New Methods in Nuclear Structure Calculations*

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Methods for evaluating shell-model matrix elements in the $SU3$ classification scheme of Elliott are given. There are no restrictions on the number of particles. The relationship of the $SU3$ coupling scheme to the model of a rotating asymmetric nucleus is investigated in detail.

I. INTRODUCTION

THE empirical success of the shell model[1] combined with recent theoretical progress[2] in understanding the basis of this approach are certainly impressive. Nevertheless, the mathematical techniques for classifying states and evaluating matrix elements are not yet sufficiently well developed to apply the shell-model program to most nuclei. In fact, outside the $1p$ shell, calculations have been carried out for at most four-particle configurations. The matrices required to deal with more than four particles are much too complicated to construct, even with the aid of modern high-speed computers. This limitation of the shell-model approach may be overcome by means of the mathematical techniques for systematically evaluating many-body matrix elements developed in the present paper. Numerical calculations and generalizations to various deformed fields and vibrating systems will be discussed in a later publication.

Before proceeding to the formal development, a semiquantitative discussion of the main physical ideas behind the new approach is in order. Our point of departure is the intermediate coupling shell model,[1] which considers the interactions among nucleons moving in the field of a spherically symmetric core. The particles outside the core usually are confined to the lowest unfilled major shell. For example, in O^{18} and F^{19}, the nucleons outside the O^{16} core are restricted to the $2s$ and $1d$ orbitals. The radial dependence of the two-particle interaction generally is expressed in the form of a Gaussian or a Yukawa, well characterized by a range and depth consistent with the effective range and scattering length of nuclear forces. Various combinations of exchange mixtures have been used in such calculations.

Recently, progress toward understanding the structure of nuclear wave functions has emerged from a comparison between the intermediate coupling shell model and the rotator model of Bohr and Mottelson.[3] In the rotator model, the nucleus assumes an axially symmetric deformation in response to a self-consistent Hartree field generated by the nucleons in nonspherical orbitals. The deformed nucleus then rotates like a quantum-mechanical top. Nilsson and Mottelson[4] proposed a specific form for the potential, consisting of an axially symmetric deformed harmonic oscillator with spin-orbit interaction. It has recently been pointed out[5] that the rotator model, as well as the shell model, may be applied to F^{19}. In fact, the phenomenological parameters of a rotator[6] can be successfully fitted to the observed spectra throughout the sd oscillator shell. It is, therefore, natural to ask if the wave functions given by the rotator model are related in some respect to the shell-model wave functions. Redlich[7] noted that such a relationship indeed exists in the case of O^{18} and F^{19}, while Kurath and Picman[8] verified that the shell-model wave functions for the $1p$ orbital also have a Nilsson-Mottelson structure to good approximation. The next step was taken by Elliott,[9] who constructed a shell-model representation for particles in mixed orbital configurations which gives rise naturally to a rotational band structure. As an intuitive aid to the mathematical treatment to follow, we give here a simplified version of the Elliott treatment.

We ask first, what are the properties of a shell-model Hamiltonian which will yield wave functions corresponding to a rotating system of independent particles in a deformed well? As in the Nilsson-Mottelson treatment, we assume a deformed harmonic oscillator potential, with spring constants $k_x=k_y$ but $k_x\neq k_z$. The ith particle in this "intrinsic" deformed well is subject to a Hamiltonian:

$$H_i=\frac{-\hbar^2\nabla_i^2}{2m}+\tfrac{1}{2}kr_i^2+\eta q_0^i.$$

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The first two terms in H_i constitute the Hamiltonian for a spherical harmonic oscillator with spring constant k, and eigenvalues $(n_i+\frac{3}{2})\hbar\omega$, where $\omega=(k/m)^{1/2}$ and $n_1=1, 2, 3, \cdots$ labels the various major shells. If we take

$$q_\mu{}^i=2(4\pi/5)^{1/2}r_i{}^2Y_\mu{}^2(\Omega_1),$$

where $Y_\mu{}^2$ is the spherical harmonic of order 2, then

$$q_0{}^i=2z^2-x^2-y^2,$$

and the potential energy becomes

$$V(r_i)=\tfrac{1}{2}kr_i{}^2+\eta q_0{}^i=(\tfrac{1}{2}k-\eta)(x^2+y^2)+(\tfrac{1}{2}k+2\eta)z^2,$$

which is the form required. Since we wish to confine the particles to a single major shell, we impose the condition that all matrix elements of $q_0{}^i$ vanish between states in different oscillator levels. The Hamiltonian H_i then commutes with $q_0{}^i$ as well as with the z component of the orbital angular momentum $L_z{}^i$, but not with the square of the total angular momentum. Hence, the eigenstates of H_i can be made simultaneously eigenstates of $q_0{}^i$ and $L_z{}^i$ but not of $\mathbf{L}^2$. Denoting these eigenstates as $\phi_{\epsilon_iK_i}$, we have

$$H_i\phi_{\epsilon_iK_i}=[(n_i+\tfrac{3}{2})\hbar\omega+\eta\epsilon_i]\phi_{\epsilon_iK_i},$$
$$q_0{}^i\phi_{\epsilon_iK_i}=\epsilon_i\phi_{\epsilon_iK_i},$$
$$L_z{}^i\phi_{\epsilon_iK_i}=K_i\phi_{\epsilon_iK_i}.$$

The Hamiltonian for N particles moving independently in the deformed well is

$$H=\sum_{i=1}^{N}\left[\frac{-\hbar^2\nabla_i^2}{2m}+\tfrac{1}{2}kr_i^2\right]+\eta Q_0.$$

The quantity in braces is again the spherical harmonic oscillator Hamiltonian, which assumes a constant eigenvalue equal to $N(n+\frac{3}{2})\hbar\omega$ for N particles in the nth oscillator shell owing to the degeneracy of the orbitals within the shell. The last term in H is the many-body quadrupole operator

$$Q_\mu=\textstyle\sum_i q_\mu{}^i.$$

The eigenfunctions of H are products of eigenfunctions of H_i:

$$H\prod_i\phi_{\epsilon_iK_i}=[N(n+\tfrac{3}{2})\hbar\omega+\eta\sum_i\epsilon_i]\prod_i\phi_{\epsilon_iK_i},$$

$$(\sum_i L_z{}^i)\prod_i\phi_{\epsilon_iK_i}=(\sum_i K_i)\prod_i\phi_{\epsilon_iK_i}.$$

In practice, these solutions must be antisymmetrized. We designate such antisymmetric solutions as $\Phi_{\epsilon K}$, where

$$K=\textstyle\sum_i K_i,\quad L_z\Phi_{\epsilon K}=K\Phi_{\epsilon K},$$
$$\epsilon=\textstyle\sum_i\epsilon_i,\quad H\Phi_{\epsilon K}=[N(n+\frac{3}{2})\hbar\omega\cdots]\Phi_{\epsilon K},$$
$$\eta Q_0\Phi_{\epsilon K}=\left\{H-\sum_i\left[\frac{-\hbar^2\nabla_i^2}{2m}+\tfrac{1}{2}kr_i^2\right]\right\}\Phi_{\epsilon K}=\eta\epsilon\Phi_{\epsilon K}.$$

The last equation follows from the fact that $\Phi_{\epsilon K}$ is an eigenfunction of the spherical harmonic oscillator and is, therefore, also an eigenfunction of Q_0 alone.

The functions $\Phi_{\epsilon K}$ represent the "intrinsic" states of a deformed nucleus. The true nuclear wave function corresponds to a spinning intrinsic system, which may be represented by a superposition of the $\Phi_{\epsilon K}$ in various orientations. We write the nuclear wave function for the state with angular momentum L and z component M as

$$\Psi_{\epsilon K}{}^{LM}=\int a_{MK}{}^L(\alpha,\beta,\gamma)\mathcal{R}(\alpha,\beta,\gamma)\Phi_{\epsilon K}d\gamma\,\sin\beta d\beta d\alpha.\quad\text{(I.1)}$$

Here the quantum number M refers to the projection of the orbital angular momentum in the laboratory coordinate system, whereas K denotes the projection of the angular momentum along the symmetry axis of the deformed nucleus. The rotation operator[10]:

$$\mathcal{R}(\alpha,\beta,\gamma)=e^{-i\alpha L_z}e^{-i\beta L_y}e^{-i\gamma L_z}$$

rotates the intrinsic function $\Phi_{\epsilon K}$ through the Euler angles α, β, γ. The amplitude of the rotated functions in the orientation defined by α, β, γ is given by the coefficients $a_{MK}{}^L$, which we now determine from the condition that $\Psi_{\epsilon K}{}^{LM}$ must be a simultaneous eigenfunction of $\mathbf{L}^2$ and L_z with eigenvalues L and M.

The intrinsic state $\Phi_{\epsilon K}$ is an eigenstate of L_z but not of $\mathbf{L}^2$. We designate a set of simultaneous eigenstates of $\mathbf{L}^2$ and L_z as $\psi_K{}^L$, and write:

$$\Phi_{\epsilon K}=\textstyle\sum_L\psi_K{}^L.\quad\text{(I.2)}$$

This equation simply expands the $\Phi_{\epsilon K}$ in a set of functions (not normalized) which transform like spherical harmonics under rotation. $\psi_K{}^L$ is the function obtained by projecting out of $\Phi_{\epsilon K}$ the component with orbital angular momentum L. As is well known[10]:

$$\mathcal{R}(\alpha,\beta,\gamma)\psi_K{}^L=\textstyle\sum_{M'}D_{M'K}{}^L(\alpha,\beta,\gamma)\psi_{M'}{}^L,\quad\text{(I.3)}$$

where $D_{M'K}{}^L$ is the rotation matrix:

$$D_{M'K}{}^L(\alpha,\beta,\gamma)=\langle LM'|e^{-i\alpha L_z}e^{-i\beta L_y}e^{-i\gamma L_z}|LK\rangle.$$

Substituting Eqs. (I.2) and (I.3) into Eq. (I.1), we obtain

$$\Psi_{\epsilon K}{}^{LM}=\int a_{MK}{}^L(\alpha,\beta,\gamma)\sum_{L'M'}D_{M'K}{}^{L'}(\alpha,\beta,\gamma)\psi_{M'}{}^{L'}d\gamma\,\sin\beta d\beta d\alpha$$
$$=\sum_{L'M'}\left[\int a_{MK}{}^L(\alpha,\beta,\gamma)D_{M'K}{}^{L'}(\alpha,\beta,\gamma)d\gamma\,\sin\beta d\beta d\alpha\right]\psi_{M'}{}^{L'}.\quad\text{(I.4)}$$

[10] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957).

Since L and M are good quantum numbers for $\Psi_{\epsilon K}{}^{LM}$, the integral in braces vanishes if $L'\neq L$ and $M'\neq M$. Hence,

$$\Psi_{\epsilon K}{}^{LM}=A_{MK}{}^{L}\psi_{M}{}^{L},$$

where $A_{MK}{}^{L}$ is the integral in braces evaluated for $L'=L$ and $M'=M$. We now compare this integral,

$$\int a_{MK}{}^{L}(\alpha,\beta,\gamma)D_{M'K}{}^{L'}(\alpha,\beta,\gamma)d\gamma\,\sin\beta d\beta d\alpha =A_{MK}{}^{L}\delta_{LL'}\delta_{MM'},$$

with the orthonormality relation for the rotation matrices,

$$\int D_{MK}{}^{L*}(\alpha,\beta,\gamma)D_{M'K}{}^{L'}(\alpha,\beta,\gamma)d\gamma\,\sin\beta d\beta d\alpha =\frac{8\pi^2}{2L+1}\delta_{LL'}\delta_{MM'},$$

and conclude that $a_{MK}{}^{L}$ is proportional to $D_{MK}{}^{L*}$. The proportionality constant is, of course, fixed by the normalization of $\psi_{M}{}^{L}$. Finally, we define the projection operator:

$$P_{M}{}^{L}=\sum_{K}\int\frac{D_{MK}{}^{L*}(\alpha,\beta,\gamma)}{8\pi^2/(2L+1)}\mathcal{R}(\alpha,\beta,\gamma)d\gamma\,\sin\beta d\beta d\alpha.$$

Restating the preceding relations in terms of $P_{M}{}^{L}$, we have

$$\psi_{M}{}^{L}=\Psi_{\epsilon K}{}^{LM}/A_{MK}{}^{L}=P_{M}{}^{L}\Phi_{\epsilon K}. \qquad \text{(I.5)}$$

The operator $P_{M}{}^{L}$ operating on an eigenstate of L_z with eigenvalue K first projects out the component which transforms as the spherical harmonic of order L and then changes K to M.

