Nucleon Form Factors in the Strong-Coupling Meson Theory*†

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Expressions for the electric form factors of the proton and neutron have been given in the charged-scalar and symmetrical pseudoscalar meson theories. Calculations have been done in the strong-coupling limit. Both theories explain the experimental data given by the Stanford and Cornell groups very well. In the pseudoscalar theory, however, we have to take an effective meson mass which is about one-half of the rest mass. Small effective mass can be understood if a cubic term is introduced in the equation of motion for mesons. In the numerical integration of this equation, the strength $g\sqrt{\lambda}$ has been obtained from the boundary conditions, g and λ being the coupling constants of the pion-nucleon and pion-pion fields, respectively. With numerical solutions, the nucleon form factors have been recalculated. They fit the experimental data quite well.

1. INTRODUCTION

As is well known, the physical nucleons consist of a bare nucleon part and a surrounding meson cloud. The electric charge of the physical nucleon is, therefore, not a point charge but has a spatial distribution given by the charge density $\rho(\mathbf{x})$. We define the electric form factor for a nucleon by the Fourier transform of $\rho(\mathbf{x})$,[1–3]

$$F_1(q^2)=\int \rho(\mathbf{x})e^{i\mathbf{q}\cdot\mathbf{x}}d\mathbf{x},$$

in the static approximation. Similarly, the physical nucleons have a magnetic charge distribution, whose Fourier transform is called the magnetic form factor, $F_2(q^2)$.[3] The structure of the nucleons is studied in terms of these two kinds of form factors. The form factors can be studied by measuring the electron-nucleon scattering cross section, and in fact, they have been measured in the last few years by the Stanford[4] and Cornell[5] groups. In the present paper, we study the electric form factors for proton and neutron theoretically. The experimental data on $F_1(q^2)$ are shown in Fig. 1, where the $\mathbf{q}$ is the momentum transfer from electron to nucleon. Now let us consider a quantity

$$F_1(q^2)-\tfrac{1}{2}[1/(1+\Lambda^2q^2)].$$

Here the Λ is the Compton wavelength of the nucleon. The factor $(1+\Lambda^2q^2)^{-1}$ is the Fourier transform of the Yukawa well which is assumed for the source function in the static-extended bare nucleon. In the data given in Fig. 1 there exists a remarkable relation,

$$\left(F_1{}^{\text{proton}}(q^2)-\frac{1}{2}\frac{1}{1+\Lambda^2q^2}\right)\approx-\left(F_1{}^{\text{neutron}}(q^2)-\frac{1}{2}\frac{1}{1+\Lambda^2q^2}\right).$$

That is, a part of the meson cloud is almost equal in magnitude and opposite in sign for the proton and neutron. In 1942, Pauli and Dancoff calculated the magnetic moment for nucleons using the strong-coupling meson theory.[6] They found that the nucleon magnetic moment is the same in magnitude but different in sign for the proton and neutron. Since the situation is very similar in the data for the electric

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[1] The Fourier transform is defined without a factor $(2\pi)^{-3/2}$ throughout this work. The natural units $\hbar=c=1$ are adopted. The κ is the meson mass divided by $\hbar c$.

[2] Reviews of the works in this field were done by R. Hofstadter, Ann. Rev. Nucl. Sci. **7**, 231 (1957); D. R. Yennie, M. M. Lévy, and D. G. Ravenhall, Rev. Mod. Phys. **29**, 144 (1957); and R. Hofstadter, F. Bumiller, and M. R. Yearian, *ibid.* **30**, 482 (1958).

[3] We write the current density of the nucleon in terms of two invariant functions $F_1(q^2)$ and $F_2(q^2)$ which are defined as follows:

$$\langle p'|j_\mu(x)|p\rangle=\frac{i}{(2\pi)^3}\frac{Me^{-iq_\nu x_\nu}}{(p_0p_0')^{1/2}}\bar{u}(\mathbf{p}')[\gamma_\mu F_1(q^2)-\sigma_{\mu\nu}q_\nu F_2(q^2)]u(\mathbf{p}),$$

where $p=(\mathbf{p},ip_0)$ and $q_\lambda=p_\lambda'-p_\lambda$. The form factors $F_1(q^2)$ and $F_2(q^2)$ describe, in some sense, the distribution of charge and magnetization in the nucleon. We also define F_{ch} and F_{mag} by

$$F_{\text{ch}}(q^2)=\int \rho e^{i\mathbf{q}\cdot\mathbf{x}}d\mathbf{x},$$

and

$$i(\boldsymbol{\sigma}\times\mathbf{q})F_{\text{mag}}(q^2)=\int \mathbf{J}e^{i\mathbf{q}\cdot\mathbf{x}}d\mathbf{x},$$

where $\mathbf{J}$ and ρ are the current and charge distributions in the nucleon, respectively. Then the relations between F_{ch}, F_{mag} and F_1, F_2 are

$$F_{\text{ch}}(q^2)=F_1(q^2)-(q^2/2M)F_2(q^2),$$
$$F_{\text{mag}}(q^2)=(1/2M)F_1(q^2)+F_2(q^2),$$

for small q^2. In the static model where $M\to\infty$, we have

$$F_{\text{ch}}(q^2)=F_1(q^2),\quad F_{\text{mag}}(q^2)=F_2(q^2).$$

Since we are treating the problem in the static limit, we consider that F_1 describes the charge distribution hereafter. The detail of the above relations is given by F. J. Ernst, R. G. Sachs, and K. C. Wali, Phys. Rev. **119**, 1105 (1960); R. G. Sachs, *ibid.* **126**, 2256 (1962).

[4] R. Hofstadter, F. Bumiller, and M. Croissiaux, Phys. Rev. Letters **5**, 263 (1960); R. Hofstadter, C. de Vries, and R. Herman, *ibid.* **6**, 290 (1961); R. Hofstadter and R. Herman, *ibid.* **6**, 293 (1961); F. Bumiller, M. Croissiaux, E. Dally, and R. Hofstadter, Phys. Rev. **124**, 1623 (1961). *Note added in proof.* After completion of this work, new data on F_1 and F_2 have been published by C. de Vries, R. Hofstadter, and R. Herman, Phys. Rev. Letters, **8**, 381 (1962). There, F_1 for the neutron may have negative values, whose explanation has not been attempted in this work.

[5] D. N. Olson, H. F. Schopper, and R. R. Wilson, Phys. Rev. Letters **6**, 286 (1961); R. M. Littauer, H. F. Schopper, and R. R. Wilson, *ibid.* **7**, 141 (1961).

[6] W. Pauli and S. M. Dancoff, Phys. Rev. **62**, 85 (1942).

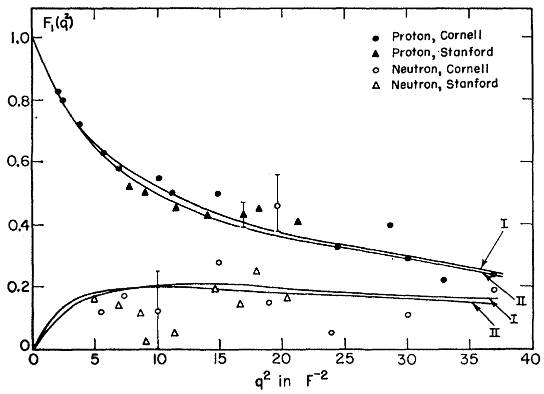

FIG. 1. Electric form factors $F_1(q^2)$ in the charged-scalar theory. The theoretical curves are: (I) for $\kappa=0.71\text{F}^{-1}$ and $\Lambda=0.20\text{F}$, (II) for $\kappa=0.71\text{F}^{-1}$ and $\Lambda=0.21\text{F}$. The experimental data are given by the Stanford[4] and Cornell[5] groups. The errors for $F_1(q^2)$ are not independent of those for $F_2(q^2)$, which are not given here. Typical deviations are indicated by arrows, see the second paper of reference 5.

form factor, we could hope to explain these data by using the strong-coupling meson theory. This is the motivation of the present work.

When the meson-nucleon coupling constant g is large ($g^2\gg1$), we cannot use perturbation calculation in the conventional theory of mesons. We have to make a representation in which the meson-nucleon interaction is diagonalized and the calculation is performed in power series of g^{-1}, instead of g as in the conventional perturbation theory. This has been done by Pauli and Dancoff in their theory of the strong coupling.[6] Recently, Pais and Serber have introduced a more general transformation than the Pauli-Dancoff transformation.[7,8] The Pais-Serber transformation can be applied not only in the strong coupling but also in the variational method and others. In the present work, we adopt this Pais-Serber transformation to calculate $F_1(q^2)$ in the strong-coupling approximation. The formulas of $F_1(q^2)$ are given in the charged-scalar and symmetrical pseudoscalar theory. Both theories explain the experimental data of $F_1(q^2)$ very well. In the pseudoscalar theory, however, we have to take an effective mass for the meson which is about one-half of the rest mass. A possible reduction of the effective mass can be understood, e.g., if a cubic term is introduced in the equation of motion for mesons,

$$[-\Delta+(\kappa^2-\lambda\varphi_\alpha{}^2)]\varphi_\alpha=\frac{(2\pi)^{1/2}g}{\kappa}\frac{\partial U}{\partial x_\alpha},\quad \lambda \text{ positive.}$$

In the zeroth approximation, the effective mass $(\kappa^2-\lambda\varphi_\alpha{}^2)^{1/2}$ is smaller than the rest mass κ. The cubic term can be obtained by taking into account the π-π interaction.[9] The existence of the π-π interaction has been emphasized in the so-called π-π resonances.[10] The effect of the π-π interaction is examined numerically and it can explain the experimental data on $F_1(q^2)$ quite well.

In Sec. 2, the Pais-Serber transformation is reviewed in the charged-scalar meson theory. Using this theory, the $F_1(q^2)$ is calculated and compared with experimental data in Sec. 3. In Secs. 4–12, we discuss the symmetrical pseudoscalar meson theory. In Sec. 4, the Pais-Serber transformation is reviewed. In Sec. 5, the p-wave functions for mesons are given in the strong-coupling limit. In Sec. 6, the $F_1(q^2)$ is calculated and compared with experimental data. To fit the data, the effective mass for mesons must be about one-half of the rest mass. Therefore, we introduce the π-π interaction in the symmetrical pseudoscalar theory in Sec. 7, where the equation of motion is given in the partial-wave representation. In Sec. 8, explicit forms for the equation of motion are given for s and p waves. We prove that the p waves couple with the s waves but the effect of s waves is negligible. In Sec. 9, we also prove that the higher partial waves have negligible effect on the p-wave equation. In Sec. 10, we give a relation between the mean square radius and the meson mass. The mean square radius is proportional to the product of the meson mass and nucleon mass in the symmetrical pseudoscalar theory, but to the square of the meson mass in the charged-scalar theory. In Sec. 11, a procedure of numerical integration is given for the differential equation for the meson wave function with the cubic term due to the π-π interaction. In Sec. 12, the results are shown. The form factors are calculated with the meson wave functions obtained by the numerical integration. In the Appendix, the meson wave function is studied with the square-well source function in the symmetrical pseudoscalar theory.

