PHYSICAL REVIEW VOLUME 130, NUMBER 2 15 APRIL 1963

Radiation Gauge Electrodynamics. I. The Two-Point Function*†

C. R. HAGEN‡
Laboratory for Nuclear Science and Physics Department, Massachusetts Institute of Technology, Cambridge, Massachusetts

(Received 12 November 1962)

The Lehmann representations of the spin zero and spin one-half Green's functions are studied with particular emphasis upon infrared effects. By means of a Bloch-Nordsieck type model the infrared parts of these functions are calculated and the analytic properties of the solution examined. A double integral representation of the two-point function is proposed and a possible application suggested.

I. INTRODUCTION

THE great difficulty of handling the equations for the Green's functions of a quantum field theory has given rise to an appreciable interest in mathematically soluble models which might provide insight into the structure of their more realistic counterparts. Such models all too frequently fall into one of two extreme classes according to their relativistic or nonrelativistic character. The former of these is characterized by its adherence to a fully covariant Lagrangian formalism but is generally lacking in any scattering or production phenomena. On the other hand, while a nonrelativistic model such as that proposed by Lee[1] has the distinct advantage of giving rise to a nontrivial S matrix, the pathological features (ghosts) which it contains have been a source of considerable anxiety to field theorists. It remains unclear as to whether these structural defects are to be considered as generally characteristic of real field theories rather than mere reminders of the inadequacies of such models.

In this paper we propose to investigate this question by means of a Bloch-Nordsieck[2] type model of electrodynamics. There are at least two good reasons for choosing electrodynamics as the framework for such a discussion. First there exists an extensive perturbative analysis of the infrared structure of electrodynamics in the literature, the results of which will be seen to be duplicated by our model. While the correct functional form of the two-point function will be found for a certain domain of momentum space, the expected analyticity properties are not obtained. Secondly and more important is the fact that in electrodynamics we have available a powerful tool in the existence of the radiation gauge formalism.[3] While this formulation is generally to be shunned for calculational purposes, it here provides an essential criterion that must be imposed upon the transform of the two-point function. Since the singularities of this matrix element correspond to permissible eigenstates of the operator $-P_\mu^2$,[4] the position of the relevant poles and cuts must be independent of the chosen coordinate system. As the Green's function itself is not manifestly covariant, the appearance of singularities whose location depends upon the three-dimensional momentum must be indicative of the breakdown of the model itself rather than a characteristic of electrodynamics.

In the following section we briefly develop, by way of introduction to the radiation gauge, the perturbation theoretical treatment of the spin zero and spin one-half Green's functions. Section III presents the model which we use for the calculation of the exact infrared structure and includes an examination of the singularities of the Green's function of the charge field. Finally, in Sec. IV a double dispersion relation is proposed for the radiation gauge Green's functions and is subsequently used to establish the connection between spin and statistics.

II. PERTURBATION THEORY AND RENORMALIZATION

The Lehmann representations[5] of the spin zero and spin one-half Green's functions in the radiation gauge have been used by Johnson[6] to extract information concerning the high-energy domain of electrodynamics. He has shown in the spin zero case that the transform of the function

$$\mathcal{G}(x,x')=i\langle 0|(\phi(x)\phi^\dagger(x'))_+|0\rangle \tag{2.1}$$

has the form:

$$\mathcal{G}(p^2,\mathbf{p}^2)=\int_m^\infty \frac{B(\kappa,\mathbf{p}^2)}{p^2+\kappa^2-i\epsilon}d\kappa. \tag{2.2}$$

The mass operator $M(p)$ can thus be written as

$$\begin{aligned}M(p)&\equiv\mathcal{G}^{-1}(p)-p^2\\&=m_0^2-\int_m^\infty \frac{s(\kappa,\mathbf{p}^2)}{p^2+\kappa^2-i\epsilon}d\kappa,\end{aligned} \tag{2.3}$$

where m_0 is the unrenormalized mass of the charge field and

$$s(\kappa,\mathbf{p}^2)=B(\kappa,\mathbf{p}^2)\,|\mathcal{G}(p^2=-\kappa^2,\,\mathbf{p}^2)|^{-2}.$$

* This work is supported in part through funds provided by the U. S. Atomic Energy Commission under Contract AT(30-1)-2098.
† Based upon a thesis submitted to the Physics Department of the Massachusetts Institute of Technology in partial fulfillment of the requirements for the degree of Doctor of Philosophy.
‡ National Science Foundation Cooperative Predoctoral Fellow.

[1] T. D. Lee, Phys. Rev. **95**, 1329 (1954).
[2] F. Bloch and A. Nordsieck, Phys. Rev. **52**, 54 (1937).
[3] A development of the radiation gauge operator formalism has been given by J. Schwinger, Phys. Rev. **91**, 713 (1953).
[4] We use the metric (1, 1, 1, −1).
[5] H. Lehmann, Nuovo Cimento **11**, 342, (1954).
[6] K. Johnson, Ann. Phys. (N. Y.) **10**, 536 (1960).

Since the Hilbert space of the radiation gauge has a positive metric, $B(\kappa,\mathbf{p}^2)\geqslant 0$.