Equation (I.5) gives the relation between the intrinsic states $\Phi_{\epsilon K}$ of a system of particles in a deformed harmonic oscillator and the eigenstates $\Psi_{\epsilon K}{}^{LM}$ of the rotating system. Since the $\psi_{\epsilon K}{}^{LM}$ describe a rotator, they must be eigenfunctions of a Hamiltonian with eigenvalues proportional to $L(L+1)$. Designating this Hamiltonian as $\mathcal{H}$, we write

$$\mathcal{H}\Psi_{\epsilon K}{}^{LM}=[a(\epsilon,K)+bL(L+1)]\Psi_{\epsilon K}{}^{LM}. \qquad \text{(I.6)}$$

Since

$$\mathbf{L}^2\Psi_{\epsilon K}{}^{LK}=L(L+1)\Psi_{\epsilon K}{}^{LK},$$

we can rewrite Eq. (I.6) as

$$(\mathcal{H}-b\mathbf{L}^2)\Psi_{\epsilon K}{}^{LK}=a(\epsilon,K)\Psi_{\epsilon K}{}^{LK}. \qquad \text{(I.7)}$$

Dividing both sides of Eq. (I.7) by $A_{KK}{}^{L}$ and summing over L, we see from Eq. (I.2) that

$$(\mathcal{H}-b\mathbf{L}^2)\Phi_{\epsilon K}=a(\epsilon,K)\Phi_{\epsilon K}.$$

Since the operators $\mathcal{H}-b\mathbf{L}^2$ and Q_0 are both diagonal in the $\Phi_{\epsilon K}$ representation, they must commute

$$[(\mathcal{H}-b\mathbf{L}^2),Q_0]=0. \qquad \text{(I.8)}$$

We now obtain additional commutation relations for $\mathcal{H}-b\mathbf{L}^2$ by applying the rotation operator to Eq. (I.8). Rotation of Q_0 yields the linear combination:

$$\mathcal{R}Q_0\mathcal{R}^{-1}=\sum_{\mu}D_{\mu 0}{}^{2}(\alpha,\beta,\gamma)Q_{\mu}.$$

Since the scalar operator $\mathcal{H}-b\mathbf{L}^2$ is invariant with respect to rotations:

$$\mathcal{R}[(\mathcal{H}-b\mathbf{L}^2),Q_0]\mathcal{R}^{-1} =\sum_{\mu}[(\mathcal{H}-b\mathbf{L}^2),Q_{\mu}]D_{\mu 0}{}^{2}(\alpha,\beta,\gamma)=0. \qquad \text{(I.9)}$$

Multiplying Eq. (I.9) by $D_{\mu'0}{}^{2*}(\alpha,\beta,\gamma)$, integrating over the three Euler angles, and recalling the orthogonality relation of the rotation matrices, we arrive at the relations:

$$[(\mathcal{H}-b\mathbf{L}^2),Q_{\mu}]=0,\quad \mu=0,\pm1,\pm2. \qquad \text{(I.10a)}$$

We have also the analogous commutation relations for the components of angular momentum:

$$[(\mathcal{H}-b\mathbf{L}^2),L_{\mu}]=0,\quad \mu=0,\pm1 \qquad \text{(I.10b)}$$

which hold for all scalar Hamiltonians.

The eight commutation relations summarized in Eqs. (I.10) have been deduced from the conditions imposed on the Hamiltonian, namely, that it must represent a deformed harmonic oscillator with eigenstates corresponding to a rotator. As shown in the main body of the paper, these commutation relations make it possible to find new quantum numbers which characterize both the intrinsic states $\Phi_{\epsilon K}$ of the deformed oscillator and the rotational states $\Psi_{\epsilon K}{}^{LM}$. We introduce here the Casimir operator

$$C\equiv\frac{\mathbf{L}^2}{12}+\sum_{\mu}(-)^{\mu}\frac{Q_{\mu}Q_{-\mu}}{36},$$

which obeys the commutation rules:

$$[C,Q_{\mu}]=0,$$
$$[C,L_{\mu}]=0.$$

Since C is constructed from the eight components of L_{μ} and Q_{μ}, it follows from Eqs. (I.10) that

$$[(\mathcal{H}-b\mathbf{L}^2),C]=0,$$

which leads immediately to the desired relation

$$[\mathcal{H},C]=0,$$

since the scalar operator C commutes with $\mathbf{L}^2$.

We now have three simultaneously diagonal operators in the "intrinsic" Φ representation—Q_0 with eigenvalue ϵ, L_z with eigenvalue K, and the Casimir operator with eigenvalue C. The projected states

$$P_{M}{}^{L}\Phi_{\epsilon K}{}^{C}=(1/A_{MK}{}^{L})\Psi_{C\epsilon K}{}^{LM}$$

are also eigenstates of C since the projection operation depends only on the operators L_{μ}, which commute with C. We have, thus, arrived at a new quantum number C

which is a constant of the motion for the intrinsic states of a deformed harmonic oscillator as well as for the projected states.

Since a realistic nuclear Hamiltonian does not yield an exact rotator, it can be expected to conform only approximately to the conditions prescribed above. In that event, C is only approximately a good quantum number. Numerical calculations for the *sd* oscillator shell nevertheless confirm that a representation labeled by C has physical meaning. We wish also to call attention to the flexibility of such a representation. Consider any scalar operator S, with an inverse S^{-1}. Then the class of transformed Hamiltonians $S\mathcal{H}S^{-1}$ with eigenstates $P_K{}^LS\Phi_{\epsilon K}{}^C$ still yields a rotational spectrum, and the transformed operator SCS^{-1} still has eigenvalue C. If S is a product of identical one-body operators, $S=\prod_i S_i$, the nonspherical term in the transformed independent-particle Hamiltonian for the intrinsic system becomes $SQ_0S^{-1}=\sum_i S_iQ_0{}^iS_i{}^{-1}$. Thus, we see that the representation outlined in this section is not restricted to a special form for the intrinsic Hartree field. A program is now in progress to discover and apply the scalar transformations S appropriate to various deformed fields and vibrating systems of physical interest.

II. THE ELLIOTT CLASSIFICATION OF STATES

In a series of two papers, Elliott[9] has discussed the classification of the many-particle wave functions of a degenerate harmonic oscillator level according to irreducible representations of the group $SU3$. First, the states are labeled by the partition $[f]$, which describes the symmetry under space permutations. Associated with $[f]$ are the isotopic spin quantum number T and the spin quantum number S. Next, the orbital states belonging to a given partition are classified according to the irreducible representations of $SU3$, labeled by the two numbers (λ,μ). Finally, three additional quantum numbers (K,ϵ,Λ) are introduced which uniquely characterize the states within an irreducible representation of $SU3$. This classification scheme applies to the so-called "intrinsic" states discussed in Sec. I. Although the intrinsic states comprise a complete orthonormal set, they have no convenient transformation properties with respect to rotations. To remedy this situation, Elliott projects from the intrinsic states a new set of states for which the total angular momentum is a constant of the motion. The wave functions associated with the final representation, thus, have L as a good quantum number and at the same time refer directly to the intrinsic system. In Sec. IIA, we show that the group $SU3$ is generated by a set of operators which commute with the harmonic oscillator Hamiltonian. It follows that the intrinsic states of a degenerate harmonic oscillator level can be classified according to the irreducible representations of $SU3$. Section IIB concerns the characterization of the intrinsic states belonging to a given irreducible representation, while Sec. IIC deals with the projected wave functions.

A. Generators of the Group $SU3$

Consider a single-particle harmonic oscillator Hamiltonian

$$H=p^2/2m+\tfrac{1}{2}m\omega^2r^2,$$

where the frequency parameter ω is expressed in terms of a length parameter b:

$$\omega=\hbar/mb^2.$$

Introduce operators that annihilate energy quanta:

$$u_{\pm1}=\mp\frac{1}{2b}\left[(x\mp iy)+\frac{i}{\hbar}b^2(p_x\mp ip_y)\right],$$

$$u_0=\frac{1}{\sqrt{2}b}\left(z+\frac{i}{\hbar}b^2p_z\right),$$

and operators that create energy quanta:

$$u_{\pm1}{}^\dagger=u_{\pm1}{}^*=\mp\frac{1}{2b}\left[(x\pm iy)-\frac{i}{\hbar}b^2(p_x\pm ip_y)\right],$$

$$u_0{}^\dagger=u_0{}^*=\frac{1}{\sqrt{2}b}\left(z-\frac{i}{\hbar}b^2p_z\right),$$

where

$$p_x=\frac{\hbar}{i}\frac{\partial}{\partial x},\quad p_y=\frac{\hbar}{i}\frac{\partial}{\partial y},\quad p_z=\frac{\hbar}{i}\frac{\partial}{\partial z}.$$

These operators have the commutation rules:

$$[u_\mu,u_\nu{}^\dagger]=\delta_{\mu\nu},$$
$$[u_\mu,u_\nu]=[u_\mu{}^\dagger,u_\nu{}^\dagger]=0.$$

The transformation properties of the $u_\mu{}^\dagger$ with respect to rotations are like those of the first-order spherical harmonics $Y_\mu{}^1$. In terms of the u_μ operators,

$$H=\tfrac{3}{2}\hbar\omega+\hbar\omega\textstyle\sum_\mu u_\mu{}^\dagger u_\mu,$$

where the eigenstates of H are of the form:

$$\psi_{nlm}(r,\theta,\varphi)=N_{nl}\frac{R_{nl}(r)}{(r/b)}Y_m{}^l(\theta,\varphi).$$

We give here the functions $R_{nl}(r)$ and the constants N_{nl} for $n=1$, 2, 3, in terms of the dimensionless parameter $S=(r/b)$. The functions R_{nl} are normalized to $\int_0^\infty R_{nl}{}^2dS=1$.

n	R_{nl}	$N_{nl}{}^2$
1	$e^{-S^2/2}S^{l+1}$	$\dfrac{2^{l+2}}{\pi^{1/2}(2l+1)!!}$
2	$e^{-S^2/2}S^{l+1}\left(1-\dfrac{2S^2}{2l+3}\right)$	$\dfrac{2^{l+1}(2l+3)}{\pi^{1/2}(2l+1)!!}$
3	$e^{-S^2/2}S^{l+1}\left[1-\dfrac{4S^2}{2l+3}+\dfrac{4S^4}{(2l+3)(2l+5)}\right]$	$\dfrac{2^l(2l+3)(2l+5)}{\pi^{1/2}(2l+1)!!}$

where $(2^l+1)!!\equiv 1\times 3\times 5\times \cdots (2^l+1)$.

Note that

$$u_\mu \psi_{1s}=0.$$

The 1*s* state, thus, plays the role of "vacuum" with respect to the operators u_μ. We shall henceforth use the symbol $|0\rangle$ to denote the 1*s* state. It is easily verified that

$$u_\mu^\dagger|0\rangle=\psi_{1p},$$

and that

$$\sum_\mu \begin{bmatrix} 1 & 1 & 0 \\ \mu & -\mu & 0 \end{bmatrix} u_\mu^\dagger u_{-\mu}^\dagger|0\rangle \propto \psi_{2s},$$

where $\begin{bmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{bmatrix}$ is the Clebsch-Gordan coefficient for coupling angular momenta j_1 and j_2 with z components m_1 and m_2 to a resultant j_3 and m_3. This follows from the fact that ψ_{2s} is an eigenstate of H with $E=\frac{7}{2}\hbar\omega$ and $l=m_l=0$. Similarly,

$$\sum_\mu \begin{bmatrix} 1 & 1 & 2 \\ \mu & m-\mu & m \end{bmatrix} u_\mu^\dagger u_{m-\mu}^\dagger|0\rangle \propto \psi_{1d}.$$

We now form the 9 product operators $u_\mu^\dagger u_\nu$. All of these operators commute with H; hence their matrix elements vanish between states in different oscillator levels. The commutators of the product operators are

$$[u_\mu^\dagger u_\nu, u_\chi^\dagger u_\tau]=u_\mu^\dagger u_\tau \delta_{\nu\chi}-u_\chi^\dagger u_\nu \delta_{\mu\tau}, \qquad \text{(IIA.1)}$$

where δ_{mn} is the Kronecker delta symbol.