2. PAIS-SERBER TRANSFORMATION IN THE CHARGED-SCALAR MESON THEORY

For the charged-scalar meson theory, Pais and Serber[7] have shown a sequence of transformations. It gives the Hamiltonian of the extended source model a form bringing out the strong coupling characteristics. This procedure consists of two distinct sets of transformations: The first group leads to a rigorous transformation of the Hamiltonian which brings into evidence the dependence of the Hamiltonian on the charge of the system. This result is formally valid for all values of the coupling constant and independent of any relativistic approximations. The second group of transformations are those which refer to expansions

[7] A. Pais and R. Serber, Phys. Rev. **105**, 1636 (1957).
[8] A. Pais and R. Serber, Phys. Rev. **113**, 955 (1959).

[9] Similar physical ideas are contained in W. G. Holladay, Phys. Rev. **101**, 1198 (1956); W. R. Frazer and J. R. Fulco, Phys. Rev. Letters **2**, 365 (1959); Phys. Rev. **117**, 1609 (1960).
[10] The pion-pion resonance phenomena have been discussed at the American Physical Society Meeting, New York, 1962, by H. Kraybill, Bull. Am. Phys. Soc. **7**, 81 (1962).

valid only for the extended source model and for large values of the coupling constant. We shall briefly review these two stages.

The basic idea of the first step is to split the meson fields φ_α into a bound part parallel to f and a free part orthogonal to it:

$$\varphi_\alpha=\varphi_\alpha'+F^{-1}f\int f\varphi_\alpha d\mathbf{x},$$
$$\pi_\alpha=\pi_\alpha'+F^{-1}f\int f\pi_\alpha d\mathbf{x},\quad \alpha=1,2, \tag{2.1}$$

where f is an arbitrary spherically symmetric function of $\mathbf{x}$ and $F=\int f^2d\mathbf{x}$. The bound part is characterized by collective coordinates $Q_\alpha=F^{-1/2}\int f\varphi_\alpha d\mathbf{x}$, $P_\alpha=F^{-1/2}\times\int f\pi_\alpha d\mathbf{x}$ which satisfy the canonical commutation relations. We next transform the P_α, Q_α to polar coordinates: $Q_\alpha\rightarrow(Q,\theta)$, $P_\alpha\rightarrow(P,\theta)$. The variable θ can be eliminated from the Hamiltonian by making a rotation such that φ_1' and τ_1 lie in the direction of the vector Q_α. The final procedure of the first step is to eliminate P and Q by introducing new variables φ_α'', π_α'' defined by

$$\varphi_1''=\varphi_1'+F^{-1/2}fQ,\quad \varphi_2''=\varphi_2',$$
$$\pi_1''=\pi_1'+F^{-1/2}fP,\quad \pi_2''=\pi_2'. \tag{2.2}$$

The foregoing results are now applied to the strong-coupling model. The central theme of strong coupling is that for very large values of the coupling constant g one should first diagonalize the interaction energy H_{int} which is proportional to g. This can be done by putting $f=U$, where U is the spherically symmetric function of $\mathbf{x}$ which describes the extended source and is normalized according to

$$\int Ud\mathbf{x}=1. \tag{2.3}$$

The interaction energy is in the form of

$$H_{\text{int}}=g\tau_1(2\pi)^{1/2}\int U\varphi_1''d\mathbf{x}. \tag{2.4}$$

Accordingly, we work in a representation in which τ_1 is diagonal. We next split φ_1'' into a static part, v, and a fluctuating part, φ_{fl}. The self-field is proportional to the coupling constant and by expanding in the ratio of free-field to self-field we obtain an expansion of the Hamiltonian in descending power of g. In the static approximation the ground state of $\langle\tau_1\rangle=-1$ states corresponds to the physical nucleon. The Hamiltonian for the physical nucleon is

$$H=\frac{1}{2}\int v\omega^2vd\mathbf{x}-g(2\pi)^{1/2}\int Uvd\mathbf{x}+(P_\theta^2/2V), \tag{2.5}$$

and v satisfies the equation

$$\omega^2v-g(2\pi)^{1/2}U-(P_\theta^2/V^2)v=0, \tag{2.6}$$

where $\omega^2=-\Delta+\kappa^2$, P_θ is the third component of the total isotopic spin of the system, and

$$V=\int v^2d\mathbf{x}. \tag{2.7}$$

3. ELECTRIC FORM FACTORS IN THE CHARGED-SCALAR MESON THEORY

The charge density of the system of nucleon and meson fields is given by

$$\rho(\mathbf{x})=\varphi_1\pi_2-\varphi_2\pi_1+\tfrac{1}{2}(1+\tau_3)U. \tag{3.1}$$

In order to find the expression of $\rho(\mathbf{x})$ in the strong-coupling approximation, we perform the Pais-Serber transformation given in Sec. 2. Then the old variables, π_α, φ_α, are given in terms of the new variables, v, $\mathfrak{q}$, θ, P_θ, as follows:

$$\pi_1=-\frac{P_\theta}{V}v\sin\theta+\frac{i}{2}\frac{U\cos\theta}{\mathfrak{q}}+\frac{1}{2}\frac{U\sin\theta}{\mathfrak{q}}\tau_3,$$
$$\pi_2=\frac{P_\theta}{V}v\cos\theta+\frac{i}{2}\frac{U\sin\theta}{\mathfrak{q}}-\frac{1}{2}\frac{U\cos\theta}{\mathfrak{q}}\tau_3,$$
$$\varphi_1=v\cos\theta,$$
$$\varphi_2=v\sin\theta,$$
$$\mathfrak{q}=\int Uvd\mathbf{x}. \tag{3.2}$$

Substituting (3.2) into (3.1), we have

$$\rho(\mathbf{x})=\frac{P_\theta}{V}v^2-\frac{1}{2}\frac{Uv}{\mathfrak{q}}\tau_3+\frac{1}{2}(1+\tau_3)U. \tag{3.3}$$

Since $\langle\tau_3\rangle=0$, (3.3) becomes

$$\langle\rho(\mathbf{x})\rangle=(\langle P_\theta\rangle/V)v^2+\tfrac{1}{2}U. \tag{3.3'}$$

Here $\langle P_\theta\rangle=\frac{1}{2}(-\frac{1}{2})$ for proton (neutron). U, v, and V are given as follows. We assume the Yukawa well for the nucleon source function,

$$U=\frac{1}{4\pi\Lambda^2}\frac{e^{-r/\Lambda}}{r},\quad r=|\mathbf{x}|, \tag{3.4}$$

where Λ is the Compton wavelength of the nucleon. From (2.6), the v has to satisfy

$$(-\Delta+\kappa^2)v-g(2\pi)^{1/2}\frac{1}{4\pi\Lambda^2}\frac{e^{-r/\Lambda}}{r}-\frac{P_\theta^2}{V^2}v=0. \tag{3.5}$$

The last term in the above equation can be neglected, since the $\langle P_\theta^2\rangle/V^2$ is of the order of g^{-4}. The solution of

(3.5) has a form of

$$v=\frac{g(2\pi)^{1/2}}{4\pi(1-\kappa^2\Lambda^2)}\frac{1}{r}(e^{-\kappa r}-e^{-r/\Lambda}). \tag{3.6}$$

Substituting (3.6) into (2.7), we have

$$V=\frac{g^2}{4\kappa}\frac{1}{(1+\kappa\Lambda)^3}. \tag{3.7}$$

As a result, the charge density becomes[11]

$$\rho(\mathbf{x})=\frac{\eta\kappa(1+\kappa\Lambda)}{4\pi(1-\kappa\Lambda)^2}\frac{1}{r^2}(e^{-\kappa r}-e^{-r/\Lambda})^2+\frac{1}{8\pi\Lambda^2}\frac{e^{-r/\Lambda}}{r}. \tag{3.8}$$

Here $\eta=1(-1)$ for proton (neutron).

The electric form factor, $F_1(q^2)$, is the Fourier transform of the charge distribution $\rho(\mathbf{x})$ and is defined by

$$F_1(q^2)=\int\rho(\mathbf{x})e^{i\mathbf{q}\cdot\mathbf{x}}d\mathbf{x}, \tag{3.9}$$

where $\mathbf{q}$ is the momentum transferred to the nucleon from an electron.

$$F_1(q^2)=\frac{\eta\kappa(1+\kappa\Lambda)}{q(1-\kappa\Lambda)^2}\left(\cot^{-1}\frac{2\kappa}{q}+\cot^{-1}\frac{2}{q\Lambda}-2\cot^{-1}\frac{\kappa+(1/\Lambda)}{q}\right)+\frac{1}{2}\frac{1}{1+\Lambda^2q^2},$$

$$\text{with}\quad \eta=1\ (-1)\text{ for proton (neutron)}, \tag{3.10}$$

for the charged-scalar meson theory. Here we take the principal values for $\cot^{-1}$.

Experimental data on $F_1(q^2)$ have been given by the Stanford[4] and Cornell[5] groups. In comparison with experimental data, we have adjusted κ and Λ in (3.10) as parameters. As is shown in Fig. 1, the best fit is obtained with

$$\kappa=0.71\mathrm{F}^{-1},\quad \Lambda=0.20\mathrm{F}. \tag{3.11}$$

These values are very close to the rest mass of the meson ($0.71\mathrm{F}^{-1}$) and the Compton wavelength of the nucleon (0.21F).