The condition that there be a particle of mass m,

$$M(p^2=-m^2)=m^2$$
$$=m_0^2-\int_m^\infty d\kappa\frac{s(\kappa,\mathbf{p}^2)}{\kappa^2-m^2},\qquad(2.4)$$

allows one to write the once subtracted form of (2.3) as

$$M(p)=m^2+(p^2+m^2)\int_m^\infty d\kappa\frac{s(\kappa,\mathbf{p}^2)}{(\kappa^2-m^2)(p^2+\kappa^2-i\epsilon)}.$$

An additional subtraction eliminates all the ultraviolet divergences of perturbation theory and yields the form:

$$M(p)=m^2+(p^2+m^2)\zeta_2(\mathbf{p}^2)+(p^2+m^2)^2\times\int_m^\infty d\kappa\frac{s(\kappa,\mathbf{p}^2)}{(\kappa^2-m^2)^2(p^2+\kappa^2-i\epsilon)},$$

where we have defined

$$\zeta_2(\mathbf{p}^2)=\int_m^\infty d\kappa\frac{s(\kappa,\mathbf{p}^2)}{(\kappa^2-m^2)^2}.$$

It should be noted that the condition referred to in the previous section [i.e., that the spectrum of $M(p)$ be independent of $\mathbf{p}^2$] has by virtue of (2.4) already led to the condition that the integral

$$\int_m^\infty d\kappa\frac{s(\kappa,\mathbf{p}^2)}{\kappa^2-m^2}$$

be independent of $\mathbf{p}^2$.

The second-order expression for the mass operator is given by

$$M(p)=m_0^2-ie_0^2\int\frac{d^4k}{(2\pi)^4}D_\mu{}^\mu(k)$$
$$+ie_0^2\int\frac{d^4k}{(2\pi)^4}\frac{(2p-k)^\mu D_{\mu\nu}(k)(2p-k)^\nu}{(p-k)^2+m^2},\qquad(2.5)$$

where

$$D^{\mu\nu}(k)=\left(g^{\mu\alpha}-\frac{k^\mu(k^\alpha+n^\alpha k\cdot n)}{k^2+(k\cdot n)^2}\right)\times\left(g^{\nu\beta}-\frac{k^\nu(k^\beta+n^\beta k\cdot n)}{k^2+(k\cdot n)^2}\right)g_{\alpha\beta}\frac{1}{k^2},\qquad(2.6)$$

and n^μ is the time-like vector $(\mathbf{0},1)$.[7] The quadratically divergent expression (2.5) is regularized by the replacement of $1/k^2$ in (2.6) by

$$\frac{1}{k^2+\mu^2}-\frac{1}{k^2+\Lambda^2}-\frac{1}{k^2+\Lambda'^2}+\frac{1}{k^2+\Lambda^2+\Lambda'^2},$$

where imposed on the regulator masses are the inequalities:

$$\Lambda'^2\gg\Lambda^2\gg m^2.$$

The calculations are straightforward and yield

$$B(\kappa,\mathbf{p}^2)=\frac{2\alpha}{\pi}\frac{\kappa}{\kappa^2-m^2}\int_0^1 v^{1/2}dv\frac{\mathbf{p}^2}{\kappa^2+\mathbf{p}^2(1-v)},$$

$$m^2-m_0^2=\frac{\alpha}{2\pi}\left[\frac{3}{2}\Lambda^2\left(1+\ln\frac{\Lambda'^2}{\Lambda^2}\right)+3m^2\left(\ln\frac{\Lambda}{m}+\frac{3}{4}\right)\right],$$

$$Z_2=1-\frac{\alpha}{2\pi}\left[-\ln\frac{\Lambda^2}{m^2}+\int_0^1\frac{\mathbf{p}^2v^{1/2}dv}{m^2+\mathbf{p}^2(1-v)}\times\left(\ln\frac{m^2+\mathbf{p}^2(1-v)}{\mu^2}-1\right)-\frac{9}{4}\right],$$

where we have defined

$$\alpha=e_0^2/4\pi,$$

and

$$Z_2(\mathbf{p}^2)=[1+\zeta_2(\mathbf{p}^2)]^{-1}.\qquad(2.7)$$

It is interesting to note for $\mathbf{p}^2=0$, Z_2 is not infrared divergent, a result which is valid to all orders and will be more fully exploited in a subsequent paper. Furthermore, the fact that $Z_2(\mathbf{p}^2=0)$ is greater than unity, does not contradict the commutation relations which require:

$$\int B(\kappa,\mathbf{p}^2)d\kappa>1$$

in the radiation gauge.[6]

The generalization to include spin is nontrivial solely because of the increase in the number of independent functions in the Lehmann representation of the Green's function:

$$G(x,x')=i\epsilon(x,x')\langle 0|(\psi(x)\bar\psi(x'))_+|0\rangle.\qquad(2.8)$$

This can be shown to have the form:

$$G(p)=\int_m^\infty d\kappa\frac{A\gamma^0p^0-C\boldsymbol{\gamma}\cdot\mathbf{p}+mB}{p^2+\kappa^2-i\epsilon},$$

where A, B, and C are real functions of κ and $\mathbf{p}^2$ for a Majorana representation of the Dirac algebra. Upon performing two subtractions at the mass "pole," one finds terms proportional to 1, $\gamma^\mu p_\mu$, and $\boldsymbol{\gamma}\cdot\mathbf{p}$. Since this therefore gives rise to three subtraction functions of $\mathbf{p}^2$ rather than the customary two, the usual assumed form of the Green's function near $p^2=-m^2$ is not applicable.

---
[7] J. Schwinger, Phys. Rev. **115**, 721 (1959).

