal{H} - \tfrac{1}{4}(\chi A^{(k)}\partial_k\chi - \partial_k\chi A^{(k)}\chi) - \tfrac{1}{4} i\partial_k[\chi(A^{(0)}S^{(0)(k)} - S^{(0)(k)T}A^{(0)})\chi],$$

and

$$T^{(0)}{}_{(k)} = -\tfrac{1}{4}(\chi A^{(0)}\partial_k\chi - \partial_k\chi A^{(0)}\chi) - \tfrac{1}{8} i\partial_l[\chi(A^{(0)}S^{(k)(l)} - S^{(k)(l)T}A^{(0)})\chi + \chi(A^{(k)}S^{(0)(l)} + A^{(l)}S^{(0)(k)} - S^{(0)(l)T}A^{(k)} - S^{(0)(k)T}A^{(l)})\chi]$$

are the desired densities. It would be incorrect to conclude from these results that the local energy and momentum densities are independent of g_{00} for any system, since not all the components of χ are fundamental dynamical variables in general, and the construction of the dependent components may involve g_{00} explicitly.

FUNDAMENTAL VARIABLES

The identification of the fundamental dynamical variables depends upon the structure of the generator of field variations

$$G_\chi = \int (d\mathbf{x}) \tfrac{1}{2}(\chi A^{(0)}\delta\chi - \delta\chi A^{(0)}\chi),$$

and of the field equations. According to the action principle, these are ($g_{0k}=0$)

$$A^{(0)}\partial_0\chi=e_0^{(0)}[(\partial_l\mathcal{H}/\partial\chi)-A^{(k)}\partial_k\chi] +\tfrac{1}{2}\partial_k e_0^{(0)}[-A^{(k)}\chi+i(A^{(0)}S^{(0)(k)}-S^{(0)(k)T}A^{(0)})\chi].$$

It is also useful to note the relations between the matrices $A^{(0)}$ and $A^{(k)}$ that are implied by invariance under infinitesimal Lorentz transformations,

$$-i(A^{(0)}S^{(0)(k)}+S^{(0)(k)T}A^{(0)})=A^{(k)},$$
$$-i(A_{(l)}S^{(0)(k)}+S^{(0)(k)T}A_{(l)})=\delta_l^k A^{(0)}.$$

Thus, the coefficient of $\partial_k e_0^{(0)}$ in the field equations is alternatively written

$$iA^{(0)}S^{(0)(k)}\chi.$$

We must now discuss several categories of fields.[4] In the first of these $A^{(0)}$ is a nonsingular matrix—all field equations are equations of motion for the fundamental variables—and the local energy and momentum densities are independent of g_{00}, validating the commutator condition. When $A^{(0)}$, or a completely reduced portion of it, refers to a Fermi-Dirac (F.D.) field ψ, the real symmetrical matrix $\alpha^{(0)}=-iA^{(0)}$ must be positive-definite according to the equal-time anticommutator implied by the action principle,

$$\{\psi(x),\psi(x')\}=(\alpha^{(0)})^{-1}\delta(\mathbf{x}-\mathbf{x}').$$

This is impossible for other than spin $\frac{1}{2}$ fields.

On introducing the real symmetrical Euclidean spin matrix

$$S_{k4}=iS_{0k}$$

(we omit the designation of local components), the transformation properties of the A matrices read

$$\{A^0,S_{k4}\}=A_k,\quad \{A_l,S_{k4}\}=\delta_{kl}A^0,$$

and therefore (no k summation)

$$A^0=\{S_{k4},\{S_{k4},A^0\}\}.$$

A corresponding matrix statement is

$$[(S_{k4}'+S_{k4}'')^2-1]\langle S_{k4}'|A^0|S_{k4}''\rangle=0,$$

and all such matrix elements of A^0 vanish, unless

$$S_{k4}'+S_{k4}''=\pm 1.$$

When the spin is $\frac{1}{2}$, the eigenvalues of S_{k4} are $\pm\frac{1}{2}$ and the diagonal elements of A^0, $S_{k4}'=S_{k4}''=\pm\frac{1}{2}$, differ from zero (in fact, $\alpha^0=1$). But with higher spin values all other diagonal matrix elements must vanish and α^0 cannot be positive-definite.