4. PAIS-SERBER TRANSFORMATION IN THE SYMMETRICAL PSEUDOSCALAR MESON THEORY

The transformations for symmetrical pseudoscalar theory are almost parallel to those for the charged-scalar theory.[8] We first perform a transformation on meson fields φ_α and their canonical conjugates π_α, leading to new fields (marked by a prime)

$$\varphi_\alpha=\varphi_\alpha'+F^{-1/2}\textstyle\sum_k Q_{\alpha k}\partial f/\partial x_k,$$
$$\pi_\alpha=\pi_\alpha'+F^{-1/2}\textstyle\sum_k P_{\alpha k}\partial f/\partial x_k,\quad \alpha=1,2,3, \tag{4.1}$$

where f is again a spherically symmetric function of $\mathbf{x}$ and $F=\frac{1}{3}\int(\nabla f)^2d\mathbf{x}$. The bound part is parallel to the gradient of f and characterized by collective coordinates

$$Q_{\alpha k}=F^{-1/2}\int\frac{\partial f}{\partial x_k}\varphi_\alpha d\mathbf{x},\quad P_{\alpha k}=F^{-1/2}\int\frac{\partial f}{\partial x_k}\pi_\alpha d\mathbf{x}.$$

The free part is orthogonal to the gradient of f and characterized by φ_α', π_α'. One next transforms the nine collective coordinates $Q_{\alpha k}$ into a set of three angular variables in space, a similar set in isotopic space, and three radial variables. For this purpose we introduce the orthogonal matrix A_{kl} which corresponds to the solid rotation in the ordinary space. Similarly we introduce an orthogonal matrix $B_{\alpha\beta}$ for the isotopic variables. The new variables q_r and p_{rs} are related to $Q_{\alpha k}$, $P_{\alpha k}$ by $Q_{\alpha k}=\sum_r B_{r\alpha}A_{rk}q_r$, $P_{\alpha k}=\sum_{s,r}B_{r\alpha}A_{sk}p_{rs}$. Here the transformation for q_{rs} is a principal axis transformation, that is, $q_{rs}=q_r\delta_{rs}$. The p_{rs} is specified by p_r, canonical conjugate of q_r, L_{rs}, ordinary angular momentum, and I_{rs}, isotopic angular momentum. The angular variables contained in A and B can be eliminated from the Hamiltonian by a canonical transformation as a result of which I_α represent the components of the total isotopic spin vector along the axes of the rotating system. Finally, it is possible to eliminate entirely the radial oscillator variables p_r, q_r from the Hamiltonian by introducing new fields

$$\pi_\alpha''=\pi_\alpha'+F^{-1/2}p_\alpha\partial f/\partial x_\alpha,$$
$$\varphi_\alpha''=\varphi_\alpha'+F^{-1/2}q_\alpha\partial f/\partial x_\alpha.$$

π'', φ'' satisfy the anomalous commutation relations

$$[\pi_\alpha''(\mathbf{x}),\varphi_\beta''(\mathbf{x}')]=-i\delta_{\alpha\beta}\left[\delta(\mathbf{x}-\mathbf{x}')-F^{-1}\sum_k{}'\frac{\partial f(r)}{\partial x_k}\frac{\partial f(r')}{\partial x_k'}\right], \tag{4.2}$$

where the prime on the summation over k means that the term $k=\alpha$ is to be excluded. The orthogonality relations are

$$\int\varphi_\alpha''\frac{\partial f}{\partial x_\beta}d\mathbf{x}=\int\pi_\alpha''\frac{\partial f}{\partial x_\beta}d\mathbf{x}=0,\quad \alpha\neq\beta. \tag{4.3}$$

In the strong-coupling treatment a particular choice of the distribution function f is made: the choice $f=U$ diagonalizes the interaction energy. In the case in hand, the interaction is

$$H_{\mathrm{int}}\sim\sum_r\tau_r\sigma_r\int\frac{\partial U}{\partial x_r}\varphi_r''d\mathbf{x}.$$

For the minimum interaction energy, $\langle\tau_r\sigma_r\rangle=-1$.[6]

[11] More precisely, $\rho(\mathbf{x})$ should be read as $\langle\rho(\mathbf{x})\rangle$, that is, the expectation value of $\rho(\mathbf{x})$. Throughout this paper, we drop the symbol $\langle\ \rangle$ if there is no confusion.

After dropping the small contributions in the strong-coupling limit the Hamiltonian becomes

$$H=\frac{1}{2}\sum_\alpha\int(\pi_\alpha''^2+\varphi_\alpha''\omega^2\varphi_\alpha'')d\mathbf{x}-\frac{g(2\pi)^{1/2}}{\kappa}\sum_r\int\frac{\partial U}{\partial x_r}\varphi_r''d\mathbf{x}.\qquad(4.4)$$

The relations between the unprimed variables φ, π and double primed ones φ'', π'' are

$$\begin{aligned}\varphi_\alpha&=\sum_\beta B_{\beta\alpha}\varphi_\beta'',\\ \pi_\alpha&=\sum_\beta B_{\beta\alpha}\pi_\beta'',\end{aligned}\qquad(4.5)$$

in the strong-coupling limit. This strong-coupling approximation is valid if $g\gg\kappa\Lambda$ or $g^2\gg0.01$.[6]

5. p-WAVE MESONS

In this section, we give the equation of motion for φ_α'', and the explicit forms of the wave functions. As is well known, the canonical equation of motion is obtained from the general rule

$$\partial f/\partial t=i[H,f],\qquad(5.1)$$

for $f=\varphi_\alpha''$ and π_α''. Using the Hamiltonian (4.4) and the commutation relations (4.2), we obtain

$$\left(\frac{\partial^2}{\partial t^2}-\Delta+\kappa^2\right)\varphi_\alpha''-\frac{g(2\pi)^{1/2}}{\kappa}\frac{\partial U}{\partial x_\alpha}-\frac{1}{F}\sum_k{}'\frac{\partial U}{\partial x_k}\int\varphi_\alpha''(\mathbf{x}')\omega^2\frac{\partial U(r')}{\partial x_k'}d\mathbf{x}'=0.\qquad(5.2)$$

Here the prime on the summation over k means that the term $k=\alpha$ is to be excluded. Since we are interested in the time-independent solution, we drop the $\partial^2/\partial t^2$ terms from (5.2) hereafter. The meson wave function φ_α'' can be decomposed into partial waves as

$$\varphi_\alpha''=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{\chi_{l,m;\alpha}(r)}{r}Y_{l,m}(\theta,\phi).\qquad(5.3)$$

Here the subscripts l, m, and α for $\chi_{l,m;\alpha}$ are the orbital angular momentum, its magnetic quantum number, and the type of fields, respectively. We use the phase of the spherical harmonics $Y_{l,m}$ defined by Condon and Shortley.[12] If we assume the p waves to be dominant,

$$\varphi_\alpha''=\sum_{m=-1}^{m=1}\frac{\chi_{1,m;\alpha}(r)}{r}Y_{1,m}(\theta,\phi).\qquad(5.4)$$

The nine radial wave functions $\chi_{1,m;\alpha}$ ($\alpha=1$, 2, 3 and $m=1$, 0, -1) are not independent because of the boundary conditions

$$\int\varphi_\alpha''\frac{\partial U}{\partial x_k}d\mathbf{x}=0\quad\text{for}\quad\alpha\neq k.\qquad(5.5)$$

Since the nucleon source function is spherically symmetric, the derivatives of U are given by

$$\begin{aligned}\frac{\partial U}{\partial x_\alpha}&=\left(\frac{2\pi}{3}\right)^{1/2}[Y_{1,-1}(\theta,\phi)-Y_{1,1}(\theta,\phi)]\frac{dU}{dr}\quad\text{for}\quad\alpha=1,\\ &=i\left(\frac{2\pi}{3}\right)^{1/2}[Y_{1,-1}(\theta,\phi)+Y_{1,1}(\theta,\phi)]\frac{dU}{dr}\quad\text{for}\quad\alpha=2,\\ &=\left(\frac{4\pi}{3}\right)^{1/2}Y_{1,0}(\theta,\phi)\frac{dU}{dr}\quad\text{for}\quad\alpha=3.\end{aligned}\qquad(5.6)$$

Equations (5.5) and (5.6) together with the orthogonality of the spherical harmonics give

$$\begin{aligned}&\chi_{1,0;1}=\chi_{1,0;2}=\chi_{1,\pm1;3}=0,\\ &-\chi_{1,1;1}=\chi_{1,-1;1}\equiv\chi_{1;1},\\ &\chi_{1,1;2}=\chi_{1,-1;2}\equiv\chi_{1;2},\\ &\chi_{1,0;3}\equiv\chi_{1;3}.\end{aligned}\qquad(5.7)$$

The symbols $\chi_{1;\alpha}$ are introduced for convenience. Equation (5.7) will hold at any r where $rdU/dr\neq0$. For the Yukawa well of U, this is true for all r. For the square well, $rdU/dr\neq0$ only at $r=a$. Since, however, we are looking for regular solutions ($\chi_{1;\alpha}=0$ at $r=0$), we conclude that (5.7) can always hold for all r, if these relations are once satisfied at a given r ($r\neq0$).

The differential equations for three p-wave functions are given below explicitly. We notice that the integration term in (5.2) vanishes by (5.7).

$$\left(-\frac{d^2}{dr^2}+\frac{2}{r^2}+\kappa^2\right)\chi_{1;\alpha}-C_\alpha\frac{2\pi}{\sqrt{3}}\frac{g}{\kappa}\frac{dU}{dr}r=0,\qquad(5.8)$$

with $C_\alpha=1$, i, and $\sqrt{2}$ for $\alpha=1$, 2, and 3, respectively. From (5.8), one can easily see that the differential equations for $\chi_{1;1}$, $\chi_{1;2}/i$ and $\chi_{1;3}/\sqrt{2}$ are identical. Since the second-order differential equation has only one solution which is regular at the origin, we can put

$$\chi_{1;1}=\chi_{1;2}/i=\chi_{1;3}/\sqrt{2}\equiv\chi_1.\qquad(5.9)$$

Here χ_1 satisfies (5.8) with $C_\alpha=1$. Thus, the p-wave functions are

$$\begin{aligned}\varphi_1''&=\left(\frac{3}{2\pi}\right)^{1/2}\frac{\chi_1}{r}\sin\theta\cos\phi,\\ \varphi_2''&=\left(\frac{3}{2\pi}\right)^{1/2}\frac{\chi_1}{r}\sin\theta\sin\phi,\\ \varphi_3''&=\left(\frac{3}{2\pi}\right)^{1/2}\frac{\chi_1}{r}\cos\theta.\end{aligned}\qquad(5.10)$$

[12] E. U. Condon and G. H. Shortley, in *Theory of Atomic Spectra* (Cambridge University Press, New York, 1935), Chap. 3.

The meson wave functions in the original system are

$$\varphi_\alpha=\sum_\beta B_{\beta\alpha}\varphi_\beta''. \tag{4.5}$$

Here the transformation B is the rotation matrix in the isotopic space. It is obviously real and orthogonal.