A sufficiently general expression in this domain is

$$G(p)\sim\exp(w\boldsymbol{\gamma}\cdot\mathbf{p})\frac{Z_2(\mathbf{p}^2)}{\gamma\cdot p+m}\exp(w\boldsymbol{\gamma}\cdot\mathbf{p}),$$

with $w(\mathbf{p}^2)=w^*(\mathbf{p}^2)$. Since the matrix residue at the "pole" gives the single-particle wave functions, one makes the identification:

$$\sum_\lambda \langle 0|\psi(0)|p^2=-m^2,\mathbf{p},\lambda\rangle\langle p^2=-m^2,\mathbf{p},\lambda|\bar{\psi}(0)|0\rangle,$$

$$=Z_2(\mathbf{p}^2)\exp(w\boldsymbol{\gamma}\cdot\mathbf{p})\frac{m-\gamma\cdot p}{2m}\exp(w\boldsymbol{\gamma}\cdot\mathbf{p}),$$

$$\equiv Z_2(\mathbf{p}^2)\textstyle\sum_{\lambda=1}^2 u_\lambda'(p)\bar{u}_\lambda'(p),$$

where we have defined the radiation gauge spinors $u_\lambda'(p)$:

$$u_\lambda'(p)\equiv\exp(w\boldsymbol{\gamma}\cdot\mathbf{p})u_\lambda(p).$$

To second order, the mass operator can now be written as

$$M(p)=m_0+ie_0^2\int\frac{d^4k}{(2\pi)^4}D^{\mu\nu}(k)\gamma_\mu\frac{1}{\gamma\cdot(p-k)+m}\gamma_\nu$$

$$=m_0+\delta m-\{\gamma\cdot p+m,\boldsymbol{\gamma}\cdot\mathbf{p}\}w(\mathbf{p}^2)+\zeta_2(\gamma\cdot p+m)$$

$$-(\gamma\cdot p+m)\int_m^\infty d\kappa\,\frac{A\gamma^0p^0+mB-C\boldsymbol{\gamma}\cdot\mathbf{p}}{p^2+\kappa^2-i\epsilon}(\gamma p+m),$$

where Z_2 is once again defined by (2.7). Upon performing the integration and taking the imaginary part, one finds

$$A=\frac{2\alpha}{\pi}\left[\frac{\kappa^2-m^2}{4\kappa^3}+\frac{\kappa}{\kappa^2-m^2}\int_0^1 v^{1/2}dv\,\frac{\mathbf{p}^2}{\kappa^2+\mathbf{p}^2(1-v)}\right],$$

$$B=\frac{2\alpha}{\pi}\left[\frac{\kappa}{\kappa^2-m^2}\int_0^1 v^{1/2}dv\,\frac{\mathbf{p}^2}{\kappa^2+\mathbf{p}^2(1-v)}\right],$$

$$C=\frac{2\alpha}{\pi}\left[\frac{\kappa^2-m^2}{4\kappa^3}+\left(\frac{\kappa}{2}+\frac{\mathbf{p}^2\kappa}{\kappa^2-m^2}\right)\int_0^1 v^{1/2}dv\,\frac{1}{\kappa^2+\mathbf{p}^2(1-v)}\right].$$

Finally by expansion around $p^2=-m^2$, the following are extracted:

$$Z_2=1-\frac{\alpha}{2\pi}\left\{\ln\frac{\Lambda}{m}-\frac{3}{4}+\int_0^1\frac{\mathbf{p}^2v^{1/2}dv}{m^2+\mathbf{p}^2(1-v)}\times\left[\ln\frac{m^2+\mathbf{p}^2(1-v)}{\mu^2}-2\right]\right\},$$

$$\delta m=3m\frac{\alpha}{2\pi}\left(\ln\frac{\Lambda}{m}+\frac{1}{4}\right),$$

$$w=-\frac{\alpha}{2\pi}\int_0^1\frac{mv^{1/2}dv}{m^2+\mathbf{p}^2(1-v)}.$$

One notes that as for spin zero the infrared divergence vanishes for the case of $\mathbf{p}^2=0$.

Because of the infrared divergences which tend to appear in electrodynamics, it is necessary to make here some remarks concerning the validity of the above techniques. In a theory for which no null-mass particle exists the expected singularities of the Green's functions (2.1) and (2.8) consist of poles corresponding to the stable single-particle states, together with a branch cut starting at the two- (or three-) particle continuum threshold. The vanishing of the photon mass means that this cut moves down to the single-particle pole and one is, in fact, faced with the question of whether the pole persists in the zero-mass limit. Thus, it becomes possible that the charge field operator acting on the vacuum cannot create an electron without the simultaneous emission of an infinite number of soft quanta. Such a phenomenon manifests itself in perturbation theory in the appearance of infrared divergences in the wave function renormalization. These divergences serve to indicate the disappearance of the single-particle state from the spectrum of $\mathcal{G}(p)$ and should be clearly distinguished from the ultraviolet divergences whose origin is more obscure. Thus, the usual renormalization procedure of performing two subtractions in the mass operator at the position of the single-particle state is at best a questionable one in electrodynamics. Nonetheless, because any finite order of perturbation theory can describe only a limited number of production processes, it has been necessary to employ this technique for perturbative calculations. The inadequacies of such an approach are remedied by the treatment of the following section.

III. A MODEL FOR ELECTRODYNAMICS

The equation for the Green's function (3.1) can be written as

$$\left[\left(\frac{1}{i}\partial_\mu-e\mathcal{A}_\mu(x)-e\frac{1}{i}\frac{\delta}{\delta J^\mu(x)}\right)^2+m_0^2\right]\times\mathcal{G}(x,x')=\delta(x-x'),\quad(3.1)$$

where we have defined

$$\mathcal{A}_\mu(x)=\frac{\langle 0\sigma_1|A_\mu(x)|0\sigma_2\rangle}{\langle 0\sigma_1|0\sigma_2\rangle},$$

and introduced a source function J_μ. The equations of motion for the Maxwell field imply

$$\mathcal{A}_\mu(x)=\int D_{\mu\nu}(x-x')dx'\times\left[J^\nu(x')+\frac{\langle 0\sigma_1|j^\nu(x')|0\sigma_2\rangle}{\langle 0\sigma_1|0\sigma_2\rangle}\right],\quad(3.2)$$

where $j^\mu(x)$ is the current vector formed from the field ϕ.