Equation (IIA.1) is a special case of the standard form

$$[X_\rho, X_\sigma]=\sum_\tau C_{\rho\sigma}{}^\tau X_\tau. \qquad \text{(IIA.2)}$$

A set of operators X_μ subject to the commutation rules (IIA.2) constitute the generators of a Lie group. We call attention here to two properties of generators which, together with their simple commutation rules, will prove extremely useful in the following development. First, consider a wave function belonging to a given irreducible representation of a group. A generator of the group operating on the wave function can change it only into a linear combination of wave functions belonging to the same irreducible representation. Secondly, by repeated application of a suitable linear combination of the generators, one can change any member of the irreducible representation into any other member. These properties may be illustrated in the case of $R3$, the group of rotations in three-dimensional space generated by the operators L_x, L_y, L_z. The irreducible representation in this case is labeled by the total angular momentum number L. Operation on any state $\psi_M{}^L$ by one of the components of **L** does not change the total angular momentum, and any state $\psi_M{}^L$ can be transformed into any other state $\psi_{M'}{}^L$ with $-L\leqslant M\leqslant L$ by applying the operators $L_x\pm iL_y$ which are linear combinations of the generators. For a formal discussion of these properties, we refer the reader to Racah's notes[11] on group theory and spectroscopy, in which the theory of continuous groups is developed from the point of view of generators. These notes provide the mathematical foundation of much of the present work and will prove helpful in elucidating the discussion.

We now show that the product operators $u_\mu^\dagger u_\nu$ generate the group $U3$. Moreover, since the form of the commutation rules (IIA.1) is invariant under any linear transformation of the operators, any nine linear combinations of the $u_\mu^\dagger u_\nu$ also generate the group $U3$. We select as generators the following linear combinations:

$$\begin{aligned} H&=\tfrac{3}{2}\hbar\omega+\hbar\omega \textstyle\sum_\mu u_\mu^\dagger u_\mu, \\ L_{\pm 1}&=\mp(u_0^\dagger u_{\mp 1}+u_{\pm 1}^\dagger u_0), \\ L_0&=u_1^\dagger u_1-u_{-1}^\dagger u_{-1}, \\ Q_{\pm 2}&=-6^{1/2}u_{\pm 1}^\dagger u_{\mp 1}, \\ Q_{\pm 1}&=-3^{1/2}(u_0^\dagger u_{\mp 1}-u_{\pm 1}^\dagger u_0), \\ Q_0&=2u_0^\dagger u_0-u_1^\dagger u_1-u_{-1}^\dagger u_{-1}. \end{aligned} \qquad \text{(IIA.3)}$$

The first of these nine generators is again the harmonic oscillator Hamiltonian, which commutes with L_μ and Q_μ since it commutes with all the $u_\mu^\dagger u_\nu$. The next three generators are the components of orbital angular momentum, which also generate the group $R3$. Our group, thus, contains $R3$ as a subgroup, as it must if the projected states $P_K{}^L\Phi_{\epsilon K}$ are to belong to the same irreducible representation as the intrinsic states $\Phi_{\epsilon K}$. The five operators Q_μ, introduced by Elliott,[9] are related to quadrupole distortions. In terms of the operators **r** and **p**, in units of $1/b$, we have

$$Q_\mu=(4\pi/5)^{1/2}[r^2Y_\mu{}^2(\Omega_r)+\hbar^2p^2Y_\mu{}^2(\Omega p)],$$

where $Y_\mu{}^2$ is the second-order spherical harmonic. Within a major shell, Q_μ is equivalent to the operator,

$$2(4\pi/5)^{1/2}r^2Y_\mu{}^2(\Omega r),$$

defined in Sec. I. The commutation rules of the L_μ and Q_μ are:

$$\begin{aligned} [L_\eta, L_{\eta'}]&=-2^{1/2}\begin{bmatrix} 1 & 1 & 1 \\ \eta & \eta' & \eta+\eta' \end{bmatrix} L_{\eta+\eta'}, \\ [L_\eta, Q_{\eta'}]&=-6^{1/2}\begin{bmatrix} 1 & 2 & 2 \\ \eta & \eta' & \eta+\eta' \end{bmatrix} Q_{\eta+\eta'}, \\ [Q_\eta, Q_{\eta'}]&=3\times 10^{1/2}\begin{bmatrix} 2 & 2 & 1 \\ \eta & \eta' & \eta+\eta' \end{bmatrix} L_{\eta+\eta'}. \end{aligned} \qquad \text{(IIA.4)}$$

Again $\begin{bmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{bmatrix}$ denotes the Clebsch-Gordan coefficient. We note that the commutation rules (IIA.4) are of the form (IIA.2). The eight operators L_μ, Q_μ,

[11] G. Racah, "Group Theory and Spectroscopy," Spring 1951 Lectures at the Institute for Advanced Study, Princeton, New Jersey (unpublished).

therefore, generate a subgroup of the group generated by the nine operators $u_\mu{}^\dagger u_\nu$.

Of the generators (IIA.3), only H, L_0, and Q_0 are Hermitian. We construct Hermitian combinations of $L_{\pm 1}$, $Q_{\pm 1}$, $Q_{\pm 2}$, denoted by $G_1, G_2, \cdots G_6$. Then

$$e^{i\alpha H}e^{i\beta L_0}e^{i\gamma Q_0}e^{i\delta G_1}\cdots e^{i\mu G_6}\equiv S(\alpha,\beta,\gamma,\cdots,\mu),$$

where the infinite set of unitary operators $S(\alpha,\beta,\cdots,\mu)$ for all values of $\alpha,\beta\cdots,\mu$ form a continuous group. Similarly, we define:

$$e^{-i\alpha H}S(\alpha,\beta,\gamma,\cdots,\mu)\equiv T(\beta,\gamma,\cdots,\mu),$$

where $T(\beta,\gamma,\cdots,\mu)$ is a subgroup of $S(\alpha,\beta,\cdots,\mu)$. The development here is analogous to the definition of the rotation operators $e^{i\alpha L_z}e^{i\beta L_y}e^{i\gamma L_z}$, which form a continuous group generated by the three components of orbital angular momentum. Carrying this analogy further, we define matrices $S(\alpha,\beta,\cdots,\mu)$ and $T(\beta,\gamma,\cdots,\mu)$ which represent the operators S and T, just as the matrix $D(\alpha,\beta,\gamma)$ represents the rotation operators. The S matrices are unitary and depend on nine parameters, α, β, $\cdots$, μ. Since an arbitrary unitary matrix of three dimensions likewise depends on nine arbitrary parameters, it is evident that the group generated by the operators (IIA.3) is isomorphic to $U3$. We note further that the matrices of the eight operators L_μ, Q_μ all have zero trace. Hence, the T matrices must have determinant unity since:

$$\begin{aligned}\det T &= \det e^{i\beta L_0}\det e^{i\gamma Q_0}\det e^{i\delta G_1}\cdots\det e^{i\mu G_6}\\ &= e^{i\beta\,\mathrm{Tr}L_0}e^{i\gamma\,\mathrm{Tr}Q_0}e^{i\delta\,\mathrm{Tr}G_1}\cdots e^{i\mu\,\mathrm{Tr}G_6}=1.\end{aligned}$$

The subgroup of $U3$ generated by the L_μ, Q_μ is, therefore, $SU3$, the group of three-dimensional matrices with determinant unity. In physical problems, we are not concerned with the transformations $e^{i\alpha H}$ which merely induce an over-all change of phase. Consequently, we restrict our attention hereafter to the subgroup $SU3$ and the corresponding matrices $T(\beta,\alpha,\cdots,\mu)$.

The generalization of the preceding considerations to many-particle systems is simple. We define the operators

$$\sum_{i=1}^{N} u_\mu{}^{i\dagger}u_\nu{}^i$$

for a system of N particles, where the generators $u_\mu{}^{i\dagger}u_\nu{}^i$ refer to the ith particle. Then,

$$[u_\mu{}^i,u_\nu{}^{j\dagger}]=\delta_{\mu\nu}\delta_{ij}.$$

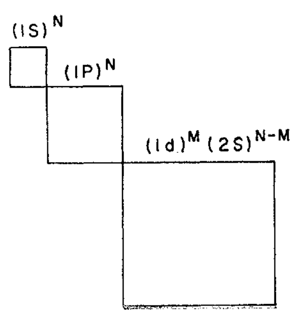

FIG. 1. Reduction of the matrix using harmonic oscillator quantum numbers.

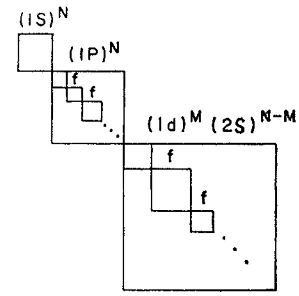

FIG. 2. Reduction of the matrix using permutation symmetry quantum numbers $[f]$.

The N-particle generators corresponding to Eqs. (IIA.3) are similarly defined as a sum of one-particle operators, in which case the commutation rules (IIA.4) hold as well for the N-particle system. Finally, the N-particle operators $T(\beta,\gamma,\cdots,\mu)$ are constructed from the N-particle generators.

In a harmonic oscillator representation, the matrices of T assume the form given in Fig. 1, that is, the nonvanishing elements of T reduce to square matrices referring to each major shell of energy $(n+\frac{3}{2})\hbar\omega$. We now consider the N-particle submatrices of $T(\beta,\gamma,\cdots,\mu)$ for all values of β, γ, $\cdots$, μ referring to a given oscillator level. Since the operator $T(\beta,\gamma,\cdots,\mu)$ is symmetric, all matrix elements of T between states of different permutation symmetry vanish. The submatrices of T are, therefore, diagonal in the quantum number $[f]$, which describes the permutation symmetry of the N-particle wave functions, and each submatrix can be factored accordingly as shown in Fig. 2. The submatrices labeled by n and $[f]$ correspond to the irreducible representations of $SU3$ only in the case of the $1p$ shell. For $n>1$, these submatrices can be factored into still smaller boxes of nonvanishing elements along the diagonal by applying suitable unitary transformations. When no further factorization is possible, the representation spanning each box is by definition irreducible and can be characterized by two numbers (λ,μ) determined by group-theoretical methods.[9] It is clear that the matrices of the generators can be factored in the same manner as the T matrices. Consequently, a generator acting on a state belonging to one irreducible representation cannot transform it into a state of another irreducible representation.

B. The Intrinsic (K,ϵ,Λ) Representation of Elliott

Our N-particle system is now classified according to the oscillator level n, the permutation symmetry $[f]$, and the irreducible representation (λ,μ) of $SU3$ to which it belongs. The problem remains to distinguish among states in the same oscillator level with the same permutation symmetry belonging to the same irreducible representation of $SU3$. We note first that the operators Q_0 and L_0 commute with each other and do not join states of different n, $[f]$, or (λ,μ). Hence, these operators can be simultaneously diagonalized and their eigenvalues ϵ, K, used to characterize the states. With

TABLE I. Commutators $[A,B]$ of the operators H_α, F_β.