The explicit form of the radial wave function, χ_1, for the p-wave meson can be given in the case where the Yukawa well is the nucleon source function. We introduce nondimensional quantities

$$\kappa r=\rho,\quad \kappa\Lambda=a. \tag{5.11}$$

The differential equation (5.8) with $C_\alpha=1$ becomes

$$\left(-\frac{d^2}{d\rho^2}+\frac{2}{\rho^2}+1\right)\chi_1-\frac{2\pi}{\sqrt{3}}\frac{g}{\kappa^3}\frac{dU}{d\rho}\rho=0, \tag{5.12}$$

with the source function

$$U=\frac{\kappa}{4\pi\Lambda^2}\frac{e^{-\rho/a}}{\rho}. \tag{5.13}$$

Equation (5.12) with (5.13) is

$$\left(-\frac{d^2}{d\rho^2}+\frac{2}{\rho^2}+1\right)\chi_1+\frac{1}{2\sqrt{3}}\frac{g}{a^2}\left(\frac{1}{\rho}+\frac{1}{a}\right)e^{-\rho/a}=0. \tag{5.14}$$

The solution of this equation with the following two boundary conditions

$$\chi_1=0 \quad \text{at} \quad \rho=0, \tag{5.15}$$

$$\chi_1 \text{ does not diverge at } \rho\to\infty, \tag{5.16}$$

is given by

$$\chi_1=\frac{1}{2\sqrt{3}}\frac{g}{1-a^2}\left[\left(\frac{1}{\rho}+\frac{1}{a}\right)e^{-\rho/a}-\left(\frac{1}{\rho}+1\right)e^{-\rho}\right] \quad \text{for all } \rho. \tag{5.17}$$

Near the origin,

$$\chi_1=-\frac{g\rho}{4\sqrt{3}a^2}+O(\rho^2) \quad \text{for} \quad \rho\to 0. \tag{5.18}$$

6. ELECTRIC FORM FACTORS OF NUCLEON IN THE SYMMETRICAL PSEUDOSCALAR THEORY

In the last section, we have studied the meson wave functions in the strong-coupling limit for the pseudoscalar mesons. Using these meson wave functions, we first give the charge distribution $\rho(\mathbf{x})$ of the system of meson fields and nucleon. Then we take the Fourier transform of $\rho(\mathbf{x})$ to find the electric form factors (3.9). The definition of the charge distribution of the system is given by

$$\rho(\mathbf{x})=(\varphi_1\pi_2-\varphi_2\pi_1)+\tfrac{1}{2}(1+\tau_3)U. \tag{3.1}$$

The meson part, $(\varphi_1\pi_2-\varphi_2\pi_1)$, can be written as

$$\varphi_1\pi_2-\varphi_2\pi_1=\omega(\varphi_1^2+\varphi_2^2),$$

where ω is the total energy of the meson. Using the fact that

$$\int\langle\rho(\mathbf{x})\rangle d\mathbf{x}=1 \quad \text{for protons,}$$
$$=0 \quad \text{for neutrons,}$$

we can eliminate ω and we have the charge distribution,

$$\rho(\mathbf{x})=\frac{\eta}{2}\frac{\varphi_1^2+\varphi_2^2}{\int(\varphi_1^2+\varphi_2^2)d\mathbf{x}}+\tfrac{1}{2}(1+\tau_3)U. \tag{6.1}$$

Here $\eta=1(-1)$ for protons (neutrons) and the expectation value $\langle\tau_3\rangle=0$ in the strong-coupling limit. Using (4.5), we have

$$\varphi_1^2+\varphi_2^2=\sum_\alpha(\varphi_\alpha'')^2-\sum_{\alpha,\beta}B_{\alpha3}B_{\beta3}\varphi_\alpha''\varphi_\beta'', \tag{6.2}$$

where $B_{\alpha3}$ transform as the components of the vector $B^{(3)}$ in the isotopic space. The expectation values of $B_{\alpha3}$ in the state specified i, n, j, m can be obtained by the method of angular momentum as given by Condon and Shortley.[12] Here i, n, j, and m are the isotopic spin, its third component, the total angular momentum, and magnetic quantum number, respectively. For the nucleons, $i=\frac{1}{2}$, $n=\pm\frac{1}{2}$, $j=\frac{1}{2}$, $m=\frac{1}{2}$. In these cases

$$\varphi_1^2+\varphi_2^2=\tfrac{2}{3}\sum_\alpha(\varphi_\alpha'')^2=\frac{1}{\pi}(\chi_1^2/r^2). \tag{6.3}$$

Therefore, the charge distribution is given by

$$\langle\rho(\mathbf{x})\rangle=\frac{\eta}{2}\frac{\chi_1^2/r^2}{\int(\chi_1^2/r^2)d\mathbf{x}}+\tfrac{1}{2}U. \tag{6.4}$$

Finally, the electric form factor defined by (3.9) is

$$F_1(q^2)=\frac{\eta}{2}\frac{\kappa^2\Lambda(1+\kappa\Lambda)}{(1-\kappa\Lambda)^2q}\left\{\left[\frac{2q^2}{\kappa^2}+\frac{2(1+\kappa\Lambda)^2}{\kappa^2\Lambda^2}-\frac{4}{\kappa\Lambda}\right]\times\tan^{-1}\left(\frac{\Lambda q}{1+\kappa\Lambda}\right)-\left(\frac{q^2}{\kappa^2}+\frac{2}{\kappa^2\Lambda^2}\right)\tan^{-1}\left(\frac{\Lambda q}{2}\right)-\left(\frac{q^2}{\kappa^2}+2\right)\tan^{-1}\left(\frac{q}{2\kappa}\right)\right\}+\frac{1}{2}\frac{1}{1+\Lambda^2q^2}, \tag{6.5}$$

with $\eta=1(-1)$ for protons (neutrons). Here we take the principal values for $\tan^{-1}$.

The formula (6.5) has been compared with the data given by the Stanford[4] and Cornell[5] groups. We have adjusted κ and Λ as parameters. As is shown in Fig. 2, the best fit is obtained with

$$\kappa=0.35\text{F}^{-1},\quad \Lambda=0.22\text{F}, \tag{6.6}$$

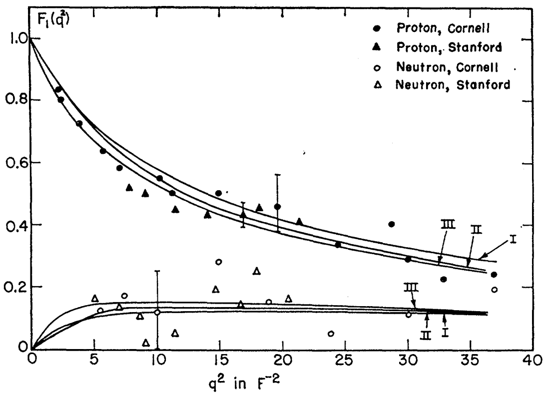

FIG. 2. Electric form factors $F_1(q^2)$ in the symmetrical pseudoscalar theory. The theoretical curves are: (I) for $\kappa=0.71\text{F}^{-1}$ and $\Lambda=0.21\text{F}$, (II) for $\kappa=0.60\text{F}^{-1}$ and $\Lambda=0.22\text{F}$, (III) for $\kappa=0.35\text{F}^{-1}$ and $\Lambda=0.22\text{F}$.

The adjusted value $\Lambda=0.22\text{F}$ is very close to the Compton wavelength of the nucleon (0.21F). On the other hand, the adjusted value $\kappa=0.35\text{F}^{-1}$ is about one-half of the rest mass of the π meson which is 0.71F^{-1}. This means that the κ in $F_1(q^2)$ is not the rest mass itself but some sort of effective mass. Therefore, to find some reason why the effective mass should be so small is a further problem. The reduction of this effective mass can be explained if we take into account the π-π interaction whose existence has been recently suggested by the so-called π-π resonances.[10] In order to investigate the effect of the π-π interaction on the charge distribution of the nucleon in detail, we will study the equation of motion for meson fields including the effect of the π-π interaction in the next section.

7. PARTIAL-WAVE REPRESENTATION OF MESON WAVE FUNCTION WITH THE π-π INTERACTION

If the π-π interaction is present, the Hamiltonian H has an additional term $H_{\pi-\pi}$:

$$H=H_0''+H_{\text{int}}''+H_{\pi-\pi}. \tag{7.1}$$

Here the π-π interaction Hamiltonian is assumed to be rotationally invariant in the isotopic space,

$$H_{\pi-\pi}=-\frac{\lambda}{4}\int\Big[\sum_\alpha(\varphi_\alpha)^2\Big]^2d\mathbf{x}\quad\text{with positive }\lambda. \tag{7.2}$$

That is, we are considering the four-pion vertices where either four pions are the same fields or two pairs of two different fields. Under the Pais-Serber transformation, the meson fields obey the rule

$$\varphi_\alpha=\textstyle\sum_\beta B_{\beta\alpha}\varphi_\beta''. \tag{4.5}$$

The $H_{\pi-\pi}$ in terms of the fields with double prime is

$$H_{\pi-\pi}=-\frac{\lambda}{4}\int\Big[\sum_\alpha(\varphi_\alpha'')^2\Big]^2d\mathbf{x}. \tag{7.3}$$

The total Hamiltonian (7.1) becomes

$$H=\tfrac{1}{2}\sum_\alpha\int(\pi_\alpha''^2+\varphi_\alpha''\omega^2\varphi_\alpha'')d\mathbf{x}-\frac{g(2\pi)^{1/2}}{\kappa}\times\sum_\alpha\int\frac{\partial U}{\partial x_\alpha}\varphi_\alpha''d\mathbf{x}-\frac{\lambda}{4}\int\Big[\sum_\alpha(\varphi_\alpha'')^2\Big]^2d\mathbf{x}. \tag{7.4}$$

From this Hamiltonian, we have the equation of motion for the meson fields φ_α'' by the same procedure as in Sec. 5.

$$\Big(-\Delta+\kappa^2+\frac{\partial^2}{\partial t^2}\Big)\varphi_\alpha''-\frac{g(2\pi)^{1/2}}{\kappa}\frac{\partial U}{\partial x_\alpha}-\lambda\varphi_\alpha''\Big(\sum_\beta\varphi_\beta''^2\Big)-\frac{1}{F}\sum_k{}'\frac{\partial U}{\partial x_k}\int\Big[\varphi_\alpha''(\mathbf{x}')\omega^2\frac{\partial U(r')}{\partial x_k'}-\lambda\varphi_\alpha''(\mathbf{x}')\Big(\sum_\beta\varphi_\beta''^2(\mathbf{x}')\Big)\frac{\partial U(r')}{\partial x_k'}\Big]d\mathbf{x}'=0, \tag{7.5}$$

with

$$F=\frac{1}{3}\int(\nabla U)^2d\mathbf{x}.$$

Since we are interested in the time-independent solution, we drop the $\partial^2\varphi_\alpha''/\partial t^2$ term from (7.5) hereafter. As we expect, the differential equation for φ_α'' has cubic terms due to the effect of the π-π interaction. Furthermore, these cubic terms involve the coupling of different fields φ_β''. An extra complication comes from the coupling of various partial waves in the cubic terms, as will be shown below.