The approximation in which vacuum polarization is neglected consists in omitting the second of the two terms in the square brackets of (3.2). Since this deletion should not affect the singularities of the Green's function in the neighborhood of $p^2=-m^2$, the infrared structure will continue to be correctly described in this approximation. It is convenient to transcribe the resulting equation:

$$\exp\left\{-\frac{i}{2}\int J_\mu D^{\mu\nu}J_\nu\right\}\left\{\left(\frac{1}{i}\partial_\mu - e\frac{1}{i}\frac{\delta}{\delta J^\mu}\right)^2+m_0^2\right\}\times\exp\left\{\frac{i}{2}\int J_\mu D^{\mu\nu}J_\nu\right\}\mathcal{G}(x,x';J)=\delta(x-x'),$$

immediately into momentum space. This is facilitated by noting that the operator I defined by

$$I(x)=\exp\left\{\int d^4k\, J_\mu(k)(e^{ik\cdot x}-1)\,;\frac{\delta}{\delta J_\mu(k)}\right\}$$

satisfies the equation:

$$\frac{\delta}{\delta J_\mu(k)}I(x)=e^{ik\cdot x}I(x)\frac{\delta}{\delta J_\mu(k)},$$

or

$$\frac{\delta}{\delta J_\mu(\xi)}I(x)=I(x)\frac{\delta}{\delta J_\mu(x+\xi)}.$$

Here the semicolon signifies that in the expansion of I all variational derivatives appear to the right of all J's. An alternative form for I is found by taking a derivative:

$$\frac{\partial I}{\partial x_\mu}=\int d^4k\, e^{ik\cdot x}J^\nu(k)I(x)\frac{\delta}{\delta J^\nu(k)}ik^\mu$$
$$=i\int k^\mu J^\nu(k)\frac{\delta}{\delta J^\nu(k)}I(x)d^4k,$$

whence

$$I=\exp\left\{ix^\mu\int k_\mu J^\nu(k)\frac{\delta}{\delta J^\nu(k)}d^4k\right\}.$$

One now readily deduces the equation:

$$\exp\left\{-\frac{i}{2}\int JDJ\right\}\left[\left(p^\mu-\int d^4k\, k^\mu J\frac{\delta}{\delta J}-e\int\frac{d^4k}{(2\pi)^4}\frac{1}{i}\frac{\delta}{\delta J_\mu(k)}\right)^2+m_0^2\right]\times\exp\left\{\frac{i}{2}\int JDJ\right\}\bar{\mathcal{G}}(p,J)=1, \quad (3.3)$$

where we have defined

$$I\mathcal{G}\equiv\bar{\mathcal{G}}.$$

Ultimately we shall only be interested in the case for which $J=0$ and $I=1$ and so shall omit the bar notation. A formal solution of (3.3) is:

$$\mathcal{G}(p,J=0)=i\int_0^\infty dx\, e^{-ix(m_0^2-i\epsilon)}\times\exp\left\{-ix\left[p-\int kJ\frac{\delta}{\delta J}-e\int\frac{1}{i}\frac{\delta}{\delta J}\right]^2\right\}\times\exp\left(\frac{i}{2}\int JDJ\right)\Bigg|_{J=0}. \quad (3.4)$$

One further approximation is necessary before the solution of (3.4) becomes possible. The replacement of the square bracket in (3.4) by

$$p^2-2p^\mu\int k_\mu J\frac{\delta}{\delta J}-2ep^\mu\int\frac{1}{i}\frac{\delta}{\delta J^\mu},$$

together with the identity proved in the Appendix:

$$e^{A+B}=e^B\exp\left(A\frac{1-e^{-\lambda}}{\lambda}\right),$$

where

$$[A,B]=-\lambda A,$$

readily leads to the result:

$$\mathcal{G}(p)=i\int_0^\infty dx\, e^{-ix(p^2+m_0^2-i\epsilon)}\times\exp\left\{ie^2\int\frac{d^4k}{(2\pi)^4}\frac{1-e^{2ip\cdot kx}}{(p\cdot k)^2}p_\mu D^{\mu\nu}(k)p_\nu\right\}. \quad (3.5)$$

The approximation made above is gauge covariant at the single particle "pole" and is equivalent to the following modifications of the perturbation theory of spin-zero electrodynamics:

(i) The term $(2p-k)^\mu D_{\mu\nu}(k)(2p-k)^\nu$ is replaced by $4p^\mu D_{\mu\nu}(k)p^\nu$.

(ii) Diagrams containing vertices at which two photons are emitted are neglected.

(iii) Propagators $[m^2+(p-k)^2]^{-1}$ are replaced by $[m^2+p^2-2p\cdot k]^{-1}$.

These approximations happen to be the ones generally made in the usual treatment of infrared effects and we shall therefore not dwell on them. Their justification is suggested in the extensive literature dealing with the infrared problem.[8]

[8] An extensive treatment of the infrared problem by perturbative techniques has been given by D. Yennie, S. Frautschi, and H. Suura, Ann. Phys. (N. Y.) **13**, 379 (1961). This work contains references to most of the previous work on the subject.