Can a^0, the reduced submatrix of A^0 associated with Bose-Einstein (B.E.) fields, be nonsingular? An argument denying this possibility is based upon the physical existence of the vacuum state. The equal-time commutation relations for the B.E. field $\phi(x)$ would be

$$[\phi(x),\phi(x')]=i(a^0)^{-1}\delta(\mathbf{x}-\mathbf{x}'),$$

where a^0 is real and antisymmetrical. We combine these with the equations of motion ($-g_{00}=1$)

$$a^0\partial_0\phi=-a^k\partial_k\phi+(\partial\mathcal{H}/\partial\phi)$$

to get

$$i[a^0\partial_0\phi(x),a^0\phi(x')] =-a^k\partial_k\delta(\mathbf{x}-\mathbf{x}')+(\partial^2\mathcal{H}/\partial\phi\partial\phi)\delta(\mathbf{x}-\mathbf{x}'),$$

and thereby the vacuum expectation value

$$\langle a^0\phi(x)P^0a^0\phi(x')+a^0\phi(x')P^0a^0\phi(x)\rangle =-a^k\partial_k\delta(\mathbf{x}-\mathbf{x}')+\langle\partial^2\mathcal{H}/\partial\phi\partial\phi\rangle\delta(\mathbf{x}-\mathbf{x}').$$

The left-hand member of the latter equation must be positive definite. But the term $-a^k\partial_k\delta(\mathbf{x}-\mathbf{x}')$ does not have this character and can lead to arbitrarily large negative values of a corresponding quadratic form (provided $\langle\partial^2\mathcal{H}/\partial\phi\partial\phi\rangle$ is finite, which is assured at least for the usual dynamical applications of B.E. fields with linear couplings). The following example is cited to indicate that purely structural considerations will not suffice, in general:

$$\mathcal{L}=-\tfrac{1}{2}F^{\mu\nu}(\partial_\mu A_\nu-\partial_\nu A_\mu)-\tfrac{1}{2}F_d{}^{\mu\nu}(\partial_\mu B_\nu-\partial_\nu B_\mu) -F_A\partial_\mu A^\mu-F_B\partial_\mu B^\mu-\mathcal{H},$$

in which $F_d{}^{\mu\nu}$ is the tensor dual to $F_{\mu\nu}$,

$$F_d{}^{01}=F_{23},\cdots.$$

The matrix a^0 is nonsingular, since every field variable appears in the time derivative term.

The matrix A^0 must be singular, then, apart from spin $\frac{1}{2}$ fields. This matrix can be reduced completely into a non-singular submatrix and a null submatrix, corresponding to the field decomposition

$$\chi=\chi_n+\chi_s,$$

where χ_s refers to the singular subspace. The latter imply field equations that are not equations of motion but equations of constraint,

$${}_s[-A^k\partial_k\chi+\partial_l\mathcal{H}/\partial\chi]=0,$$

and these are independent of g_{00}. The second category consists of those physical systems for which the constraint equations suffice to determine the χ_s explicitly in terms of the χ_n. Since that connection does not refer to g_{00}, the commutator condition for T^{00} will be valid.

There are no F.D. fields in this category. The action principle gives the anticommutation relations for the ψ_n in terms of $(\alpha^0)_n$, the nonsingular submatrix, which must be positive-definite. But, as a consequence of the Lorentz transformation equations,

$$0={}_s(S_{k4})_n(\alpha^0)_n\,{}_n(S_{k4})_s,$$

[4] It may be of interest to compare these remarks with some earlier ones, which were limited to uncoupled fields. They are summarized by E. M. Corson, *Introduction to Tensors, Spinors, and Relativistic Wave Equations* (Blackie and Son, Limited, London, 1953).

which is impossible unless

$$_s(S_{k4})_n = {}_n(S_{k4})_s = 0.$$

Thus, the separation into two subspaces has a Lorentz invariant meaning and

$$\alpha^\mu = (\alpha^\mu)_n,$$

which is a complete reduction to the subspace of positive-definite α^0, and spin $\frac{1}{2}$.

To discuss B.E. fields we must be more explicit about the structure of $\mathcal{H}(\chi)$. Let

$$\mathcal{H} = \tfrac{1}{2}\phi B\phi + \mathcal{H}_1,$$

where $\mathcal{H}_1$ is no more than linear in the components of any particular type of B.E. field. The real symmetrical matrix B must satisfy conditions expressing the invariance of the corresponding term under infinitesimal Lorentz transformations,

$$BS_{\mu\nu} + S_{\mu\nu}{}^T B = 0.$$

With the permissible choice of imaginary antisymmetrical S_{kl} and imaginary symmetrical S_{0k}, these conditions are satisfied by

$$B = R_{\text{sp}} I,$$
$$[I, S_{\mu\nu}] = 0,$$

where R_{sp} is the geometrical space reflection matrix. It obeys

$$R_{\text{sp}}{}^T = R_{\text{sp}}{}^* = R_{\text{sp}}{}^{-1} = R_{\text{sp}}.$$

We shall adopt the general principle that the question of parity conservation is a purely dynamical one, referring only to the coupling among various fields, so that

$$[R_{\text{sp}}, I] = 0.$$

Then the invariant matrix I, like R_{sp}, is real and symmetrical and both matrices can be brought into diagonal form. Such a representation of I has a Lorentz invariant meaning, while the diagonalization of R_{sp} is automatic in the tensor description of ϕ.