The decomposition of φ_α'' into the spherical components is given by (5.3) which is abbreviated as

$$\varphi_\alpha''=\sum_{l,m}\frac{\chi_{l,m;\alpha}}{r}Y_{l,m}. \tag{5.3'}$$

The differential equations for the radial wave function, $\chi_{l,m;\alpha}$, can be obtained in the following way. As is well known, the Laplacian is given in terms of the spherical coordinates as

$$\Delta=\frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial}{\partial r}\Big)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial}{\partial\theta}\Big)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \tag{7.6}$$

From this the first term in (7.5) becomes

$$(-\Delta+\kappa^2)\varphi_\alpha''=\sum_{l,m}\frac{1}{r}\Big[-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}+\kappa^2\Big]\chi_{l,m;\alpha}Y_{l,m}=\sum_{l,m}D_{l,m;\alpha}Y_{l,m}. \tag{7.7}$$

Here we abbreviate

$$D_{l,m;\alpha}\equiv D_{l,m;\alpha}(r)=\frac{1}{r}\left[-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}+\kappa^2\right]\chi_{l,m;\alpha}. \quad (7.8)$$

The derivative of U in the second term of (7.5) is given in (5.6). The cubic terms can be expressed simply by a triple product of (5.3).

$$\varphi_\alpha''[\textstyle\sum_\beta(\varphi_\beta'')^2]$$
$$=\sum_\beta\sum_{l_1,m_1}\sum_{l_2,m_2}\sum_{l_3,m_3}\frac{1}{r^3}\chi_{l_1,m_1;\alpha}\chi_{l_2,m_2;\beta}\chi_{l_3,m_3;\beta}\times Y_{l_1,m_1}Y_{l_2,m_2}Y_{l_3,m_3}. \quad (7.9)$$

The product of three spherical harmonics is reduced to a sum of single spherical harmonics, by application of the following identity twice.[13]

$$Y_{l_1,m_1}Y_{l_2,m_2}=\sum_l\left(\frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)}\right)^{1/2}\times(l_1l_200|l0)(l_1l_2m_1m_2|lm)\delta_{m_1+m_2,m}Y_{l,m}. \quad (7.10)$$

Here $(l_1l_2m_1m_2|lm)$ is the Clebsch-Gordan coefficient for vector addition.[12,13] Applying (7.10) and taking into account the conservation of magnetic quantum number, we have

$$\lambda\varphi_\alpha''[\sum_\beta(\varphi_\beta'')^2]=\frac{\lambda}{4\pi}\sum_{l,m}S_{l,m;\alpha}Y_{l,m}, \quad (7.11)$$

with

$$S_{l,m;\alpha}=\textstyle\sum_\beta S_{l,m;\alpha,\beta}$$

and

$$S_{l,m;\alpha,\beta}\equiv S_{l,m;\alpha,\beta}(r)=\sum_{l_1,l_2,l_3,l',m_1,m_2}\left(\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{(2l+1)}\right)^{1/2}(l_1l_200|l'0)(l'l_300|l0)$$
$$\times(l_1l_2m_1m_2|l'm_1+m_2)(l'l_3m_1+m_2m-m_1-m_2|lm)\chi_{l_1,m_1;\alpha}\chi_{l_2,m_2;\beta}\chi_{l_3,m-m_1-m_2;\beta}(1/r^3). \quad (7.12)$$

In the first term of the integral in (7.5), we have

$$\frac{\partial U}{\partial x_k}\int\varphi_\alpha''\omega^2\frac{\partial U}{\partial x_k'}d\mathbf{x}'=\frac{2\pi}{3}(Y_{1,-1}\mp Y_{1,1})\frac{dU}{dr}\int(D_{1,-1;\alpha}\mp D_{1,1;\alpha})\frac{dU}{dr'}r'^2dr',\quad\text{for}\quad k=1\text{ or }2$$
$$=\frac{4\pi}{3}Y_{1,0}\frac{dU}{dr}\int D_{1,0;\alpha}\frac{dU}{dr'}r'^2dr',\quad\text{for}\quad k=3 \quad (7.13)$$

where the upper and lower signs in the first expression refer to $k=1$ and 2, respectively. Here we have used the orthogonality of $Y_{l,m}$. The argument of D and U in the integral is r'. From a similar consideration, we have

$$\frac{\partial U}{\partial x_k}\int\varphi_\alpha''[\sum_\beta(\varphi_\beta'')^2]\frac{\partial U}{\partial x_k'}d\mathbf{x}'=\tfrac{1}{6}(Y_{1,-1}\mp Y_{1,1})\frac{dU}{dr}\int(S_{1,-1;\alpha}\mp S_{1,1;\alpha})\frac{dU}{dr'}r'^2dr',\quad\text{for}\quad k=1\text{ or }2$$
$$=\tfrac{1}{3}Y_{1,0}\frac{dU}{dr}\int S_{1,0;\alpha}\frac{dU}{dr'}r'^2dr',\quad\text{for}\quad k=3 \quad (7.14)$$

where the upper and lower signs in the first expression refer to $k=1$ and 2, respectively. Here the argument of S and U in the integral is r'.

Summarizing the above results, we have the differential equations in the partial-wave representation as below.

$$\sum_{l,m}\left(D_{l,m;\alpha}-\frac{\lambda}{4\pi}S_{l,m;\alpha}\right)Y_{l,m}-(i)^{\alpha-1}\frac{2\pi g}{\sqrt{3}\kappa}\frac{dU}{dr}[Y_{1,-1}+(-1)^\alpha Y_{1,1}]-\frac{2\pi}{3F}\frac{dU}{dr}\int\{[Y_{1,-1}+(-1)^{\alpha-1}Y_{1,1}]$$
$$\times[D_{1,-1;\alpha}+(-1)^{\alpha-1}D_{1,1;\alpha}]+2Y_{1,0}D_{1,0;\alpha}\}\frac{dU}{dr'}r'^2dr'+\frac{\lambda}{6F}\frac{dU}{dr}\int\{[Y_{1,-1}+(-1)^{\alpha-1}Y_{1,1}]$$
$$\times[S_{1,-1;\alpha}+(-1)^{\alpha-1}S_{1,1;\alpha}]+2Y_{1,0}S_{1,0;\alpha}\}\frac{dU}{dr'}r'^2dr'=0,\quad\text{for}\quad\alpha=1\text{ and }2, \quad (7.15)$$

$$\sum_{l,m}\left(D_{l,m;\alpha}-\frac{\lambda}{4\pi}S_{l,m;\alpha}\right)Y_{l,m}-\frac{2\sqrt{2}\pi g}{\sqrt{3}\kappa}\frac{dU}{dr}Y_{1,0}-\frac{4\pi}{3F}\frac{dU}{dr}\int(Y_{1,-1}D_{1,-1;\alpha}+Y_{1,1}D_{1,1;\alpha})\frac{dU}{dr'}r'^2dr'$$
$$+\frac{\lambda}{3F}\frac{dU}{dr}\int(Y_{1,-1}S_{1,-1;\alpha}+Y_{1,1}S_{1,1;\alpha})\frac{dU}{dr'}r'^2dr'=0,\quad\text{for}\quad\alpha=3.$$

[13] See, for example, M. Morita, in *Lectures in Theoretical Physics*, edited by W. E. Brittin (Interscience Publishers, Inc., New York, 1962), Vol. 4.

When $D_{l,m;\alpha}$, $S_{l,m;\alpha}$, and U are inside the integral, their arguement should be read as r'. However, the $Y_{l,m}$ has no primed angular variable.

8. EXPLICIT FORMS OF DIFFERENTIAL EQUATION FOR PARTIAL WAVES

The explicit forms of the l, m component of the general differential equation, (7.15), are studied in this section, by assuming only s and p waves

$$r\varphi_\alpha''=\chi_{0;\alpha}Y_{0,0}+\chi_{1,1;\alpha}Y_{1,1}+\chi_{1,0;\alpha}Y_{1,0}+\chi_{1,-1;\alpha}Y_{1,-1}. \quad (8.1)$$

That is, we introduce $4\times 3=12$ radial wave functions. However, as is shown in Sec. 5, the boundary condition (5.5) gives some restrictions on the p-wave functions, (5.7), which give only three p-wave functions, $\chi_{1;\alpha}$, to be independent. Therefore, the initially introduced twelve radial functions reduce to three s waves $\chi_{0;\alpha}$ and three p waves $\chi_{1;\alpha}$. Integrations in (7.15) always vanish if the boundary conditions (5.5) or equivalently (5.7) are satisfied. It is noticed here that $\chi_{0;\alpha}$ with $\alpha=1, 2, 3$, $\chi_{1;1}$, $i\chi_{1;2}$, and $\chi_{1;3}$ are all real since φ_α'' are real.

Now we give the explicit forms of the differential equations, which are coupled equations for six radial functions. Some of the l, m components of (7.15) give not the differential equations but subsidiary conditions among various χ's. The differential equations for $\alpha=1$ and 2, $l=1$, $m=\pm 1$ are divided into real and purely imaginary parts.