$A \backslash B$	H_1	H_2	F_1	F_{-1}	F_5	F_{-5}	F_4	F_{-4}
H_1	0	0	$-F_1$	F_{-1}	F_5	$-F_{-5}$	$2F_4$	$-2F_{-4}$
H_2	0	0	$3F_1$	$-3F_{-1}$	$3F_5$	$-3F_{-5}$	0	0
F_1	F_1	$-3F_1$	0	$\frac{1}{2}(-H_1+H_2)$	0	F_{-4}	$-F_5$	0
F_{-1}	$-F_{-1}$	$3F_{-1}$	$\frac{1}{2}(H_1-H_2)$	0	$-F_4$	0	0	F_{-5}
F_5	$-F_5$	$-3F_5$	0	F_4	0	$\frac{1}{2}(H_1+H_2)$	0	$-F_1$
F_{-5}	F_{-5}	$3F_{-5}$	$-F_{-4}$	0	$\frac{1}{2}(-H_1-H_2)$	0	F_{-1}	0
F_4	$-2F_4$	0	F_5	0	0	$-F_{-1}$	0	H_1
F_{-4}	$2F_{-4}$	0	0	$-F_{-5}$	F_1	0	$-H_1$	0

this choice of representation, it becomes convenient to consider new linear combinations of $L_{\pm1}, Q_{\pm1}, Q_{\pm2}$, which act as step operators with respect to L_0 and Q_0. That is, we seek operators which transform simultaneous eigenstates of L_0 and Q_0 into different eigenstates of L_0 and Q_0. Following the notation of Racah[11] we define:

$$\begin{aligned}
H_1&=L_0=u_1{}^\dagger u_1-u_{-1}{}^\dagger u_{-1},\\
H_2&=Q_0=2u_0{}^\dagger u_0-u_1{}^\dagger u_1-u_{-1}{}^\dagger u_{-1},\\
F_1&=[1/(12)^{1/2}](Q_{-1}-\sqrt{3}L_{-1})=-u_0{}^\dagger u_1,\\
F_{-1}&=F_1{}^+=-[1/(12)^{1/2}](Q_1-\sqrt{3}L_1)=-u_1{}^\dagger u_0,\\
F_5&=[1/(12)^{1/2}](Q_1+\sqrt{3}L_1)=-u_0{}^\dagger u_{-1},\\
F_{-5}&=F_5{}^+=-[1/(12)^{1/2}](Q_{-1}+\sqrt{3}L_{-1})=-u_{-1}{}^\dagger u_0,\\
F_4&=(1/6^{1/2})Q_2=-u_1{}^\dagger u_{-1},\\
F_{-4}&=F_4{}^+=(1/6^{1/2})Q_{-2}=-u_{-1}{}^\dagger u_1.
\end{aligned}\tag{IIB.1}$$

The commutation rules of the operators (IIB.1) are summarized in Table I.

We note that:

$$\begin{aligned}
&[H_1,H_2]=0,\ [H_\alpha,F_\beta]=C_{\alpha\beta}F_\beta,\ [F_\beta,F_{-\beta}]=C_1H_1+C_2H_2,\\
&[F_\beta,F_{\beta'}]=\pm F_{\beta+\beta'}\quad \text{if}\quad \beta+\beta'=\pm1,\ \pm4,\ \pm5,\\
&[F_\beta,F_{\beta'}]=0\qquad\quad\ \text{if}\quad \beta+\beta'\neq\pm1,\ \pm4,\ \pm5.
\end{aligned}$$

From the commutation rules in Table I, we see that F_β acts as a step operator with respect to eigenstates of H_1 and H_2; that is, $F_{\pm1}$ and $F_{\pm5}$ change K by one unit and change ϵ by three units, whereas $F_{\pm4}$ change K by two units without changing ϵ.

As observed by Elliott,[9] the three operators,

$$-\tfrac{1}{2}(F_4+F_{-4}),\quad \tfrac{1}{2}i(F_4-F_{-4}),\quad \tfrac{1}{2}L_0,\tag{IIB.2}$$

have commutation rules identical with those of the three components of angular momentum, L_x, L_y, L_z. Denoting the operators (IIB.2) as $\Lambda_x, \Lambda_y, \Lambda_z$, we define a new operator:

$$\begin{aligned}
\Lambda^2&=\Lambda_z{}^2+\Lambda_x{}^2+\Lambda_y{}^2\\
&=\tfrac{1}{4}L_0{}^2+\tfrac{1}{2}(F_4F_{-4}+F_{-4}F_4)=\tfrac{1}{4}L_0{}^2+\tfrac{1}{2}L_0+F_{-4}F_4\\
&=\tfrac{1}{4}L_0{}^2-\tfrac{1}{2}L_0+F_4F_{-4}.
\end{aligned}\tag{IIB.3}$$

The last two equalities in Eq. (IIB.3) follow from the commutation rule $[F_4,F_{-4}]=L_0$. We note that Λ^2 commutes with L_0 and Q_0, and has no matrix elements joining states of different $[f]$ or (λ,μ). The eigenvalues of Λ^2 can then be used together with the eigenvalues of L_0 and Q_0 to classify the states spanning an irreducible representation of $SU3$. In fact, Elliott has shown that the states belonging to a given irreducible representation (λ,μ) are uniquely characterized by the three quantum numbers ϵ, K, Λ.

Following the notation in Sec. I, we designate the single-particle simultaneous eigenstates of L_0, Q_0, Λ^2 as $\varphi(\lambda\mu,K\epsilon\Lambda)$ and the corresponding many-particle eigenstates as $\Phi([f](\lambda\mu)K\epsilon\Lambda)$. Hence,

$$\begin{aligned}
L_0\Phi([f](\lambda\mu)K\epsilon\Lambda)&=K\Phi([f](\lambda\mu)K\epsilon\Lambda),\\
Q_0\Phi([f](\lambda\mu)K\epsilon\Lambda)&=\epsilon\Phi([\nu](\lambda\mu)K\epsilon\Lambda),\\
(L_0{}^2/4+\tfrac{1}{2}F_4F_{-4}+\tfrac{1}{2}F_{-4}F_4)\Phi([f](\lambda\mu)K\epsilon\Lambda)&\\
=\Lambda(\Lambda+1)\Phi([f](\lambda\mu)K\epsilon\Lambda),&
\end{aligned}\tag{IIB.4}$$

where $F_{\pm4}$, L_0, Q_0 are the many-particle operators obtaining by summing the single-particle operators. We have suppressed the quantum number n designating the oscillator level to which the states belong. The quantum numbers in the argument of Φ fail to uniquely distinguish the states only if an irreducible representation of given (λ,μ) occurs more than once for given n and $[f]$. Although such irreducible representations are encountered in dealing with many-particle configurations, they correspond to states of high energy and may generally be disregarded in calculations involving low excited states.

The allowed values of ϵ, K, and Λ for the various irreducible representations of $SU3$ have been determined by Elliott. For a given (λ,μ), ϵ may assume the values

$$\epsilon=2\lambda+\mu,\ 2\lambda+\mu-3,\ \cdots,\ -\lambda-2\mu,\tag{IIB.5}$$

while for each value of ϵ, Λ takes on the values

$$\begin{aligned}
\Lambda=\tfrac{1}{6}|2\lambda-2\mu-\epsilon|,\ \ \tfrac{1}{6}|2\lambda-2\mu-\epsilon|+1,\ \cdots,&\\
\min[\tfrac{1}{6}(2\lambda+4\mu-\epsilon),\tfrac{1}{6}(4\lambda+2\mu+\epsilon)]&
\end{aligned}\tag{IIB.6}$$

and for given Λ,

$$K=2\Lambda,\ 2\Lambda-2,\ \cdots,\ -2\Lambda.\tag{IIB.7}$$

Equation (IIB.7) follows from the definition of Λ_z as the operator $L_0/2$. For the maximum and minimum values of ϵ, the value of Λ is unique:

$$\Lambda=\mu/2 \quad \text{for} \quad \epsilon=\epsilon_{\max}=2\lambda+\mu,$$
$$\Lambda=\lambda/2 \quad \text{for} \quad \epsilon=\epsilon_{\min}=-\lambda-2\mu,$$

and, hence, for $\epsilon=2\lambda+\mu$, we have $K_{\max}=\mu$. The state with $(\epsilon,K)=(\epsilon_{\max},K_{\max})=(2\lambda+\mu,\mu)$ is called the state of highest weight in the irreducible representation (λ,μ).

Since the F_β operators cannot change (λ,μ), and since states with $\epsilon>\epsilon_{\max}$ and $K>K_{\max}$ do not exist, we have the following relations involving the step-up operators F_1, F_5, F_4:

$$\begin{aligned} F_\beta\Phi(\epsilon_{\max},K_{\max})&=0 \quad \text{if} \quad \beta>0,\\ F_\beta\Phi(\epsilon_{\max},K)&=0 \quad \text{if} \quad \beta=1,5. \end{aligned} \tag{IIB.8}$$

Equations (IIB.8) are analogous to the relation

$$(L_x+iL_y)\psi_{M_l=L}{}^L=0,$$

for the angular momentum operators in the (L,M) representation.

We return now to the Casimir operator C, introduced in Sec. I:

$$C=\frac{L^2}{12}+\frac{1}{36}Q\cdot Q=\frac{L_0^2}{12}+\frac{Q_0^2}{36}+\tfrac{1}{6}\sum_\beta F_\beta F_{-\beta}$$
$$=\frac{L_0^2+2L_0}{12}+\frac{Q_0^2+6Q_0}{36}+\tfrac{1}{3}\sum_{\beta>0}F_{-\beta}F_\beta, \tag{IIB.9}$$

where

$$[C,L_0]=[C,Q_0]=[C,F_\beta]=0. \tag{IIB.10}$$

The last equality in (IIB.9) and the commutation rules (IIB.10) follow from the commutation properties of the H_α and F_β. In general, a Casimir operator, C, which commutes with all the generators of any group can be formed from a bilinear combination of the generators. It follows that the matrix of C within an irreducible representation of the group is a constant. For example, the Casimir operator for the group $R3$ of rotations is the square of the total-angular momentum, $\mathbf{L}^2$, which has the same eigenvalue for all states $\psi_M{}^L$ belonging to a given irreducible representation of $R3$. Similarly, all states $\Phi([f](\lambda\mu)K\epsilon\Lambda)$ belonging to a given irreducible representation (λ,μ) of $SU3$ must be eigenstates of C with the same eigenvalues. This eigenvalue is readily determined from Eqs. (IIB.5), (IIB.6) and (IIB.7), (IIB.8), and (IIB.9). Since

$$C\Phi([f](\lambda\mu)K_{\max},\epsilon_{\max},\Lambda)$$
$$=\left[\frac{K_{\max}{}^2+2K_{\max}}{12}+\frac{\epsilon_{\max}{}^2+6\epsilon_{\max}}{36}\right]\times\Phi([f](\lambda\mu)K_{\max},\epsilon_{\max},\Lambda),$$

and

$$K_{\max}=\mu, \quad \epsilon_{\max}=2\lambda+\mu,$$

TABLE II. The allowed values of L and K for the irreducible representation $(\lambda,\mu)=(10,6)$.

$K \backslash L$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
6							x	x	x	x	x	x	x	x	x	x	x
4					x	x	x	x	x	x	x	x	x	x	x		
2			x	x	x	x	x	x	x	x	x	x	x				
0	x		x		x		x		x		x						
	1	0	2	1	3	2	4	3	4	3	4	3	3	2	2	1	1

we conclude that the eigenvalue of C associated with an irreducible representation (λ,μ) of $SU3$ is

$$\frac{\mu^2+2\mu}{12}+\frac{(2\lambda+\mu)^2+6(2\lambda+\mu)}{36}$$
$$=\frac{(\lambda+\mu)(\lambda+\mu+3)-\lambda\mu}{9}. \tag{IIB.11}$$

It follows from (IIB.11) that

$$\begin{aligned} Q\cdot Q\Phi([f](\lambda\mu)K\epsilon\Lambda)&=36(C-\tfrac{1}{12}L^2)\Phi([f](\lambda\mu)K\epsilon\Lambda)\\ &=[4(\lambda+\mu)(\lambda+\mu+3)-4\lambda\mu-3\mathbf{L}^2]\\ &\quad\times\Phi([f](\lambda\mu)K\epsilon\Lambda). \end{aligned}$$

The operator $Q\cdot Q$ is, therefore, an example of a Hamiltonian which yields a pure rotator spectrum.