The equations of motion $(-g_{00}=1)$ and constraint equations are

$$(a^0)_n \partial_0 \phi_n = -(a^k)_n \partial_k \phi_n - {}_n(a^k)_s \partial_k \phi_s + B\phi_n + \partial\mathcal{H}_1/\partial\phi_n,$$

and

$$0 = -{}_s(a^k)_n \partial_k \phi_n + B_s \phi_s + \partial\mathcal{H}_1/\partial\phi_s,$$

respectively. The ϕ_s are explicitly determined by the latter equations provided B_s is nonsingular. In that situation the commutation relations of the fundamental variables are

$$[\phi_n(x), \phi_n(x')] = i(a^0)_n{}^{-1}\delta(\mathbf{x}-\mathbf{x}').$$

We can now derive the commutator

$$i[(a^0)_n \partial_0 \phi_n(x), (a^0)_n \phi_n(x')] = -(a^k)_n \partial_k \delta(\mathbf{x}-\mathbf{x}') + B_n \delta(\mathbf{x}-\mathbf{x}') + {}_n(a^k)_s B_s{}^{-1}\, {}_s(a^l)_n \partial_k \partial_l{}' \delta(\mathbf{x}-\mathbf{x}'),$$

with the assumption that

$$[\phi_n(x), \partial\mathcal{H}_1/\partial\phi(x')] = 0.$$

This result also gives the vacuum expectation value of the commutator. The corresponding positive-definite quadratic form can be dominated by either of the last two terms if one considers sufficiently slowly or rapidly varying functions. Accordingly, it is necessary that

$$B_n > 0, \quad -B_s > 0.$$

But $B = R_{\text{sp}} I$, where I has a common value for all components of a given type of tensor field. Every component of ϕ_n selected from a particular tensor must, therefore, have the same value of the space parity while the ϕ_s components possess the opposite value. If we now observe that

$$\{a^\mu, R_{\text{st}}\} = 0$$
$$R_{\text{st}} = e^{\pi i S_{12}} e^{\pi i S_{34}},$$

which refers to the invariance property of B.E. fields with respect to the proper transformation $x^\mu \to -x^\mu$, it follows that the matrices a^μ only connect different tensors, characteristized by opposite values of R_{st}. And, since

$$[a^0, R_{\text{sp}}] = 0$$

we can conclude, for an irreducible field, that all components of ϕ_n have a common parity. In that circumstance,

$$(a^k)_n = 0.$$

For which B.E. fields can a^0 and $R_{\text{sp}} a^0$ be identical, or differ merely by a sign? If we compare

$$a^0 = \{S_{k4}, \{S_{k4}, a^0\}\}$$

with

$$R_{\text{sp}} a^0 = [S_{k4}, [S_{k4}, R_{\text{sp}} a^0]],$$

it is seen that nonvanishing matrix elements $\langle S_{k4}' | a^0 \times | S_{k4}'' \rangle$ can only occur when both of the following conditions are satisfied:

$$(S_{k4}' + S_{k4}'')^2 = 1,$$
$$(S_{k4}' - S_{k4}'')^2 = 1.$$

Thus, one of the quantum numbers must be zero and the other of unit magnitude. Since no larger value can appear, the spins of such fields are limited to zero and unity.

There is a third category, of spin one fields, for which B_s is singular corresponding to the intrinsic arbitrariness of vector fields that admit gauge transformations. We shall not consider these systems here since the commutator condition has already been verified (indeed, discovered) in connection with the general non-Abelian vector gauge field coupled to a spin $\frac{1}{2}$ field.

The fourth category contains F.D. fields of spin $\geq\frac{3}{2}$ and B.E. fields with spins ≥ 2. The matrices $(\alpha^0)_n$ of F.D. fields and B_n for B.E. fields will be indefinite and not all of the χ_n can be fundamental variables. Hence, the constraint equations must act to diminish the space of independent variables by introducing relations among the components of χ_n. Since all of these obey equations of motion, the introduction of the restrictions on the χ_n will yield further constraint equations. But the latter, being consequences of the equations of motion, involve g_{00} and the needed reassurance that $(-g_{00})T^{00}$ and $(-g_{00})^{1/2}T^{0k}$ are independent of g_{00} can no longer be given.