For $\alpha=1$, $l=m=0$:

$$(-d^2/dr^2+\kappa^2)\chi_{0;1}-\frac{\lambda}{4\pi}(\chi_{0;1}{}^2+\chi_{0;2}{}^2+\chi_{0;3}{}^2+6\chi_{1;1}{}^2-2\chi_{1;2}{}^2+\chi_{1;3}{}^2)\frac{\chi_{0;1}}{r^2}=0. \quad (8.2)$$

For $\alpha=2$, $l=m=0$:

$$(-d^2/dr^2+\kappa^2)\chi_{0;2}-\frac{\lambda}{4\pi}(\chi_{0;1}{}^2+\chi_{0;2}{}^2+\chi_{0;3}{}^2+2\chi_{1;1}{}^2-6\chi_{1;2}{}^2+\chi_{1;3}{}^2)\frac{\chi_{0;2}}{r^2}=0. \quad (8.3)$$

For $\alpha=3$, $l=m=0$:

$$(-d^2/dr^2+\kappa^2)\chi_{0;3}-\frac{\lambda}{4\pi}(\chi_{0;1}{}^2+\chi_{0;2}{}^2+\chi_{0;3}{}^2+2\chi_{1;1}{}^2-2\chi_{1;2}{}^2+3\chi_{1;3}{}^2)\frac{\chi_{0;3}}{r^2}=0. \quad (8.4)$$

For $\alpha=1$, $l=1$, $m=\pm 1$:

$$\left(-\frac{d^2}{dr^2}+\frac{2}{r^2}+\kappa^2\right)\chi_{1;1}-\frac{2\pi g}{\sqrt{3}\kappa}\frac{dU}{dr}r-\frac{\lambda}{4\pi}(3\chi_{0;1}{}^2+\chi_{0;2}{}^2+\chi_{0;3}{}^2+\frac{18}{5}\chi_{1;1}{}^2-\frac{6}{5}\chi_{1;2}{}^2+\frac{3}{5}\chi_{1;3}{}^2)\frac{\chi_{1;1}}{r^2}=0, \quad (8.5)$$

and

$$\chi_{0;1}\chi_{0;2}\chi_{1;2}=0. \quad (8.6)$$

For $\alpha=2$, $l=1$, $m=\pm 1$:

$$\left(-\frac{d^2}{dr^2}+\frac{2}{r^2}+\kappa^2\right)\chi_{1;2}-i\frac{2\pi g}{\sqrt{3}\kappa}\frac{dU}{dr}r-\frac{\lambda}{4\pi}(\chi_{0;1}{}^2+3\chi_{0;2}{}^2+\chi_{0;3}{}^2+\frac{6}{5}\chi_{1;1}{}^2-\frac{18}{5}\chi_{1;2}{}^2+\frac{3}{5}\chi_{1;3}{}^2)\frac{\chi_{1;2}}{r^2}=0, \quad (8.7)$$

and

$$\chi_{0;1}\chi_{0;2}\chi_{1;1}=0. \quad (8.8)$$

For $\alpha=1$, $l=1$, $m=0$:

$$\chi_{0;1}\chi_{0;3}\chi_{1;3}=0. \quad (8.9)$$

For $\alpha=2$, $l=1$, $m=0$:

$$\chi_{0;2}\chi_{0;3}\chi_{1;3}=0. \quad (8.10)$$

For $\alpha=3$, $l=1$, $m=1$:

$$\chi_{0;3}(-\chi_{0;1}\chi_{1;1}+\chi_{0;2}\chi_{1;2})=0. \quad (8.11)$$

For $\alpha=3$, $l=1$, $m=-1$:

$$\chi_{0;3}(\chi_{0;1}\chi_{1;1}+\chi_{0;2}\chi_{1;2})=0. \quad (8.12)$$

For $\alpha=3$, $l=1$, $m=0$:

$$\left(-\frac{d^2}{dr^2}+\frac{2}{r^2}+\kappa^2\right)\chi_{1;3}-\frac{2\sqrt{2}\pi g}{\sqrt{3}\kappa}\frac{dU}{dr}r-\frac{\lambda}{4\pi}(\chi_{0;1}{}^2+\chi_{0;2}{}^2+3\chi_{0;3}{}^2+\frac{6}{5}\chi_{1;1}{}^2-\frac{6}{5}\chi_{1;2}{}^2+\frac{9}{5}\chi_{1;3}{}^2)\frac{\chi_{1;3}}{r^2}=0. \quad (8.13)$$

Since the pseudoscalar mesons are dominantly p waves, we assume the s waves $|\chi_{0;\alpha}|$ to be small and $|\chi_{0;\alpha}/\chi_{1;\beta}|$ is of the order of ϵ, $(\epsilon\ll 1)$, where α and β are any values of 1, 2, and 3. The set of twelve equations (8.2)–(8.13) shows various symmetric properties with respect to the radial wave functions. They are given below.

A. If we overlook all subsidiary conditions, (8.6), (8.8)–(8.12), then the other six coupled differential equations are satisfied by

$$\chi_{0;1}=\chi_{0;2}=\chi_{0;3}\equiv\chi_0, \quad (8.14)$$

$$\chi_{1;1}=\chi_{1;2}/i=\chi_{1;3}/\sqrt{2}\equiv\chi_1. \quad (8.14')$$

The differential equations for χ_0 and χ_1 are

$$\left(-\frac{d^2}{dr^2}+\kappa^2\right)\chi_0-\frac{\lambda}{4\pi}(3\chi_0{}^2+10\chi_1{}^2)\frac{\chi_0}{r^2}=0, \quad (8.15)$$

$$\left(-\frac{d^2}{dr^2}+\frac{2}{r^2}+\kappa^2\right)\chi_1-\frac{2\pi g}{\sqrt{3}\kappa}\frac{dU}{dr}r-\frac{\lambda}{4\pi}(5\chi_0{}^2+6\chi_1{}^2)\frac{\chi_1}{r^2}=0. \quad (8.16)$$

Here the error involved in the solution is of the order of ϵ^2, compared with those of the differential equations using a subsidiary condition properly.

B. Taking the subsidiary conditions correctly, we have four possible sets of solutions, either

$$\chi_{0;\alpha}=0 \quad \text{for all } \alpha,$$

or

$$\chi_{0;\alpha}=\chi_{0;\beta}=0 \quad \text{and} \quad \chi_{0;\gamma}\neq 0, \alpha, \beta, \gamma \text{ cyclic.}$$

For the former case, we have

$$\chi_{1;1}=\chi_{1;2}/i=\chi_{1;3}/\sqrt{2}\equiv\chi_1, \quad \chi_{0;\alpha}=0,$$

with

$$\left(-\frac{d^2}{dr^2}+\frac{2}{r^2}+\kappa^2\right)\chi_1-\frac{2\pi g}{\sqrt{3}\kappa}\frac{dU}{dr}r-\frac{3\lambda}{2\pi}\frac{\chi_1^3}{r^2}=0. \quad (8.17)$$

For the latter case, the equations of motion are dependent on which $\chi_{0;\alpha}$ is nonzero. For example,

$$\chi_{0;1}=0, \quad \chi_{0;2}=0, \quad \text{and} \quad \chi_{0;3}\neq 0, \quad (8.18)$$

$$\left(-\frac{d^2}{dr^2}+\kappa^2\right)\chi_{0;3}-\frac{\lambda}{4\pi}(\chi_{0;3}^2+4\chi_{1;1}^2+3\chi_{1;3}^2)\frac{\chi_{0;3}}{r^2}=0, \quad (8.19)$$

$$\left(-\frac{d^2}{dr^2}+\frac{2}{r^2}+\kappa^2\right)\chi_{1;3}-\frac{2\sqrt{2}\pi g}{\sqrt{3}\kappa}\frac{dU}{dr}r-\frac{\lambda}{4\pi}\left(3\chi_{0;3}^2+\frac{12}{5}\chi_{1;1}^2+\frac{9}{5}\chi_{1;3}^2\right)\frac{\chi_{1;3}}{r^2}=0, \quad (8.20)$$

$$\left(-\frac{d^2}{dr^2}+\frac{2}{r^2}+\kappa^2\right)\chi_{1;1}-\frac{2\pi g}{\sqrt{3}\kappa}\frac{dU}{dr}r-\frac{\lambda}{4\pi}\left(\chi_{0;3}^2+\frac{24}{5}\chi_{1;1}^2+\frac{3}{5}\chi_{1;3}^2\right)\frac{\chi_{1;1}}{r^2}=0, \quad (8.21)$$

$$\chi_{1;2}=i\chi_{1;1}. \quad (8.22)$$

The solutions $\chi_{1;\alpha}$ of (8.20) and (8.21) are nearly equal to χ_1 in (8.17), that is,

$$\chi_{1;\alpha}=[\chi_1 \text{ of Eq. (8.17)}][1+O(\epsilon^2)]. \quad (8.23)$$

The relation (8.14′) also holds in this approximation. Concluding these analyses, the p-wave radial wave function is always given by (8.17) to a very good approximation.

9. EFFECT OF HIGHER PARTIAL WAVES ON THE p-WAVE EQUATION

Until now we have assumed only s and p waves for the meson wave functions. In this section, we study the effect of the higher partial waves on the p-wave differential equation. We prove that

$$\left(-\frac{d^2}{dr^2}+\frac{2}{r^2}+\kappa^2\right)\chi_1-\frac{2\pi g}{\sqrt{3}\kappa}\frac{dU}{dr}r-\frac{3\lambda}{2\pi}\frac{\chi_1^3}{r^2}=0, \quad (8.17)$$

holds in a good approximation and there is no coupling of the higher partial waves if we neglect $O(\epsilon^2)$.

The proof is simply derived from the parity consideration. Let us first examine the general expression for the equation of motion of mesons, (7.15). The coupled terms of different partial waves always come through $S_{l,m;\alpha}$, which is a sum of the terms, $\chi_{l1,m1}\chi_{l2,m2}\times\chi_{l3,m3}$.[14] Here the subscripts, l_1, m_1, etc., are conventionally printed as $l1$, $m1$, etc. The $\mathbf{l}$ is the sum of the three angular momenta, $\mathbf{l}_1$, $\mathbf{l}_2$, and $\mathbf{l}_3$,

$$\mathbf{l}=\mathbf{l}_1+\mathbf{l}_2+\mathbf{l}_3. \quad (9.1)$$

The parity condition is, therefore, given by

$$l+l_1+l_2+l_3=\text{even}, \quad (9.2)$$

and, of course, it is

$$1+l_1+l_2+l_3=\text{even for the } p \text{ wave.} \quad (9.3)$$

One notices that these conditions are properly taken into account in the product of the two Clebsch-Gordan coefficients with vanishing quantum numbers for $S_{l,m;\alpha}$, (7.12). Now we assume

$$\chi_{1,m}\sim O(1),$$
$$\chi_0 \quad \text{and} \quad \chi_{2,m}\sim O(\epsilon), \quad (9.4)$$
$$\chi_{3,m}\sim O(\epsilon^2), \quad \text{and so on,}$$

where ϵ is a certain small quantity. The case where none of l_1, l_2, l_3 is unity is out of consideration, since $\chi_{l1,m1}\times\chi_{l2,m2}\chi_{l3,m3}$ is at most $O(\epsilon^3)$. Therefore, we take, e.g., $l_1=1$. The parity condition in this case is

$$l_2+l_3=\text{even}. \quad (9.5)$$

This guarantees that l_2 and l_3 are of the same parity. Possible cubic terms of $\chi_{l,m}$ are expressed by

$$\chi_{1,m1}\chi_{l2,m2}\chi_{l2+2n,m3} \quad \text{with } n=0 \text{ or positive integral.} \quad (9.6)$$