The integrals in (3.5) are readily evaluated and yield

$$\mathcal{G}(p)=i\int_0^\infty dx\, e^{-ix(p^2+m_0^2)}\exp\left\{-\frac{\alpha}{2\pi}\int_0^1\frac{\mathbf{p}^2v^{1/2}dv}{-p^2+\mathbf{p}^2(1-v)}\times\ln[p^2-\mathbf{p}^2(1-v)]x^2\Lambda^2\right\},$$

$$=\frac{1}{p^2+m_0^2}\exp\left\{-\frac{\alpha}{\pi}\int_0^1\frac{\mathbf{p}^2v^{1/2}dv}{-p^2+\mathbf{p}^2(1-v)}\times\ln\frac{-p^2+\mathbf{p}^2(1-v)}{(p^2+m_0^2)^2}\Lambda^2\right\}\times\Gamma\left(1-\frac{\alpha}{\pi}\int_0^1\frac{\mathbf{p}^2v^{1/2}dv}{-p^2+\mathbf{p}^2(1-v)}\right). \quad (3.6)$$

It is to be noted that while the infrared divergent part of the wave function renormalization is correctly duplicated to second order by (3.6), the ultraviolet effects (mass renormalization and ultraviolet part of Z_2) do not agree with perturbation theory. It is possible to consider slight variations of the model under consideration which tend to indicate that Λ^2 should be replaced by $-p^2+\mathbf{p}^2(1-v)$. It will be convenient to make this replacement although none of our subsequent results will depend critically upon it.

It should be pointed out that a result quite similar to (3.5) has been derived by Svidzinskii.[9] His treatment begins with the Dirac equation in which the gamma matrices are replaced by numbers u^μ (following the original Bloch-Nordsieck calculation). This approach has the effect of immediately reducing the Dirac equation to a first order differential equation and renders inadequate his treatment of the boundary conditions. A further disadvantage is the lack of a direct means of identification of the numbers u^μ.

One notes on inspection of the form (3.6) the fact that it does not satisfy the Lehmann representation in its usual sense. While there are clearly no complex singularities, one finds in addition to the expected cut from $p^2=-m^2$ to $p^2=-\infty$, an unphysical cut from $p^2=0$ to $p^2=\infty$ which arises from the combination $-p^2+\mathbf{p}^2(1-v)$. Since spacelike eigenvalues of the momentum operator (ghosts) are inadmissible in a physical theory, an immediate shortcoming of the model has appeared which persists for arbitrarily small though nonvanishing values of α and $\mathbf{p}^2$. In view of the fact that this spurious cut begins so close to the physical spectrum, the model must in fact be restricted in its application to the domain:

$$|p^2+m^2|\ll m^2.$$

Another serious cause of concern arises from the appearance of the gamma function in (3.6). It is well known that this function has poles when its argument is zero or a negative integer. Thus there are "bound states" which occur when

$$\frac{\alpha}{\pi}\int_0^1\frac{\mathbf{p}^2v^{1/2}dv}{-p^2+\mathbf{p}^2(1-v)}=n,$$

where $n=1, 2, 3\cdots$. More explicitly these singularities occur for

$$\mathbf{p}^2/-p^2\sim e^{\pi n/\alpha}.$$

Thus, for arbitrarily small coupling there will be an infinite number of such bound states with an accumulation point at $p^2=0$. The dependence of these poles on $\mathbf{p}^2$ violates the basic condition of covariance that the spectrum of the Green's function shall be independent of the chosen frame of reference.

These remarks can be further strengthened by a comparison with the results of the same calculation performed within the framework of the indefinite metric formalism. In a gauge in which

$$D_{\mu\nu}=\left(g_{\mu\nu}-\gamma\frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2},$$

(3.5) yields

$$\mathcal{G}(p)=\frac{1}{p^2+m^2}\left(\frac{-p^2\Lambda^2}{p^2+m^2}\right)^{(\alpha/2\pi)(2-\gamma)}\Gamma\left(1+\frac{\alpha}{2\pi}(2-\gamma)\right).$$

The fact that the "bound states" do not occur here shows again the lack of reliability of this model in the determination of the actual singularities of $\mathcal{G}(p)$ in the entire complex plane.

Finally, we note that with the neglect of the unphysical singularities, $\mathcal{G}(p)$ can be cast into the Lehmann form (2.2) with

$$B(\kappa,\mathbf{p}^2)=\frac{2\kappa}{\kappa^2-m^2}\exp\left\{-\frac{\alpha}{\pi}\int_0^1\frac{\mathbf{p}^2v^{1/2}dv}{\kappa^2+\mathbf{p}^2(1-v)}\times\ln\frac{\kappa^2+\mathbf{p}^2(1-v)}{\kappa^2-m^2}\right\}\left\{\Gamma\left(\frac{\alpha}{\pi}\int_0^1\frac{\mathbf{p}^2v^{1/2}dv}{\kappa^2+\mathbf{p}^2(1-v)}\right)\right\}^{-1}, \quad (3.7)$$

showing clearly that $B(\kappa,\mathbf{p}^2)$ has the required positive-definite character.

The analogous treatment of the spin one-half field with the neglect of the magnetic moment interaction is entirely straightforward and will not, therefore, be explicitly considered here.

IV. A CONJECTURE

A few years ago Mandelstam[10] put forth the very appealing conjecture that two-body scattering amplitudes can simultaneously be continued into the union of the complex planes of the two scalar invariants which describe the kinematics of elastic scattering. In view of

[9] A. V. Svidzinskii, Zh. Eksperim. i Teor. Fiz. **31**, 324 (1956).