C. The Projected (L,M) Representation

Consider the degenerate states of a harmonic oscillator belonging to a given partition $[f]$ and to a given irreducible representation (λ,μ). We have seen in Sec. IIB that these states are uniquely characterized by the three quantum numbers (K,ϵ,Λ). Alternatively, the operators $\mathbf{L}^2$ and L_z may be diagonalized within each irreducible representation and the states characterized by the quantum numbers (L,M). In this case, however, the classification is not complete, as the same values of L, M may occur more than once in the same irreducible representation.

Elliott has shown that the allowed values of L for given (λ,μ) are just those of a series of rotational bands based on states with $L_z=K$, where K assumes all values consistent with $\epsilon=\epsilon_{\max}$, that is,

If $\lambda\geqslant\mu$,

$$\begin{aligned} &K=\mu,\mu-2,\cdots,0 \text{ or } 1,\\ &L=K,K+1,K+2,\cdots,(K+\lambda) \quad \text{if} \quad K\neq0,\\ &L=\lambda,\lambda-2,\lambda-4,\cdots,0 \quad \text{if} \quad K=0. \end{aligned} \tag{IIC.1}$$

If $\mu>\lambda$,

$$\begin{aligned} &K=\lambda,\lambda-2,\cdots,0 \text{ or } 1,\\ &L=K,K+1,K+2,\cdots,(K+\mu) \quad \text{if} \quad K\neq0,\\ &L=\mu,\mu-2,\mu-4,\cdots,0 \text{ or } 1 \quad \text{if} \quad K=0. \end{aligned} \tag{IIC.2}$$

An example of the rule (IIC.1) is given in Table II, which shows the allowed values of L and K for the irreducible representation $(\lambda,\mu)=(10,6)$. The last row of the table indicates the number of states of each L value in the representation. It is noteworthy that each value of L up to $L=6$ occurs the same number of times in the irreducible representation (10,6) as in an asymmetric rotator[12] made up of even numbers of neutrons and protons. This suggests that the various states in an irreducible representation can be associated with the states of motion of an asymmetric rotator, in which case different (λ,μ) symmetries may correspond to different states of intrinsic excitation. Support for such an interpretation is provided by the discussion in Sec. IIIC.

It remains to determine the transformation from the (K,ϵ,Λ) to the (L,M) scheme. For this purpose we introduce the operator P^L which projects out of an arbitrary function that part which transforms as the spherical harmonic of order L. In general, P^L can be expressed in terms of the rotation operators as discussed in Sec. I. If the function under consideration contains angular momenta only up to some maximum value, $L_{\max}$, the operator P^L becomes simply a polynomial in L^2 which vanishes for all $L\neq L'$ and equals unity for $L=L'$. In any event, the P^L operator commutes with all components of the angular momentum operator and with any scalar. It is always possible to combine P^L with the step operators $L_x \pm iL_y$ which change the eigenvalue of L_z. In the following we denote as $P_M{}^L$ the combination of operators that first projects out the part of a function with angular momentum L and next changes the eigenvalue of L_z to M.

Since no angular momenta greater than $L=\lambda+\mu$ occur in the (λ,μ) irreducible representation, the operator $P_M{}^L$ acting on the states $\Phi([f](\lambda\mu)K\epsilon\Lambda)$ can be expressed as a polynomial in $\mathbf{L}^2$ multiplied by an appropriate linear combination of L_x and L_y. Recalling that L_x, L_y, L_z are among the generators of $SU3$, we conclude that $P_M{}^L$ does not change the irreducible representation (λ,μ). The projected states can then be written as a linear combination of the Φ:

$$\Psi_M{}^L([f](\lambda\mu)K\epsilon\Lambda)\equiv P_M{}^L\Phi([f](\lambda\mu)K\epsilon\Lambda) = \sum_{\epsilon'\Lambda'} a(K\epsilon'\Lambda')\Phi([f](\lambda\mu)M\epsilon'\Lambda'), \quad \text{(IIC.3)}$$

where the functions $\Psi_M{}^L$ are unnormalized eigenfunctions of $\mathbf{L}^2$ and L_z with eigenvalues L and M. The quantum numbers (K,ϵ,Λ) in the argument of $\Psi_M{}^L$ refer to the intrinsic state from which the projected state is derived.

It is obvious that equations of the form (IIC.3) for all allowed values of L, M, K, ϵ, Λ, will yield many more functions Ψ than Φ. These functions cannot all be linearly independent since the Ψ are simply linear combinations of the Φ. The problem is to find a complete, linearly independent set of functions $\Psi_M{}^L$ which span the same space as the Φ and constitute the basis vectors of the L, M representation. Elliott has shown that this is accomplished by choosing

$$\begin{aligned}&\Psi_M{}^L([f](\lambda\mu)K\epsilon_{\max}) \quad \text{if} \quad \lambda\geqslant\mu,\\ &\Psi_M{}^L([f](\lambda\mu)K\epsilon_{\min}) \quad \text{if} \quad \mu>\lambda.\end{aligned} \quad \text{(IIC.4)}$$

The quantum number Λ has been suppressed in Eqs. (IIC.4) and the following as it is uniquely determined for the extreme values of ϵ. The allowed values of L and K are given by the rules (IIC.1) and (IIC.2).

The physical interpretation of the functions (IIC.4) is clear. The quantum numbers $[f]$, (λ,μ) describe the intrinsic state of an axially symmetric rotator composed of independent particles in a deformed well. Both ϵ and K also refer to the intrinsic system; the former relates to the quadrupole distortion of the well, whereas the latter is the projection of the angular momenta on the symmetry axis. For any given state of internal motion, states with different values of K [$\min(\lambda,\mu)$, $\min(\lambda,\mu)-2$, $\cdots 0$ or 1] can be mixed to form an asymmetric rotator. Finally, the quantum numbers L and M give the total angular momentum and its z component in the laboratory coordinate system. It should be noted that states with different quantum number K are not necessarily orthogonal in the LM representation. This causes no special problems in principle or in practice provided that the nonorthogonality of the representation is duly considered in carrying out calculations.

III. MATRIX ELEMENTS FOR THE 2s, 1d OSCILLATOR LEVEL

In this section we restrict our attention to the $2s$, $1d$ oscillator level with $n=2$, $E=\frac{7}{2}\hbar\omega$. We assume further that $\lambda\geqslant\mu$, in which case the representation of interest is spanned by the states:

$$\Psi_M{}^L([f](\lambda\mu)K\epsilon_{\max})=P_M{}^L\phi([f](\lambda\mu)K\epsilon_{\max}).$$

In Sec. IIIA, the single-particle states φ_K for the sd oscillator level are considered and a set of operators convenient for dealing with these states is introduced. Section IIIB concerns the matrix elements of the Hamiltonian in the intrinsic (K,ϵ,Λ) representation. Finally, in IIIC we apply the projection operator and arrive at a general expression for the matrix elements of the Hamiltonian in the projected (L,M) scheme. The discussion in Sec. III is limited to the $2s$, $1d$ oscillator level in order to facilitate presentation and is readily extended to other oscillator levels and to states with $\lambda<\mu$.

A. Single-Particle States

The single-particle states for the $2s$, $1d$ oscillator level are derived from the shell-model states ψ_{2s}, $\psi_{1d,m}$ by simultaneously diagonalizing the matrices of L_0 and

[12] A. S. Davydov and G. F. Filippov, Zh. Eksperim. i Teor. Fiz. 35, 440 (1958) [tranlsation: Soviet Phys.—JETP 8, 303 (1959)].

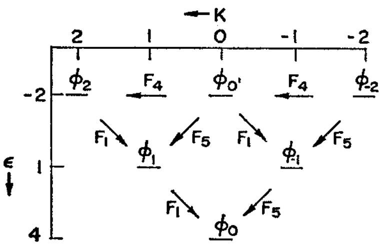

FIG. 3. Single-particle states. See text for further explanation.

Q_0. Designating the simultaneous eigenstates of L_0 and Q_0 as $\varphi(\lambda\mu,K\epsilon\Lambda)$, we find that:

$$
\begin{aligned}
&(\lambda\mu,K\epsilon\Lambda)\\
&\varphi(20,\ \ 0\ \ 4\ 0)=\varphi_0\ =(1/\sqrt{3})(-\psi_{2s}+\sqrt{2}\psi_{1d,0}),\\
&\varphi(20,\ \ 1\ \ 1\tfrac{1}{2})=\varphi_1\ =\psi_{1d,1},\\
&\varphi(20,\ -1\ \ 1\tfrac{1}{2})=\varphi_{-1}=\psi_{1d,-1},\\
&\varphi(20,\ \ 2\ -2\ 1)=\varphi_2\ =\psi_{1d,2},\\
&\varphi(20,\ \ 0\ -2\ 1)=\varphi_{0'}=(1/\sqrt{3})(\sqrt{2}\psi_{2s}+\psi_{1d,0}),\\
&\varphi(20,\ -2\ -2\ 1)=\varphi_{-2}=\psi_{1d,-2}.
\end{aligned}
\qquad \text{(IIIA.1)}
$$

The allowed eigenvalues K, ϵ, Λ, for $(\lambda,\mu)=(2,0)$ follow from Eqs. (IIB.5)–(IIB.7).

It is convenient to regard the states φ_K as levels in a $K-\epsilon$ plane, as shown in Fig. 3. The arrows in the diagram indicate the directions in which the step operators F_4, F_1, F_5 change the eigenvalues K and ϵ. The Hermitian conjugate operators change the eigenvalues in the opposite directions; that is, F_{-4} decreases K by 2 units and leaves ϵ unchanged, F_{-1} increases K by 1 unit and decreases ϵ by 3 units, F_{-5} decreases K by 1 unit and decreases ϵ by 3 units. Many-particle states Φ are constructed by filling the single-particle levels in accordance with the Pauli principle. For $\lambda\geqslant\mu$, the many-particle state of lowest energy is generally the state of highest weight; that is, the state with $\epsilon=\epsilon_{\max}$, $K=K_{\max}$. This coupling scheme may be compared with the Nilsson[4] scheme in the limit of large deformation. Since the φ_K states are eigenfunctions of Q_0, they are equivalent to the asymptotic states in the Nilsson scheme if mixing of major shells is neglected. The two schemes differ inasmuch as Elliott considers an intrinsic Hartree field given by Q_0 alone, whereas Nilsson includes also a spin-orbit force and an L^2 force.