Here, the subscript, $l2+2n$, should be read as l_2+2n. If $l_2=$ even, they are

$$\chi_{1,m1}\chi_0^2, \quad \chi_{1,m1}\chi_{2,m2}\chi_{2,m3}, \quad \chi_{1,m1}\chi_0\chi_{2,m2}, \quad \chi_{1,m1}\chi_{2,m2}\chi_{4,m3}, \text{ etc.,} \quad (9.7)$$

of which the magnitude for the largest terms is $O(\epsilon^2)$. If $l_2=$ odd, they are

$$\chi_{1,m1}\chi_{1,m2}\chi_{1,m3}, \quad \chi_{1,m1}\chi_{1,m2}\chi_{3,m3}, \quad \chi_{1,m1}\chi_{3,m2}\chi_{3,m3}, \text{ etc.,} \quad (9.8)$$

of which $\chi_{1,m1}\chi_{1,m2}\chi_{1,m3}$ is the largest and the others are at most $O(\epsilon^2)$. Therefore, the summation of the

[14] In this Section throughout, $\chi_{l,m;\alpha}$ are replaced by $\chi_{l,m}$. This does not change the conclusion.

cubic terms, $S_{l,m;\alpha}$ with $l=1$, has only one term, $\chi_{1,m1}\chi_{1,m2}\chi_{1,m3}$, if we neglect $O(\epsilon^2)$ and higher powers of ϵ. Finally, the differential equation for the p wave does not couple with any other waves and Eq. (8.17) holds generally up to $O(\epsilon)$.

10. MASS DEPENDENCE OF MESON MEAN SQUARE RADIUS

The dimension of mass is the inverse first power of length. Therefore, it is natural to assume that the mean square radius of the meson wave function, $\langle R^2\rangle$, is inversely proportional to the second power of the mass. We must be careful that in the theory of mesons there are two kinds of mass, namely, the meson mass and the nucleon mass. One may take the mean square radius to be inversely proportional to the square of the meson mass. In fact, this is true in the case of the scalar theory. For example, the dominant term in the expression of the mean square radius, $\langle R^2\rangle$, is

$$\langle R^2\rangle\approx 1/2\kappa^2,$$

for the scalar meson with the Yukawa well source function.[15] Here the π-π interaction is not considered.

In the case of the pseudoscalar theory, the situation is different. The mean square radius is defined by

$$\langle R^2\rangle=\int(\varphi_1^2+\varphi_2^2)r^4drd\Omega\bigg/\int(\varphi_1^2+\varphi_2^2)r^2drd\Omega$$
$$=\int\chi_1^2\rho^2d\rho\bigg/\left(\kappa^2\int\chi_1^2d\rho\right). \quad (10.1)$$

Here χ_1 is given in (5.17), when the source function is the Yukawa well and the π-π interaction is not present. The result of integration is

$$\langle R^2\rangle=\frac{\frac{5}{2}a(1+a)^2-8a^2[1+a/(1+a)^2]}{\kappa^2(1-a)^2} \quad (10.2)$$

with

$$a=\kappa\Lambda.$$

The dominant term is

$$\langle R^2\rangle\approx\frac{5}{2}\frac{a}{\kappa^2}=\frac{5}{2}\frac{\Lambda}{\kappa}=\frac{5}{2}\frac{1}{M\kappa}. \quad (10.3)$$

The mean square radius in the pseudoscalar theory is linear in meson mass as well as in nucleon mass. We assume that this proportionality to $(M\kappa)^{-1}$ holds approximately in the pseudoscalar theory with the π-π interaction. The difference between the mass proportionality of the mean square radius in the scalar theory and the pseudoscalar theory seems to originate from the nucleon source function in the meson wave function. We have the derivative of the source function in the pseudoscalar theory, while we have the source function itself in the scalar theory.

In the end of Sec. 6, we studied the electric form factors for the nucleons by adjusting the Λ and κ as parameters. We have found that Λ is nearly equal to the Compton wavelength of the nucleon and κ is about half of the rest mass of the meson. This can be seen explicitly. Expanding $F_1(q^2)$, (6.5), in the power series of q^2 and using $\langle R^2\rangle$, (10.2), we have

$$F_1(q^2)=1-\tfrac{1}{6}q^2(\tfrac{1}{2}\langle R^2\rangle+3\Lambda^2)+\cdots\text{for protons}, \quad (10.4)$$

for low momentum transfer. The coefficient of $\frac{1}{6}q^2$ is experimentally known and it is[4,5]

$$\tfrac{1}{2}\langle R^2\rangle+3\Lambda^2=0.64\text{–}0.72\text{F}^2. \quad (10.5)$$

With $\Lambda=0.22$F, we have

$$\langle R^2\rangle_{\text{exp}}=1.0\text{–}1.2\text{F}^2. \quad (10.6)$$

From (10.2) and (10.6), the effective meson mass is

$$\kappa=0.46\text{–}0.55\text{F}^{-1}, \quad (10.7)$$

which is about 0.6–0.8 times the rest mass. This value gives the best fit of $F_1(q^2)$ at the low momentum transfer, while the best fit in the over-all energy region is given by

$$\kappa=0.35\text{F}^{-1}, \quad (6.6)$$

which is about half of the rest mass.

The mean square radius of the nucleon source function is

$$\langle R^2\rangle_p=6\Lambda^2\quad\text{for the Yukawa well.} \quad (10.8)$$

Thus

$$\tfrac{1}{2}\langle R^2\rangle+3\Lambda^2=\tfrac{1}{2}\langle R^2\rangle+\tfrac{1}{2}\langle R^2\rangle_p. \quad (10.9)$$

This is equal to the mean square radius of the charge distribution defined by

$$\int r^2\rho(\mathbf{x})d\mathbf{x},$$

$\rho(\mathbf{x})$ being given by (6.4). Consequently, the form factor becomes

$$F_1(q^2)=1-\tfrac{1}{6}q^2\left[\int r^2\rho(\mathbf{x})d\mathbf{x}\right]+\cdots. \quad (10.10)$$

This expression is also given by a power series expansion of (3.9) directly. The factor $\frac{1}{2}$ in (10.4) and (10.9) is essential for the symmetrical pseudoscalar theory in the strong-coupling approximation.

11. NUMERICAL SOLUTIONS OF MESON WAVE FUNCTIONS WITH THE π-π INTERACTION

The meson wave functions with π-π interaction are given by (5.10), where χ_1 satisfies the differential

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[15] In the charged-scalar theory, the radial wave function (the quantity corresponding to χ_1 of the pseudoscalar theory) is analytically solvable, if we take the Yukawa well for the nucleon source function. The mean square radius is given by

$$\langle R^2\rangle=[(1+a)^3(1+a^3)-16a^3]/2(1-a^2)^2\kappa^2.$$

equation (8.17). For convenience, we introduce

$$\rho=\kappa r \quad \text{and} \quad a=\Lambda\kappa,$$
$$\chi_1=(2\pi/3\lambda)^{1/2}X_1, \tag{11.1}$$
$$U=\frac{\kappa}{4\pi\Lambda^2}\frac{e^{-\rho/a}}{\rho}.$$

Then (8.17) becomes

$$\left(-\frac{d^2}{d\rho^2}+\frac{2}{\rho^2}+1\right)X_1-\frac{X_1^3}{\rho^2}+\frac{\lambda^{1/2}}{2(2\pi)^{1/2}}\frac{g}{a^2}\left(\frac{1}{\rho}+\frac{1}{a}\right)e^{-\rho/a}=0. \tag{11.2}$$

The asymptotic form of X_1 in (11.2) is

$$X_1=\alpha e^{-\rho} \quad \text{at} \quad \rho\rightarrow\infty, \tag{11.3}$$

where α is constant. Here the nucleon source function, π-π interaction, and the centrifugal force have no effect. In the region where the former two are almost zero but the centrifugal force is still effective, the solution of (11.2) is ρ times the spherical Hankel function of the first kind,[16]

$$\text{const}\,\rho h_1^{(1)}(i\rho)=\alpha(1+1/\rho)e^{-\rho}. \tag{11.4}$$

The $h_1^{(1)}$ is defined by

$$h_1^{(1)}(i\rho)=j_1(i\rho)+in_1(i\rho). \tag{11.5}$$

The behavior of X_1 near the origin is examined by expanding X_1 in terms of ρ.

$$X_1=a_1\rho+a_2\rho^2+a_3\rho^3+\cdots. \tag{11.6}$$

The results are

$$a_1=-C/2, \tag{11.7}$$

$$a_2\neq 0, \tag{11.8}$$

$$a_3=-\frac{C}{8}\left(1+\frac{1}{a^2}-\frac{C^2}{4}\right), \tag{11.9}$$

with

$$C=\frac{\lambda^{1/2}}{2(2\pi)^{1/2}}\frac{g}{a^2}. \tag{11.10}$$

The coefficients a_1 and a_2 are determined by continuity of X_1 and its derivative $\partial X_1/\partial\rho$ at $\rho=a$. The a_1 determines the magnitude of $g\lambda^{1/2}$.

Numerical solutions of (11.2) have been obtained by the following procedure.

A. We assume the asymptotic form (11.4) in large ρ. Without loss of generality, we take a positive value for α. With this X_1, we integrate (11.2) from large ρ to small ρ. At $\rho=a$, the wave function is connected with the appropriate inner solution, (11.6)–(11.10). The strength of the π-π interaction (in a form of $g\lambda^{1/2}$) will be given by this connection. A different value of α corresponds to a different value of $g\lambda^{1/2}$.

TABLE I. Mean square radius and coupling constant as functions of α.

U	Cutoff Yukawa				Yukawa		Exp.
α	1	1.4	1.75	3	1.2	1.4	
$\langle R^2\rangle$ in F²	0.90	1.16	1.38	2.2	1.08	1.24	1.0–1.2
$g\lambda^{1/2}$	−3.3	−2.5	−2.2	−1.94	−1.01	−1.13	

B. We first solve (11.2) with the cutoff Yukawa for the nucleon source function. The tail of the Yukawa well is simply omitted in the numerical integration.[17] Using the solution, we calculate the mean square radius $\langle R^2\rangle$. We perform similar calculations with different value of α. From the curve, α vs $\langle R^2\rangle$, we can find the value of α which gives the mean square radius $\langle R^2\rangle=1.0$–1.2F^2.