[10] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).

the considerable emphasis now being placed upon double dispersion relations, it requires little imagination to speculate upon the possibility of such a relation being valid for the two-point functions of quantum electrodynamics. Thus, ignoring the question of subtractions, we ask whether the spin-zero Green's function could satisfy a relation of the form:

$$\mathcal{G}(p^2,\mathbf{p}^2)=\int_m^\infty d\kappa\int_m^\infty d\lambda\,\frac{\rho(\kappa,\lambda)}{(p^2+\kappa^2-i\epsilon)(\mathbf{p}^2+\lambda^2)}. \quad (4.1)$$

This is equivalent to requiring that

$$B(\kappa,\mathbf{p}^2)=\int_m^\infty d\lambda\,\frac{\rho(\kappa,\lambda)}{\mathbf{p}^2+\lambda^2},$$

i.e., that $B(\kappa,\mathbf{p}^2)$ be an analytic function of $\mathbf{p}^2$. In the case of nonzero spin, a dispersion relation in $\mathbf{p}^2$ would, of course, be expected to hold for each of the independent weight functions.

It is easy to verify that such a relation does in fact hold to lowest order in α. For the spin-zero case one can write:

$$B(\kappa,\mathbf{p}^2)=\mathbf{p}^2\int_m^\infty d\lambda\,\frac{4\alpha}{\pi}\,\frac{\kappa}{\kappa^2-m^2}\left(1-\frac{\kappa^2}{\lambda^2}\right)^{1/2}\frac{1}{\lambda}\theta_+(\lambda-\kappa)\frac{1}{\mathbf{p}^2+\lambda^2}$$

and

$$Z_2(\mathbf{p}^2)=1+\frac{\alpha}{2\pi}\left(\ln\frac{\Lambda^2}{m^2}+\frac{9}{4}\right)-\frac{\alpha}{\pi}\mathbf{p}^2\int_m^\infty d\lambda\,\frac{\sigma(\lambda)}{\mathbf{p}^2+\lambda^2},$$

where

$$\sigma(\lambda)=\left(1-\frac{m^2}{\lambda^2}\right)^{1/2}\frac{1}{\lambda}\left[\ln\frac{4(\lambda^2-m^2)}{\mu^2}-3\right],$$

and $\theta_+(x)$ is the usual unit step function. While a proof of the conjectured analyticity properties is at present unknown to the author, the utility of such a representation would seem to suggest the desirability of an investigation of its validity.

The second order result is not the only basis for supposing the analyticity of $B(\kappa,\mathbf{p}^2)$. The model of the previous section with the spectral weight given by (3.7) also displays this property. Casting it into the form of (4.1), one finds:

$$\rho(\kappa,\lambda)=\frac{4}{\pi}\frac{\kappa\lambda}{\kappa^2-m^2}\theta_+(\lambda-\kappa)$$
$$\times\left[\Gamma\left(\pi\frac{\partial}{\partial\ln(\kappa^2-m^2)}\right)\right]^{-1}\left(\frac{\kappa^2-m^2}{m^2}\right)^{\alpha/2\pi}$$
$$\times\exp\left[\frac{\alpha}{2\pi}\,\mathrm{Re}\int_0^1 dx\,\ln^2\frac{\kappa^2-\lambda^2(1-x^2)}{\kappa^2-m^2}\right]$$
$$\times\sin\left\{\alpha\left(1-\frac{\kappa^2}{\lambda^2}\right)^{-1/2}\left[\ln\frac{\kappa^2-m^2}{4(\lambda^2-\kappa^2)}+2\right]\right\}.$$

Since the physical content of the assumed analyticity property of $B(\kappa,\mathbf{p}^2)$ is more obscure than the corresponding statement of Mandelstam it is worthwhile to demonstrate how a slight extension of our assumption leads to the correct relation between spin and statistics.[11] Our treatment corresponds closely to that of Brown and Schwinger[12] who considered the case of a manifestly covariant theory.

The two-point function for the Hermitian field χ can be written:

$$\langle 0|\chi(x)\chi(x')|0\rangle=\int\frac{d^4p}{(2\pi)^4}e^{ip\cdot(x-x')}\theta_+(p)B(\kappa,p)\delta(p^2+\kappa^2)d\kappa, \quad (4.2)$$

where $B(\kappa,p)$ is a Hermitian non-negative matrix. Equation (4.2) leads to the following representation of the (anti) commutator:

$$\langle 0|[\chi(x),\chi(x')]_\pm|0\rangle=\int\frac{d^4p}{(2\pi)^4}e^{ip\cdot(x-x')}[\theta_+(p)B(\kappa,p)\pm\theta_-(p)B^*(\kappa,-p)]\times\delta(p^2+\kappa^2)d\kappa. \quad (4.3)$$

The relation[12]

$$B(\kappa,-p)=(-1)^{2S}R_{st}B(\kappa,p)R_{st}^{T*}, \quad (4.4)$$

where

$$R_{st}=e^{\pi iS_{12}}e^{\pi iS_{34}}$$

implies that for S integer [half-integer] the matrix $B(\kappa,p)+B(\kappa,-p)^*[\epsilon(p)(B(\kappa,p)-B(\kappa,-p)^*)]$ is also a non-negative matrix. Equation (4.3) is more conveniently written in the form

$$\langle 0|[\chi(x),\chi(x')]_\pm|0\rangle=\int\frac{d^4p}{(2\pi)^4}e^{ip\cdot(x-x')}d\kappa\,\delta(p^2+\kappa^2)\tfrac{1}{2}[B(\kappa,p)\pm B(\kappa,-p)^*]$$
$$+\int\frac{d^4p}{(2\pi)^4}e^{ip\cdot(x-x')}d\kappa\,\delta(p^2+\kappa^2)\tfrac{1}{2}\epsilon(p)\times[B(\kappa,p)\mp B(\kappa,-p)^*]. \quad (4.5)$$

Since R_{st} is a Hermitian matrix, one can bring it to diagonal form by a unitary transformation upon the fields χ;

$$\chi_a'=\textstyle\sum_b U_{ab}\chi_b,$$

---

[11] This application has been suggested by L. Brown though his proof is unknown to the author.