We next introduce a set of operators $\chi_\mu{}^\dagger$, χ_ν, with μ, $\nu=0,\pm1,\pm2,0'$, which create and annihilate the φ_K states. The commutation rules of the $\chi_\mu{}^\dagger$, χ_ν are:

$$[\chi_\mu,\chi_\nu{}^\dagger]=\delta_{\mu\nu},\quad [\chi_\mu,\chi_\nu]=0,\quad [\chi_\mu{}^\dagger,\chi_\nu{}^\dagger]=0. \qquad \text{(IIIA.2)}$$

From the $\chi_\mu{}^\dagger$, χ_ν, 36 product operators $\chi_\mu{}^\dagger\chi_\nu$ may be formed which generate the group $U6$ just as the nine $u_\mu{}^\dagger u_\nu$ generate $U3$. The eight operators H_α, F_β, can be expressed as linear combinations of the $\chi_\mu{}^\dagger\chi_\nu$ which generate $SU3$. In order to find the appropriate linear combinations, the matrices of the H_α, F_β are evaluated in the φ_K representation using shell-model methods. For example, the matrix of the single-particle operator F_1 is

$$
\begin{array}{c|cccccc|}
 & \phi_0 & \phi_1 & \phi_{-1} & \phi_2 & \phi_{0'} & \phi_{-2}\\
\phi_0 & 0 & -\sqrt{2} & 0 & 0 & 0 & 0\\
\phi_1 & 0 & 0 & 0 & -\sqrt{2} & 0 & 0\\
\phi_{-1} & 0 & 0 & 0 & 0 & -1 & 0\\
\phi_2 & 0 & 0 & 0 & 0 & 0 & 0\\
\phi_{0'} & 0 & 0 & 0 & 0 & 0 & 0\\
\phi_{-2} & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\qquad \text{(IIIA.3)}
$$

and, hence,

$$F_1=-[\sqrt{2}\chi_0{}^\dagger\chi_1+\chi_{-1}{}^\dagger\chi_{0'}+\sqrt{2}\chi_1{}^\dagger\chi_2]. \qquad \text{(IIIA.4a)}$$

Similarly, we arrive at expressions for the remaining operators:

$$
\begin{aligned}
F_{-1}&=-[\sqrt{2}\chi_1{}^\dagger\chi_0+\chi_{0'}{}^\dagger\chi_{-1}+\sqrt{2}\chi_2{}^\dagger\chi_1], && \text{(IIIA.4b)}\\
F_4&=-[\chi_1{}^\dagger\chi_{-1}+\sqrt{2}\chi_{0'}{}^\dagger\chi_{-2}+\sqrt{2}\chi_2{}^\dagger\chi_{0'}], && \text{(IIIA.4c)}\\
F_{-4}&=-[\chi_{-1}{}^\dagger\chi_1+\sqrt{2}\chi_{-2}{}^\dagger\chi_{0'}+\sqrt{2}\chi_{0'}{}^\dagger\chi_2], && \text{(IIIA.4d)}\\
F_5&=-[\sqrt{2}\chi_0{}^\dagger\chi_{-1}+\chi_1{}^\dagger\chi_{0'}+\sqrt{2}\chi_{-1}{}^\dagger\chi_{-2}], && \text{(IIIA.4e)}\\
F_{-5}&=-[\sqrt{2}\chi_{-1}{}^\dagger\chi_0+\chi_{0'}{}^\dagger\chi_1+\sqrt{2}\chi_{-2}{}^\dagger\chi_{-1}], && \text{(IIIA.4f)}\\
H_1&=L_0=\chi_1{}^\dagger\chi_1-\chi_{-1}{}^\dagger\chi_{-1}+2\chi_2{}^\dagger\chi_2-2\chi_{-2}{}^\dagger\chi_{-2}, && \text{(IIIA.4g)}\\
H_2&=Q_0=4\chi_0{}^\dagger\chi_0+\chi_1{}^\dagger\chi_1+\chi_{-1}{}^\dagger\chi_{-1}-2\chi_2{}^\dagger\chi_2\\
&\qquad -2\chi_{0'}{}^\dagger\chi_{0'}-2\chi_{-2}{}^\dagger\chi_{-2}. && \text{(IIIA.4h)}
\end{aligned}
$$

In the next section we shall see that calculations involving the operators (IIIA.4) can be simplified by invoking the time-reversal transformation, under which

$$
\begin{aligned}
\mathbf{L}&\rightarrow-\mathbf{L},\\
Y_M{}^L&\rightarrow(-1)^M Y_{-M}{}^L.
\end{aligned}
\qquad \text{(IIIA.5)}
$$

The five operators Q_μ behave like $Y_\mu{}^2$ and the F_β operators can be expressed in terms of L and Q. Hence, the transformation properties under time reversal of all the operators (IIIA.4) follow from (IIIA.5). The transformations of interest are:

$$
\begin{array}{llll}
Q_\mu\rightarrow(-1)^\mu Q_{-\mu}, & F_{\pm1}\rightarrow-F_{\pm5}, & \chi_0\leftrightarrow\chi_0, & \\
L_\mu\rightarrow(-1)^{\mu+1}L_{-\mu}, & F_{\pm5}\rightarrow-F_{\pm1}, & \chi_1\leftrightarrow-\chi_{-1}, & \\
L_+\rightarrow-L_-, & F_{\pm4}\rightarrow F_{\mp4}, & \chi_2\leftrightarrow\chi_{-2}, & \text{(IIIA.6)}\\
 & & \chi_{0'}\leftrightarrow\chi_{0'}, & \\
L_-\rightarrow-L_+, & \Lambda_\pm\rightarrow\Lambda_\mp, & C\rightarrow C. &
\end{array}
$$

We also need the transformation properties under time reversal of the Elliott wave functions $\Phi([f](\lambda\mu)K\epsilon\Lambda)$. In the absence of spin, the time-reversal transformation is simply complex conjugation. Therefore

$$\Phi([f](\lambda\mu)K\epsilon\Lambda)\rightarrow\Phi^*([f](\lambda\mu)K\epsilon\Lambda).$$

The eigenvalue equations for L_0, Q_0, and Λ^2 transform under time reversal as follows:

$$(L_0\Phi=K\Phi)\rightarrow(L_0\Phi^*=-K\Phi^*),$$
$$(Q_0\Phi=\epsilon\Phi)\rightarrow(Q_0\Phi^*=\epsilon\Phi^*),$$
$$[\Lambda^2\Phi=\Lambda(\Lambda+1)\Phi]\rightarrow[\Lambda^2\Phi^*=\Lambda(\Lambda+1)\Phi^*].$$

Since the generators of $SU3$ transform into each other under time reversal, the transformed functions must span an irreducible representation of $SU3$. Moreover, the transformed functions must span the same irreducible representation as the Φ since they have the same values of $\epsilon_{\max}$ and $K_{\max}$. Hence, under the time reversal,

$$\Phi([f](\lambda\mu)K\epsilon\Lambda)\rightarrow e^{i\alpha}\Phi([f](\lambda\mu)-K\epsilon\Lambda),$$

where α is a real phase.

B. Matrix Elements in the Intrinsic (K,ϵ,Λ) Scheme

We write the Hamiltonian for a system of N particles in the $2s$, $1d$ oscillator level as a general sum of one-body and two-body operators:

$$\mathcal{H}=\sum_{i=1}^{N}\sum_{\mu,\nu}a_{\mu\nu}\chi_\mu^{\dagger(i)}\chi_\nu^{(i)}+\sum_{i,j=1}^{N}\sum_{\mu\nu\chi\sigma}b_{\mu\nu\chi\sigma}\chi_\mu^{\dagger(i)}\chi_\nu^{\dagger(j)}\chi_\chi^{(i)}\chi_\sigma^{(j)},\quad\text{(IIIB.1)}$$

where the $\chi_\sigma^{\dagger(i)}$, $\chi_\tau^{(i)}$ create and annihilate the states $\varphi_K(r_i)$ of the ith particle. The restriction to one-body and two-body forces is made for convenience only and is not essential to the following development. Since $\mathcal{H}$ is a scalar symmetric in particles, the partition $[f]$ is a good quantum number. Then the function $\mathcal{H}\Phi\times([f](\lambda\mu)K,\epsilon_{\max})$ can be expanded in the complete orthonormal set of Φ's:

$$\mathcal{H}\Phi([f](\lambda\mu)K,\epsilon_{\max})=\sum_{\lambda'\mu'\epsilon'\Lambda'}C_{\epsilon'\Lambda'}{}^{\lambda'\mu'K}\Phi([f](\lambda'\mu')K\epsilon'\Lambda').\quad\text{(IIIB.2)}$$

Since a suitable combination of generators can change any member of an irreducible representation into any other member, we have

$$C_{\epsilon'\Lambda'}{}^{\lambda'\mu'K}\Phi([f](\lambda'\mu')K\epsilon'\Lambda')=E_{\epsilon'\Lambda'}{}^{\lambda'\mu'K}(H_\alpha,F_\beta)\Phi([f](\lambda'\mu')K,\epsilon_{\max}),\quad\text{(IIIB.3)}$$

where $E_{\epsilon'\Lambda'}{}^{\lambda'\mu'K}(H_\alpha,F_\beta)$ is a sum of products of the eight generators of $SU3$ which changes ϵ from $\epsilon_{\max}$ to ϵ' and changes Λ from $\mu/2$ to Λ' without changing K. Substituting (IIIB.3) into (IIIB.2), we obtain

$$\mathcal{H}\Phi([f](\lambda\mu)K,\epsilon_{\max})=\sum_{\epsilon'\Lambda'}E_{\epsilon'\Lambda'}{}^{\lambda\mu K}\Phi([f](\lambda\mu)K,\epsilon_{\max})+\sum_{\epsilon'\Lambda'}\sum_{(\lambda'\mu')\neq(\lambda\mu)}E_{\epsilon'\Lambda'}{}^{\lambda'\mu'K}\Phi([f](\lambda'\mu')K,\epsilon_{\max}).\quad\text{(IIIB.4)}$$

The term in (IIIB.4) with $(\lambda',\mu')\neq(\lambda,\mu)$ is small compared with the first sum if the classification of states according to irreducible representations of $SU3$ is a good zero-order scheme. We, therefore, deal first with the operators $E_{\epsilon'\Lambda'}{}^{\lambda\mu K}(H_\alpha,F_\beta)$.

Referring to the diagram (IIIA.3) of the φ_K states, we note that a Hamiltonian composed of one-body and two-body operators can change the eigenvalue ϵ by three, six, nine, or twelve units. For example, a twelve-unit transition would be obtained by operating with $\chi_{0'}{}^{\dagger(i)}\chi_0{}^{(i)}\chi_{0'}{}^{\dagger(j)}\chi_0{}^{(j)}$. We also note that ΔK is odd if $\Delta\epsilon$ is odd, whereas ΔK is even if $\Delta\epsilon$ is even. Since $\mathcal{H}$ is a scalar, it cannot change K. Hence, the first sum on the right in (IIIB.4) includes only terms with $\epsilon=\epsilon_{\max}$, $\epsilon_{\max}-6$, and $\epsilon_{\max}-12$. For $\epsilon=\epsilon_{\max}$, the sum over Λ' reduces to a single term with $\Lambda'=\mu/2$ and

$$E_{\epsilon_{\max},\Lambda'}{}^{\lambda\mu K}=C_{\epsilon_{\max},\mu/2}{}^{\lambda\mu K}.\quad\text{(IIIB.5)}$$

For fixed K and $\epsilon=\epsilon_{\max}-6$, there are three allowed values of Λ'. The corresponding three operators $E_{\epsilon_{\max}-6,\Lambda'}{}^{\lambda\mu K}(H_\alpha,F_\beta)$ are sums of products of H_α, F_β, which change ϵ by six units without changing K. Since

$$F_1\Phi([f](\lambda\mu)K,\epsilon_{\max})=F_5\Phi([f](\lambda\mu)K,\epsilon_{\max})=0,\quad\text{(IIIB.6)}$$

the operators F_1 and F_5 may be eliminated from the argument of the E's by commuting them to the right where they annihilate the wave function. This can be accomplished regardless of the form of the E's by repeatedly writing:

$$F_\beta F_{\beta'}=[F_\beta,F_{\beta'}]+F_{\beta'}F_\beta,\qquad F_\beta H_\alpha=[F_\beta,H_\alpha]+H_\alpha F_\beta,\quad\text{(IIIB.7)}$$

where $\beta=1$, 5, and using the rules in Table I to evaluate the commutators. In the same manner, H_1 and H_2 can be commuted to the right where they are replaced with their eigenvalues, K, and $\epsilon_{\max}=2\lambda+\mu$. The only combinations of the remaining four operators capable of changing ϵ by six units without changing K are

$$F_{-1}F_{-5},\quad F_{-1}F_{-1}F_{-4},\quad F_{-5}F_{-5}F_4,\quad\text{(IIIB.8)}$$

and, hence,

$$\sum_{\Lambda'}E_{\epsilon_{\max}-6,\Lambda'}{}^{\lambda\mu K}\Phi([f](\lambda\mu)K\epsilon_{\max})=[a_1{}^{\lambda\mu K}F_{-1}F_{-5}+a_2{}^{\lambda\mu K}F_{-1}F_{-1}F_{-4}+a_3{}^{\lambda\mu K}F_{-5}F_{-5}F_4]\times\Phi([f](\lambda\mu)K\epsilon_{\max}),$$

where the $a_i{}^{\lambda\mu K}$ are constants for fixed (λ,μ) and K.