C. We integrate (11.2), taking into account the tail of the Yukawa well. As a trial wave function we take α and the strength $g\lambda^{1/2}$, both given in process B to obtain $\langle R^2\rangle=1.0$–1.2F^2. In the end of integration, the strength can be calculated with the boundary conditions at $\rho=a$. The consistency for $g\lambda^{1/2}$ has to be obtained by a trial and error method. Assuming the constructive effect of the π-π interaction and the nucleon source function, the required value of $g\lambda^{1/2}$ is little smaller than that of the cutoff Yukawa.

D. We find a solution X_1 of (11.2) which gives $\langle R^2\rangle=1.0$–1.2F^2 in the above method. Using this wave function we calculate the nucleon form factor as a function of momentum transfer q.

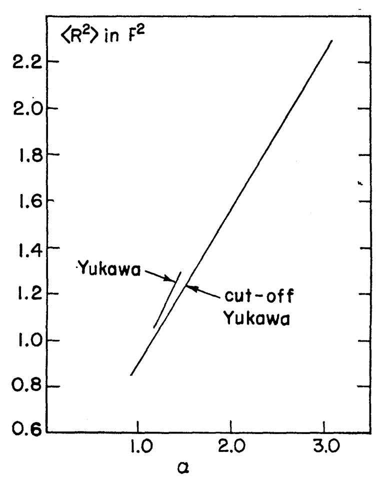

FIG. 3. Mean square radius as a function of α.

[16] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), p. 78.

[17] The normalization of U, (2.3), holds approximately. If we adopt the square well for U, the calculation is rigorous. In this case, there is no tail effect, since the derivative of the source function has nonzero value at $\rho=a$ only. The mean square radius is nearly equal to that given by the cutoff Yukawa, since the wave functions for both cases are equal in the region $\rho>a$, if α is the same. The meson wave function for the square-well case is also studied in the Appendix.

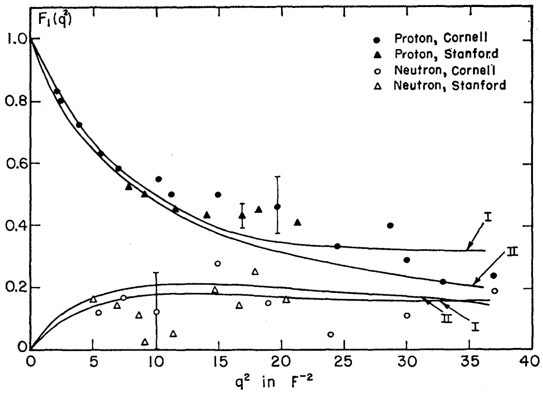

FIG. 4. Electric form factors $F_1(q^2)$ in the symmetrical pseudoscalar theory. The effect of the π-π interaction is taken into account. The numerical solutions of the differential equation (11.2) have been used. The theoretical curves are: (I) for $\alpha=1.2$ ($g\lambda^{1/2}=-1.01$), (II) for $\alpha=1.4$ ($g\lambda^{1/2}=-1.13$).

12. RESULTS AND CONCLUSION

The mean square radius, the product of coupling constants, $g\lambda^{1/2}$, and the form factors have been calculated in the manner described in Sec. 11. The adjustable parameter involved in our calculation is only α. [That is the κ and Λ are chosen as 0.71F^{-1} (rest mass of meson) and 0.22F (Compton wavelength of nucleon), respectively.] Calculated values of the mean square radius and the coupling constant $g\lambda^{1/2}$ are summarized in Table I. There the error involved in $\langle R^2\rangle$ is of the order of a few percent, while that of $g\lambda^{1/2}$ is about 30%. This large error comes from the uncertainty of the slope of the wave function near the origin. For the cutoff Yukawa well, the $\langle R^2\rangle$ is almost linear in α in the region $\alpha=1-3$, see Fig. 3. With the tail of the Yukawa well taken into account correctly, the solutions are given for $\alpha=1.2$ and 1.4. For these two solutions, the electric form factors of the nucleons are calculated and shown in Fig. 4. The solution $\alpha=1.2$ [corresponding to $g\lambda^{1/2}=-1.01$] reproduces the form factors quite well. This solution has much better fit than the wave function with $\kappa=0.35\text{F}^{-1}$, $\Lambda=0.22\text{F}$ and no π-π interaction, [compare I in Fig. 4 and III in Fig. 2]. The calculated $\langle R^2\rangle$ are 1.08F^2 for the former and 1.66F^2 for the latter, while the experimental value is 1.0–1.2F^2. This supports the existence of the π-π interaction. The wave function with $\alpha=1.2$ is shown in Fig. 5.

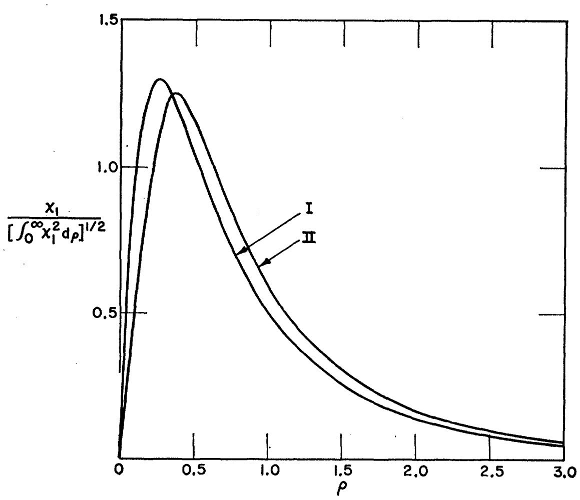

FIG. 5. Meson wave functions: (I) No π-π interaction, (5.17) with $\kappa=0.71\text{F}^{-1}$. (II) With π-π interaction, solution of (11.2) with $\alpha=1.2$. One notices that the wave function is, in fact, pushed out with the effect of the π-π interaction.

In conclusion, the electric form factors for proton and neutron can be explained by the charged-scalar as well as symmetrical pseudoscalar meson theory in the strong-coupling limit. In the latter theory, the effective meson mass is about one-half of the rest mass of the meson. The introduction of the π-π interaction is consistent with this reduction of the effective meson mass.

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APPENDIX

Radial Wave Functions in the Symmetrical Pseudoscalar Meson Theory with Square Well for Nucleon Source Function

As is easily seen, the radial wave function of p-wave meson depends on the nucleon source function. In this Appendix, we study the radial wave function with the square-well nucleon source function, for the purpose of comparison. The differential equation for the p-wave mesons is

$$\left(-\frac{d^2}{dr^2}+\frac{2}{r^2}+\kappa^2\right)\chi_1-\frac{3\lambda}{2\pi}\frac{\chi_1^3}{r^2}-\frac{2\pi g}{\sqrt{3}\kappa}\frac{dU}{dr}r=0, \quad (8.17)$$

with the square well

$$U=3/4\pi\Lambda^3, \quad r\leqslant\Lambda$$
$$=0, \quad r>\Lambda. \quad (\text{A1})$$

The derivative is

$$\frac{dU}{dr}=-\frac{3}{4\pi\Lambda^3}\delta(r-\Lambda). \quad (\text{A2})$$

1. No π-π interaction ($\lambda=0$)

In this case, (8.17) becomes

$$\left(-\frac{d^2}{d\rho^2}+\frac{2}{\rho^2}+1\right)\chi_1=-s\rho\delta(\rho-a), \quad (\text{A3})$$

with $\rho = r\kappa$ and $s = (\sqrt{3}/2)(g/a^3)$. Equation (A3) is solvable. The solution is given by

$$\chi_1 = \frac{s}{2}(1+a)e^{-a}\left[-e^{-\rho}\left(\frac{1}{\rho}+1\right)+e^{\rho}\left(\frac{1}{\rho}-1\right)\right], \quad \rho < a$$

$$= \frac{s}{2}[e^{a}(1-a)-e^{-a}(1+a)]e^{-\rho}\left(\frac{1}{\rho}+1\right), \quad \rho > a. \qquad \text{(A4)}$$

This solution satisfies two boundary conditions,

$$\chi_1 = 0 \quad \text{at} \quad \rho = 0,$$
$$\chi_1 \to 0 \quad \text{at} \quad \rho \to \infty.$$

The dominant term of the mean square radius defined by (10.1) is

$$\langle R^2 \rangle \approx \frac{25}{24}\frac{a}{\kappa^2} = \frac{25}{24}\frac{1}{M\kappa}. \qquad \text{(A5)}$$

This is again linearly proportional to $(M\kappa)^{-1}$, as in the case of the Yukawa well, (10.3).

2. *With π-π interaction* (λ≠0)

Let us introduce

$$\chi_1 = (2\pi/3\lambda)^{1/2} X_1. \qquad \text{(A6)}$$

Equation (8.17) becomes

$$\left(-\frac{d^2}{d\rho^2}+\frac{2}{\rho^2}+1\right)X_1 - \frac{X_1^3}{\rho^2} = -\frac{3}{2}\left(\frac{\lambda}{2\pi}\right)^{1/2}\frac{g}{a^3}\rho\delta(\rho-a). \qquad \text{(A7)}$$

The asymptotic form is

$$X_1 = \alpha\left(1+\frac{1}{\rho}\right)e^{-\rho} \quad \text{at} \quad \rho \to \infty. \qquad \text{(A8)}$$

Near the origin,

$$X_1 = a_1\rho + a_2\rho^2 + a_3\rho^3 + a_4\rho^4 + \cdots, \qquad \text{(A9)}$$

with

$$a_1 = a_3 = 0, \qquad \text{(A10)}$$
$$a_2 \neq 0, \qquad \text{(A11)}$$
$$a_4 = a_2/10. \qquad \text{(A12)}$$

Here a_2 is determined by the continuity of X_1 at $\rho = a$,

$$a_2 a^2(1+\tfrac{1}{10}a^2) = X_1^{\text{outer}}(a). \qquad \text{(A13)}$$

The $X_1^{\text{outer}}(a)$ is known from the numerical integration of (A7) from $\rho = \infty$ to $\rho = a$. The strength $g\lambda^{1/2}$ is given by the discontinuity of the derivative of X_1 at $\rho = a$,

$$\left[\frac{dX_1^{\text{inner}}}{d\rho} - \frac{dX_1^{\text{outer}}}{d\rho}\right]_{\rho=a} = 2a_2 a(1+\tfrac{1}{5}a^2) - \left.\frac{dX_1^{\text{outer}}}{d\rho}\right|_{\rho=a},$$
$$= -\frac{3\lambda^{1/2}}{2(2\pi)^{1/2}}\frac{g}{a^2}. \qquad \text{(A14)}$$

Here X_1^{inner} and X_1^{outer} represent the solutions for $\rho < a$ and $\rho > a$, respectively.