[12] L. Brown and J. Schwinger, Progr. Theoret. Phys. (Kyoto) **26**, 917 (1961).

The relation

$$R_{st}^2=1$$

shows that the only permissible eigenvalues of R_{st} are ± 1. Defining

$$B'=UBU^\dagger$$

and using (4.4), one finds

$$B_{aa}'(\kappa,-p)=(-1)^{2S}B_{aa}'(\kappa,p),$$
$$=(-1)^{2S}B_{aa}'^{*}(\kappa,p).$$

Thus,

$$\langle 0|[\chi_a'(x),\chi_a'(x')^\dagger]_\pm|0\rangle$$
$$=\int \frac{d^4p}{(2\pi)^4}e^{ip\cdot(x-x')}d\kappa\,\delta(p^2+\kappa^2)B_{aa}'(\kappa,p)\tfrac{1}{2}[1\pm(-1)^{2S}]$$
$$+\int \frac{d^4p}{(2\pi)^4}e^{ip\cdot(x-x')}d\kappa\,\delta(p^2+\kappa^2)$$
$$\times B_{aa}'(\kappa,p)\epsilon(p)\tfrac{1}{2}[1\mp(-1)^{2S}].$$

If we now assume the "wrong" statistics, i.e., that when $x^0=x^{0\prime}$ and $\mathbf{x}\neq\mathbf{x}'$, $\{\chi'(x),\chi'^\dagger(x')\}=0$ for integral spin and $[\chi'(x),\chi'^\dagger(x')]=0$ for half-integral spin, then

$$\int \frac{d^3p}{(2\pi)^3}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}B_{aa}'(\kappa,p)(p^2+\kappa^2)^{-1/2}=0,\quad \mathbf{x}\neq\mathbf{x}'. \tag{4.6}$$

Our assumption of the analyticity of the spectral weight is now generalized by requiring that $(\mathbf{p}^2+\kappa^2)^{-1/2}B'$ be analytic in $\mathbf{p}^2$ with the same singularities as B' alone. This requirement eliminates the trivial case in which B' is proportional to $(\mathbf{p}^2+\kappa^2)^{1/2}$ but is otherwise analytic in the entire $\mathbf{p}^2$ plane.

The condition (4.6) is easily seen to be equivalent to the requirement

$$\int d\kappa\int\frac{d^3p}{(2\pi)^3}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}b(\kappa,\mathbf{p}^2)=0\quad\text{for}\quad \mathbf{x}\neq\mathbf{x}', \tag{4.7}$$

where $b(\kappa,\mathbf{p}^2)$ is a positive definite function for $\mathbf{p}^2\geqslant 0$ and

$$b(\kappa,\mathbf{p}^2)=\int_m^\infty d\lambda\frac{\sigma(\kappa,\lambda)}{\mathbf{p}^2+\lambda^2}. \tag{4.8}$$

In writing (4.8) we have assumed that no subtractions are necessary in the dispersion relation for $B'(\kappa,p)$ although this is not essential to the final result. Equations (4.7) and (4.8) imply:

$$\frac{1}{r}\int\int_m^\infty d\kappa d\lambda\, e^{-\lambda r}\sigma(\kappa,\lambda)=0\quad\text{for}\quad r>0,$$

which, in turn, requires that

$$\int d\kappa\,\sigma(\kappa,\lambda)=0. \tag{4.9}$$

From (4.9) one deduces that

$$\int_m^\infty\frac{d\lambda}{\mathbf{p}^2+\lambda^2}\int_m^\infty d\kappa\,\sigma(\kappa,\lambda)=0,$$

or, upon interchange of orders of integration, that

$$\int_m^\infty d\kappa\, b(\kappa,\mathbf{p}^2)=0,$$

thus contradicting the positivity of $b(\kappa,\mathbf{p}^2)$.

In order to put this result into proper perspective it should be remarked that the analyticity condition on the spectral weight which was used in the proof can be replaced by weaker conditions which also serve to establish the correct relation between spin and statistics. Since, however, the usual proof for the manifestly covariant case rests primarily upon the analyticity properties in p^2, the present argument has the virtue of forming a most natural extension to electrodynamics.

V. CONCLUSION

The model considered in this paper has been seen to possess pathologies quite similar to those which have been found in other field theoretical models. The advantage of our approach lies in the fact that it has clearly indicated that these unphysical singularities by virtue of their noncovariant character must be attributed solely to the inadequacies of the model as an approximation to quantum electrodynamics. While the infrared structure has been suitably described in this treatment, the extrapolation of our results in the complex p^2 plane was required to be limited to a small neighborhood of the threshold of the physical cut. The questionable character of calculations which attempt to determine the singularities of operator expectation values by means of models cannot be too highly stressed. Recent results which have involved the extraction of bound states in this way must be viewed with some suspicion.

ACKNOWLEDGMENT

The author wishes to thank Professor K. Johnson for the advice and criticism which were so essential to this work.