For fixed K and $\epsilon=\epsilon_{\max}-12$, Λ' can assume five values. Again the five operators $E_{\epsilon_{\max}-12,\Lambda'}{}^{\lambda\mu K}$ are determined by eliminating H_1, H_2, F_1, F_5, and requiring that the remaining generators change ϵ by twelve units without changing K. The only five products meeting the requirements are

$$f_1=F_{-1}F_{-1}F_{-5}F_{-5},\qquad f_2=F_{-1}F_{-1}F_{-1}F_{-5}F_{-4},$$
$$f_3=F_{-5}F_{-5}F_{-5}F_{-1}F_4,\quad f_4=F_{-1}F_{-1}F_{-1}F_{-1}F_{-4}F_{-4},$$
$$f_5=F_{-5}F_{-5}F_{-5}F_{-5}F_4F_4, \tag{IIIB.9}$$

and, hence, there exists a set of five constants $b_j^{\lambda\mu K}$ such that

$$[\sum_{\Lambda'} E_{\epsilon_{\max}-12,\Lambda'}{}^{\lambda\mu K}]\Phi([f](\lambda\mu)K\epsilon_{\max})$$
$$=[\sum_{j=1}^{5} b_j{}^{\lambda\mu K}f_j(F_\beta)]\Phi([f](\lambda\mu)K\epsilon_{\max})$$
$$=[b_1{}^{\lambda\mu K}F_{-1}F_{-1}F_{-5}F_{-5}+b_2{}^{\lambda\mu K}F_{-1}F_{-1}F_{-1}F_{-5}F_{-4}$$
$$+b_3{}^{\lambda\mu K}F_{-5}F_{-5}F_{-5}F_{-1}F_4+b_4{}^{\lambda\mu K}F_{-1}F_{-1}F_{-1}F_{-1}F_{-4}F_{-4}$$
$$+b_5{}^{\lambda\mu K}F_{-5}F_{-5}F_{-5}F_{-5}F_4F_4]\Phi([f](\lambda\mu)K\epsilon_{\max}).$$

In the rest of this paper, we suppress the partition quantum number $[f]$, and denote the state $\Phi([f](\lambda\mu)K,\epsilon_{\max})$ simply as $\Phi_{\epsilon K}{}^{\lambda\mu}$ or as $|\lambda\mu\epsilon K\rangle$, with the understanding that $\epsilon=\epsilon_{\max}=(2\lambda+\mu)$, $\Lambda=\mu/2$. Using this notation and substituting (IIIB.5), (IIIB.8), and (IIIB.9) into (IIIB.4), we have

$$\mathcal{H}|\lambda\mu\epsilon K\rangle$$
$$=\{C_\epsilon{}^{\lambda\mu K}+a_1{}^{\lambda\mu K}(F_{-1}F_{-5})+a_2{}^{\lambda\mu K}(F_{-1}F_{-1}F_{-4})$$
$$+a_3{}^{\lambda\mu K}(F_{-5}F_{-5}F_4)+b_1{}^{\lambda\mu K}(F_{-1}F_{-1}F_{-5}F_{-5})+\cdots$$
$$+b_5{}^{\lambda\mu K}(F_{-5}F_{-5}F_{-5}F_{-5}F_4F_4)\}|\lambda\mu\epsilon K\rangle$$
$$+\sum_{\epsilon'\Lambda'}\sum_{(\lambda'\mu')\neq(\lambda\mu)} E_{\epsilon'\Lambda'}{}^{\lambda\mu}(H_\alpha,F_\beta)|\lambda'\mu'\epsilon'K\rangle. \tag{IIIB.10}$$

The next step is to obtain expressions for the constants $C_\epsilon{}^{\lambda\mu K}$, $a_i{}^{\lambda\mu K}$, $b_j{}^{\lambda\mu K}$. This is easily accomplished owing to the orthonormality of the set of functions $\Phi_{\epsilon K}{}^{\lambda\mu}$. Multiplying (IIIB.10) on the left with $\langle\lambda\mu\epsilon K|$ and integrating, we obtain

$$\langle\lambda\mu\epsilon K|\mathcal{H}|\lambda\mu\epsilon K\rangle=C_\epsilon{}^{\lambda\mu K}. \tag{IIIB.11}$$

All matrix elements on the right with coefficients a_i, b_j, vanish because they involve operators that change ϵ, while all matrix elements in the sum over $(\lambda'\mu')\neq(\lambda\mu)$ vanish because the operators H_α, F_β, cannot change $(\lambda\mu)$.

We now write (IIIB.10) three times and multiply each equation from the left with the Hermitian conjugate of one of the operators (IIIB.8). Multiplying again on the left with $\langle\lambda\mu\epsilon K|$ and integrating, we obtain a set of linear equations for the a_i:

$$\langle\lambda\mu\epsilon K|F_5F_1\mathcal{H}|\lambda\mu\epsilon K\rangle=\langle\lambda\mu\epsilon K|[a_1{}^{\lambda\mu K}F_5F_1F_{-1}F_{-5}+a_2{}^{\lambda\mu K}F_5F_1F_{-1}F_{-1}F_{-4}+a_3{}^{\lambda\mu K}F_5F_1F_{-5}F_{-5}F_4]|\lambda\mu\epsilon K\rangle,$$
$$\langle\lambda\mu\epsilon K|F_4F_1F_1\mathcal{H}|\lambda\mu\epsilon K\rangle$$
$$=\langle\lambda\mu\epsilon K|[a_1{}^{\lambda\mu K}F_4F_1F_1F_{-1}F_{-5}+a_2{}^{\lambda\mu K}F_4F_1F_1F_{-1}F_{-1}F_{-4}+a_3{}^{\lambda\mu K}F_4F_1F_1F_{-5}F_{-5}F_4]|\lambda\mu\epsilon K\rangle, \tag{IIIB.12}$$
$$\langle\lambda\mu\epsilon K|F_{-4}F_5F_5\mathcal{H}|\lambda\mu\epsilon K\rangle$$
$$=\langle\lambda\mu\epsilon K|[a_1{}^{\lambda\mu K}F_{-4}F_5F_5F_{-1}F_{-5}+a_2{}^{\lambda\mu K}F_{-4}F_5F_5F_{-1}F_{-1}F_{-4}+a_3{}^{\lambda\mu K}F_{-4}F_5F_5F_{-5}F_{-5}F_4]|\lambda\mu\epsilon K\rangle.$$

The matrix elements on the right of (IIIB.12) are diagonal in all quantum numbers and involve only products of the F_β, whereas the matrix elements on the left also involve $\mathcal{H}$. We hereafter refer to matrix elements such as those on the right as "homogeneous" and we call those on the left "inhomogeneous." All other matrix elements vanish because they are off-diagonal either in ϵ or in $(\lambda\mu)$.

In order to evaluate the homogeneous matrix elements on the right of (IIIB.12), we apply the commutation rules in Table I and the relations (IIIB.6) and (IIIB.7). First, the operators F_1 and F_5 are commuted to the right, while F_{-1} and F_{-5} are commuted to the left where they annihilate the wave function. Next, H_1 and H_2 are commuted to the right or left and replaced with their eigenvalues. There remain the product operators $F_{\pm4}F_{\mp4}$, which satisfy the eigenvalue equations:

$$F_4F_{-4}|\lambda\mu\epsilon K\rangle=\left(\Lambda^2-\frac{L_0^2}{4}+\frac{L_0}{2}\right)|\lambda\mu\epsilon K\rangle$$
$$=\left[\frac{\mu(\mu+2)-K(K-2)}{4}\right]|\lambda\mu\epsilon K\rangle,$$
$$F_{-4}F_4|\lambda\mu\epsilon K\rangle=\{F_4F_{-4}+[F_{-4},F_4]\}|\lambda\mu\epsilon K\rangle$$
$$=\left[\frac{\mu(\mu+2)-K(K+2)}{4}\right]|\lambda\mu\epsilon K\rangle, \tag{IIIB.13}$$

$$a_1{}^{\lambda\mu K}=\frac{1}{\gamma}\{4[(\mu-\epsilon+2)(\mu+\epsilon)+2(\epsilon+K)(\epsilon-K)]h_1$$
$$+2[(K+\mu)(K-\mu-2)(\epsilon+K)]h_2$$
$$+2[(K-\mu)(K+\mu+2)(\epsilon-K)]h_3\}, \tag{IIIB.14}$$

TABLE III. Expressions for the matrix elements $C_{jj'}(\lambda\mu\epsilon K)$. The notation $\tilde{C}_{jj'}(\lambda\mu\epsilon K)=C_{jj'}(\lambda\mu\epsilon,\,-K)$ is used.

$$C_{11}=\tilde{C}_{11}=\tfrac{1}{4}(\mu+K)(\mu-K+2)(\mu+K-2)(\mu-K+4)+\tfrac{1}{4}(\epsilon-K-4)(\epsilon-K-6)(\epsilon+K)(\epsilon+K-2)+(\mu+K)(\mu-K+2)(\epsilon+K-2)(\epsilon-K-4)$$
$$C_{12}=\tilde{C}_{13}=\tfrac{3}{8}(\mu-K)(\mu+K+2)(\mu+K)(\mu-K+2)(\epsilon-K-2)+\tfrac{3}{8}(\mu+K)(\mu-K+2)(\epsilon-K)(\epsilon-K-2)(\epsilon+K-6)$$
$$C_{13}=\tilde{C}_{12}=\tfrac{3}{8}(\mu+K)(\mu-K+2)(\mu-K)(\mu+K+2)(\epsilon+K-2)+\tfrac{3}{8}(\mu-K)(\mu+K+2)(\epsilon+K)(\epsilon+K-2)(\epsilon-K-6)$$
$$C_{14}=\tilde{C}_{15}=\tfrac{3}{8}(\mu+K)(\mu-K+2)(\mu+K-2)(\mu-K+4)(\epsilon-K)(\epsilon-K-2)$$
$$C_{15}=\tilde{C}_{14}=\tfrac{3}{8}(\mu-K)(\mu+K+2)(\mu-K-2)(\mu+K+4)(\epsilon+K)(\epsilon+K-2)$$
$$C_{21}=\tilde{C}_{31}=\tfrac{3}{2}[(\mu-K)(\mu+K+2)(\epsilon-K-2)+(\epsilon-K)(\epsilon-K-2)(\epsilon+K-6)]$$
$$C_{22}=\tilde{C}_{33}=\tfrac{3}{8}(\epsilon-K)(\epsilon-K-2)(\epsilon-K-4)(\epsilon+K-2)+(9/8)(\mu+K-2)(\mu-K+4)(\epsilon-K)(\epsilon-K-2)$$
$$C_{23}=\tilde{C}_{32}=\tfrac{3}{8}(\mu-K)(\mu+K+2)[(\mu-K-2)(\mu+K+4)+3(\epsilon+K-4)(\epsilon-K-2)]$$
$$C_{24}=\tilde{C}_{35}=\tfrac{3}{4}(\mu+K-2)(\mu-K+4)(\epsilon-K)(\epsilon-K-2)(\epsilon-K+2)$$
$$C_{25}=\tilde{C}_{34}=\tfrac{3}{4}(\mu-K)(\mu+K+2)(\mu-K-2)(\mu+K+4)(\epsilon+K-2)$$
$$C_{31}=\tilde{C}_{21}=\tfrac{3}{2}[(\mu+K)(\mu-K+2)(\epsilon+K-2)+(\epsilon+K)(\epsilon+K-2)(\epsilon-K-6)]$$
$$C_{32}=\tilde{C}_{23}=\tfrac{3}{8}(\mu+K)(\mu-K+2)[(\mu+K-2)(\mu-K+4)+3(\epsilon-K-4)(\epsilon+K-2)]$$
$$C_{33}=\tilde{C}_{22}=\tfrac{3}{8}(\epsilon+K)(\epsilon+K-2)(\epsilon+K-4)(\epsilon-K-2)+(9/8)(\mu-K-2)(\mu+K+4)(\epsilon+K)(\epsilon+K-2)$$
$$C_{34}=\tilde{C}_{25}=\tfrac{3}{4}(\mu+K)(\mu-K+2)(\mu+K-2)(\mu-K+4)(\epsilon-K-2)$$
$$C_{35}=\tilde{C}_{24}=\tfrac{3}{4}(\mu-K-2)(\mu+K+4)(\epsilon+K)(\epsilon+K-2)(\epsilon+K+2)$$
$$C_{41}=\tilde{C}_{51}=6(\epsilon-K)(\epsilon-K-2)$$
$$C_{42}=\tilde{C}_{53}=3(\epsilon-K)(\epsilon-K-2)(\epsilon-K+2)$$
$$C_{43}=\tilde{C}_{52}=3(\mu-K)(\mu+K+2)(\epsilon-K-2)$$
$$C_{44}=\tilde{C}_{55}=\tfrac{3}{2}(\epsilon-K)(\epsilon-K-2)(\epsilon-K+2)(\epsilon-K+4)$$
$$C_{45}=\tilde{C}_{54}=\tfrac{3}{2}(\mu-K)(\mu+K+2)(\mu-K-2)(\mu+K+4)$$
$$C_{51}=\tilde{C}_{41}=6(\epsilon+K)(\epsilon+K-2)$$
$$C_{52}=\tilde{C}_{43}=3(\mu+K)(\mu-K+2)(\epsilon+K-2)$$
$$C_{53}=\tilde{C}_{42}=3(\epsilon+K)(\epsilon+K-2)(\epsilon+K+2)$$
$$C_{54}=\tilde{C}_{45}=\tfrac{3}{2}(\mu+K)(\mu-K+2)(\mu+K-2)(\mu-K+4)$$
$$C_{55}=\tilde{C}_{44}=\tfrac{3}{2}(\epsilon+K)(\epsilon+K-2)(\epsilon+K+2)(\epsilon+K+4)$$