The $\partial_k e_0{}^{(0)}$ term in the field equations is essential to this argument for otherwise the existence of linear relations among the χ_n would imply constraint equations that were still independent of g_{00}. The process of deriving new constraint conditions from the equations of motion may require several repetitions depending upon the spin of the field, and the explicit construction of the dependent variables will involve corresponding numbers of space and time derivatives of g_{00}.

SPIN $\frac{3}{2}$

We shall briefly examine the field in the fourth category with the lowest spin value, in order to obtain the simplest example of the additional terms in the commutator condition. It will suffice to consider an uncoupled field for, unlike the situation with lower spin fields, our task is no longer to show that the commutator condition is valid despite the effect of field interactions. The preceding discussion indicates that ψ_s becomes an explicit function of g_{00}, but that time derivatives of g_{00} do not yet appear, for spin $\frac{3}{2}$. Accordingly, $T^{(0)(0)}$ must remain independent of g_{00}. The explicit dependence of $T^{(0)(k)}$ is given by

$$\delta_3 T^{(0)(k)}(x)/\delta e_0{}^{(0)}(x') = -\tfrac{1}{2}i\partial_l[\psi(x)(A^kS^{0l}+A^lS^{0k})_s\delta_3\psi_s(x)/\delta e_0{}^{(0)}(x')],$$

where

$$-i(A^kS^{0l})_s = i(S^{0l}A^k)_s = (S^{0l}A^0S^{0k})_s.$$

We choose

$$\mathcal{H} = \tfrac{1}{2}\psi B\psi,$$

where B is an antisymmetrical imaginary matrix, and the constraint equation becomes

$$_s(A^k\partial_k - B)_n\psi_n = 0.$$

The further constraint equation derived from the equations of motion is

$$0 = {}_s(A^k\partial_k - B)_n(A_n{}^0)^{-1}{}_n[e_0{}^{(0)}(-A^l\partial_l + B)\psi + (\partial_l e_0{}^{(0)})iA^0S^{0l}\psi].$$

The local nature of the theory requires that this be an explicit expression for ψ_s, rather than a differential equation. It can also be anticipated that the only significant terms are those containing the second spatial derivatives of $e_0{}^{(0)}$. These considerations give, for $-g_{00}=1$,

$$(BA_n{}^0B)_s\frac{\delta_3\psi_s(x)}{\delta e_0{}^{(0)}(x')} = {}_s(S^{0m}A^0S^{0n}\psi(x))\partial_m\partial_n\delta(\mathbf{x}-\mathbf{x}'),$$

and

$$\frac{\delta_3 T^{(0)(k)}(x)}{\delta e_0{}^{(0)}(x')} = \partial_l\partial_m{}'\partial_n{}'[\delta(\mathbf{x}-\mathbf{x}')f^{kl,mn}(x)].$$

Here

$$f^{kl,mn}(x) = \psi(x)S^{0(k}A^0S^{0l)}(BA_n{}^{0-1}B)_s{}^{-1}S^{0(m}A^0S^{0n)}\psi(x),$$

which uses the notation

$$S^{0(k}A^0S^{0l)} = \tfrac{1}{2}(S^{0k}A^0S^{0l} + S^{0l}A^0S^{0k}).$$

We notice that $f^{kl,mn}$ has the symmetries

$$f^{kl,mn} = f^{lk,mn} = f^{kl,nm} = -f^{mn,kl},$$

where the last statement is a consequence of the F.D. anticommutation relations. One can verify with a specific example that $f^{kl,mn}$ is not identically zero.

The resulting equal-time energy density commutator equation is

$$-i[T^{00}(x),T^{00}(x')] = -(T^{0k}(x)+T^{0k}(x'))\partial_k\delta(\mathbf{x}-\mathbf{x}') - \partial_k\partial_l\partial_m{}'\partial_n{}'\{\delta(\mathbf{x}-\mathbf{x}')f^{kl,mn}(x)\}.$$

The additional term has the anticipated properties. It vanishes for finite $|\mathbf{x}-\mathbf{x}'|$, it is an antisymmetrical function of x and x', and it does not contribute to commutators containing the integrated quantities P^0 or J_{0k}.

The considerations of this paper apply to all fields save one—the gravitational field.