APPENDIX

Consider

$$e^{A+B},$$

where

$$[A,B]=-\lambda A.$$

An explicit representation of these operators is given by

$$A=e^{\alpha},$$
$$B=\lambda d/d\alpha.$$

Then

$$\exp\left(e^{\alpha}+\lambda\frac{d}{d\alpha}\right)=\exp\left(e^{f(\alpha)}\lambda\frac{d}{d\alpha}e^{-f(\alpha)}\right),$$

where

$$f(\alpha)=-\frac{1}{\lambda}e^{\alpha}.$$

Thus,

$$e^{A+B}=e^{f(\alpha)}e^{\lambda d/d\alpha}e^{-f(\alpha)},$$
$$=e^{f(\alpha)}e^{-f(\alpha+\lambda)}e^{\lambda d/d\alpha},$$
$$=\left[\exp\left(A\frac{e^{\lambda}-1}{\lambda}\right)\right]e^{B}.$$

The adjoint of the above yields the alternative representation:

$$e^{A+B}=e^{B}\exp\left(A\frac{1-e^{-\lambda}}{\lambda}\right).$$

PHYSICAL REVIEW VOLUME 130, NUMBER 2 15 APRIL 1963

Two Vacuum Poles and Pion-Nucleon Scattering*

Keiji Igi†
Department of Physics, University of California, Berkeley, California

(Received 6 December 1962)

A general expression is given for the pion-nucleon non-charge-exchange scattering amplitude for arbitrary energy and small momentum transfer on the assumption that only the vacuum pole P and the second vacuum pole P' exist in the upper half J plane. We derive sum rules for non-spin-flip and spin-flip amplitudes and use them, combined with the analysis of the high-energy π-N cross sections in terms of Regge poles, to investigate the behavior of P and P' trajectories near $t\approx 0$. For this purpose the importance of a precise measurement of the low-energy partial-wave phase shifts is emphasized. A sum rule for the S-wave pion-nucleon non-charge-exchange scattering length can be satisfied with $\alpha_{P'}\approx 0.5$.

I. INTRODUCTION

THERE have been many attempts to investigate the low-energy S-, P-, and D-wave pion-nucleon scattering based on the dispersion relations.[1–3] The charge-exchange scattering amplitude was successfully explained by Bowcock, Cottingham, and Lurié[2] by incorporating the $I=1$ pion-pion interaction into the analysis of CGLN.[1] However, the above method cannot be applied directly for the non-charge-exchange amplitude because the dispersion integrals diverge.

The aforementioned divergence problem which is related to the subtractions in the Mandelstam representation was greatly clarified by the Regge pole assumption[4] that all poles of the strong-interaction S matrix move in the complex J plane as a function of energy and that these poles control the asymptotic behavior. In a previous paper,[5] hereafter referred to as I, a sum rule was derived for the S-wave pion-nucleon non-charge-exchange scattering length, starting from the assumption that the amplitude can be written as the sum of two terms, the vacuum-Regge pole term which diverges at infinite energy and the remaining term which converges at infinity and satisfies an unsubtracted dispersion relation. This assumption led to a discrepancy between the observed and the calculated scattering lengths. Therefore, it was concluded that there should be another vacuum-Regge trajectory P' with $\alpha_{P'}(0)\sim 0.5$.[6] Existence of such a pole is also favored in the analysis of high-energy p-p and $\bar{p}$-p scattering,[7,8] high-energy π-p and K-p scattering.[7]

The purpose of the present paper is twofold: (a) to generalize the previous sum rule for pion-nucleon non-charge-exchange scattering, to hold for arbitrary s and small t (we assume, as in I, that only P and P' trajectories exist in the upper half J plane for t near zero); (b) to

* Research supported in part by the Air Force Office of Scientific Research.

† On leave of absence from Tokyo University of Education, Tokyo, Japan.

[1] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957), hereafter referred to as CGLN.

[2] J. Bowcock, W. N. Cottingham, and D. Lurié, Nuovo Cimento **16**, 918 (1960); **19**, 142 (1961).

[3] For detailed references, see A. Takahashi, Progr. Theoret. Phys. (Kyoto) **27**, 665 (1962).

[4] G. F. Chew, S. C. Frautschi, and S. Mandelstam, Phys. Rev. **126**, 1202 (1962); S. C. Frautschi, M. Gell-Mann, and F. Zachariasen, *ibid.* **126**, 2204 (1962). This assumption predicts a logarithmic shrinking of the p-p diffraction pattern with increasing energy. Such an effect has been observed experimentally [A. N. Diddens, E. Lillethun, G. Manning, A. E. Taylor, T. G. Walker, and A. M. Wetherell, Phys. Rev. Letters **9**, 108, 111 (1962)]. Moreover, the occurrence of Regge poles in the relativistic S matrix has been shown by Gribov, Domokos, Mandelstam, and Eden using the Mandelstam representation and elastic unitarity. See reference 7.

[5] K. Igi, Phys. Rev. Letters **9**, 76 (1962).

[6] In the previous paper I, it was concluded that there should be another vacuum trajectory in the region $1>\alpha(0)>0$. However, the notation of calling it as ABC pole has caused some confusion. It should have been noted by P' as introduced in the references 7 and 8. Detailed analysis for $\alpha_{P'}(0)$ is given in Appendix A.

[7] S. D. Drell, in *Proceedings of the 1962 Annual International Conference on High-Energy Physics at CERN* (CERN, Geneva, 1962).

[8] F. Hadjioannou, R. J. N. Phillips, and W. Rarita, Phys. Rev. Letters **9**, 183 (1962); Y. Hara, Progr. Theoret. Phys. (Kyoto) **28**, 711 (1962).