$$a_2^{\lambda\mu K}=\frac{1}{6\gamma}\{-48(\epsilon+K)h_1+12(\epsilon+K)(\epsilon+K-2)h_2 -12(K-\mu)(\mu+K+2)h_3\},$$

$$a_3^{\lambda\mu K}=\frac{1}{6\gamma}\{-48(\epsilon-K)h_1-12(K+\mu)(K-\mu-2)h_2 +12(\epsilon-K)(\epsilon-K-2)h_3\},$$

where

$$\epsilon=2\lambda+\mu,$$

$$\gamma=(\epsilon-\mu)(\epsilon+\mu+2)(\epsilon+\mu)(\epsilon-\mu-2) =16\lambda(\lambda+\mu)(\lambda+\mu+1)(\lambda-1),$$

$$h_1=\langle\lambda\mu\epsilon K|F_5F_1\mathcal{H}|\lambda\mu\epsilon K\rangle,$$

$$h_2=\frac{\langle\lambda\mu\epsilon K|F_4F_1F_1\mathcal{H}|\lambda\mu\epsilon K\rangle}{\langle\lambda\mu\epsilon K|F_4F_{-4}|\lambda\mu\epsilon K\rangle},$$

$$h_3=\frac{\langle\lambda\mu\epsilon K|F_{-4}F_5F_5|\lambda\mu\epsilon K\rangle}{\langle\lambda\mu\epsilon K|F_{-4}F_4|\lambda\mu\epsilon K\rangle}.$$

The constants $b_j^{\lambda\mu K}$ are dealt with in the same manner as the $a_i^{\lambda\mu K}$. Multiplying (IIIB.10) on the left with $\langle\lambda\mu\epsilon K|f_j^\dagger$, $j=1,\cdots 5$, where the f_j are the five operators (IIIB.9), we obtain upon integration a set of five equations:

$$t_j(\lambda\mu\epsilon K)=\sum_{j'=1}^{5} b_{j'}^{\lambda\mu K}C_{jj'}(\lambda\mu\epsilon K), \qquad \text{(IIIB.15)}$$

where the $C_{jj'}(\lambda\mu\epsilon K)$ are the homogeneous matrix elements

$$C_{jj'}(\lambda\mu\epsilon K)=\frac{\langle\lambda\mu\epsilon K|f_j^\dagger f_{j'}|\lambda\mu\epsilon K\rangle}{\langle\lambda\mu\epsilon K|g_j|\lambda\mu\epsilon K\rangle},$$

$$g_1=1,\quad g_2=F_{4-4},\quad g_3=F_{-44},\quad g_4=F_{44-4-4},\quad g_5=F_{-4-444},$$

and we adopt the notation for the inhomogeneous terms:

$$F_\alpha F_\beta F_\gamma=F_{\alpha\beta\gamma}.$$

$$t_1=\langle\lambda\mu\epsilon K|F_{1155}\mathcal{H}|\lambda\mu\epsilon K\rangle,\qquad t_2=\frac{\langle\lambda\mu\epsilon K|F_{45111}\mathcal{H}|\lambda\mu\epsilon K\rangle}{\langle\lambda\mu\epsilon K|F_{4-4}|\lambda\mu\epsilon K\rangle},$$

$$t_3=\frac{\langle\lambda\mu\epsilon K|F_{-41555}\mathcal{H}|\lambda\mu\epsilon K\rangle}{\langle\lambda\mu\epsilon K|F_{-44}|\lambda\mu\epsilon K\rangle},\qquad t_4=\frac{\langle\lambda\mu\epsilon K|F_{441111}\mathcal{H}|\lambda\mu\epsilon K\rangle}{\langle\lambda\mu\epsilon K|F_{44-4-4}|\lambda\mu\epsilon K\rangle},$$

$$t_5=\frac{\langle\lambda\mu\epsilon K|F_{-4-45555}\mathcal{H}|\lambda\mu\epsilon K\rangle}{\langle\lambda\mu\epsilon K|F_{-4-444}|\lambda\mu\epsilon K\rangle}; \qquad \text{(IIIB.16)}$$

$$\langle\lambda\mu\epsilon K|F_{4-4}|\lambda\mu\epsilon K\rangle=\tfrac{1}{4}[\mu(\mu+2)-K(K-2)],$$
$$\langle\lambda\mu\epsilon K|F_{-44}|\lambda\mu\epsilon K\rangle=\tfrac{1}{4}[\mu(\mu+2)-K(K+2)],$$
$$\langle\lambda\mu\epsilon K|F_{44-4-4}|\lambda\mu\epsilon K\rangle=\tfrac{1}{16}[(\mu+K)(\mu-K+2)(\mu+K-2)(\mu-K+4)],$$
$$\langle\lambda\mu\epsilon K|F_{-4-444}|\lambda\mu\epsilon K\rangle=\tfrac{1}{16}[(\mu-K)(\mu+K+2)(\mu-K-2)(\mu+K+4)].$$

Consider now $t_2(\lambda\mu\epsilon K)$. Applying the time-reversal operator, we have

$$t_2^*(\lambda\mu\epsilon K)=\frac{\langle\lambda\mu\epsilon,-K|F_{-41555}|\lambda\mu\epsilon,-K\rangle}{\langle\lambda\mu\epsilon,-K|F_{-44}|\lambda\mu\epsilon,-K\rangle}.\quad\text{(IIIB.17)}$$

Setting the variable K equal to $-K$ on both sides of (IIIB.17) and comparing with the definition (IIIB.16) of t_3, we conclude that

$$t_2^*(\lambda\mu\epsilon,-K)=t_2(\lambda\mu\epsilon,-K)=t_3(\lambda\mu\epsilon K).\quad\text{(IIIB.18)}$$

Similarly,

$$t_4(\lambda\mu\epsilon,-K)=t_5(\lambda\mu\epsilon K),\quad t_1(\lambda\mu\epsilon,-K)=t_1(\lambda\mu\epsilon K),\quad\text{(IIIB.19)}$$

where we have used the fact that the t_j's are real.

The matrix $C_{jj'}(\lambda\mu\epsilon K)$ is given in Table III, in which we use the notation:

$$\tilde{C}_{jj'}(\lambda\mu\epsilon K)=C_{jj'}(\lambda\mu\epsilon,-K).$$

The equalities between the various $C_{j'j}$'s listed in the table follow from time-reversal considerations similar to those discussed above.

Finally, we consider the terms in (IIIB.10) with $(\lambda'\mu')\neq(\lambda\mu)$. We note first that $\mu-\mu'$ must be an even integer since K and μ are either both even or both odd and $\mathcal{H}$ cannot change K. This restriction limits the possible values of ϵ' to $\epsilon_{\max}'$, $\epsilon_{\max}'-6$, $\epsilon_{\max}'-12$, and $\epsilon_{\max}'-18$, where $\epsilon_{\max}'=2\lambda'+\mu'$. In practice, the terms with $\epsilon'=\epsilon_{\max}'-18$ are negligible for Hamiltonians of the type generally used in intermediate shell-model calculations. We, therefore, drop these terms and rewrite (IIIB.10):

$$\mathcal{H}|\lambda\mu\epsilon K\rangle=\{\sum_{\lambda'\mu'}{}^{\lambda\mu}C^{\lambda'\mu'K}+\sum_{i=1}^{3}{}^{\lambda\mu}a_i^{\lambda'\mu'K}f_i(F_\beta)+\sum_{j=1}^{5}{}^{\lambda\mu}b_j^{\lambda'\mu'K}f_j(F_\beta)\}|\lambda'\mu'\epsilon'K\rangle,$$

where the operators $f_i(F_\beta)$, $f_j(F_\beta)$ are given in (IIIB.8) and (IIIB.9). Multiplying (IIIB.16) on the left successively with $\langle\lambda''\mu''\epsilon''K|$, with $\langle\lambda''\mu''\epsilon''K|f_i^+(F_\beta)$, and with $\langle\lambda''\mu''\epsilon''K|f_j^+(F_\beta)$, we obtain the sets of equations:

$$\langle\lambda''\mu''\epsilon''K|\mathcal{H}|\lambda\mu\epsilon K\rangle={}^{\lambda\mu}C_\epsilon^{\lambda''\mu''K},$$
$$\langle\lambda''\mu''\epsilon''K|f_i^+(F_\beta)\mathcal{H}|\lambda\mu\epsilon K\rangle=\sum_{i=1}^{3}{}^{\lambda\mu}a_1^{\lambda''\mu''K}\langle\lambda''\mu''\epsilon''K|f_i^+f_{i'}|\lambda''\mu''\epsilon''K\rangle,\quad i=1,2,3,\quad\text{(IIIB.20)}$$
$$\langle\lambda''\mu''\epsilon''K|f_j^+(F_\beta)\mathcal{H}|\lambda\mu\epsilon K\rangle=\sum_{j=1}^{5}{}^{\lambda\mu}b_j^{\lambda''\mu''K}\langle\lambda''\mu''\epsilon''K|f_j^+f_{j'}|\lambda''\mu''\epsilon''K\rangle,\quad j=1,\cdots,5.$$

Again, all off-diagonal elements on the right vanish because of the orthogonality of the $\Phi_{\epsilon K}{}^{\lambda\mu}$. We see that the expressions for the ${}^{\lambda\mu}C_\epsilon{}^{\lambda''\mu''K}$, the ${}^{\lambda\mu}a_i{}^{\lambda''\mu''K}$, and the ${}^{\lambda\mu}b_j{}^{\lambda''\mu''K}$ are the same as those for the $C_\epsilon{}^{\lambda\mu K}$, the $a_i{}^{\lambda\mu K}$ and the $b_j{}^{\lambda\mu K}$ with the eigenvalues (λ'',μ'') substituted for (λ,μ) throughout the equations except on the right side of the inhomogeneous matrix elements. If terms with $\epsilon'=\epsilon_{\max}'-18$ were included, we would obtain an additional set of seven equations of the form (IIIB.20) involving seven operators f_k which could be determined in the same manner as the f_i and f_j. The general problem of calculating matrix elements in the intrinsic (K,ϵ) scheme thus reduces to the evaluation of inhomogeneous matrix elements of the type on the left of (IIIB.20).

C. Matrix Elements in the Projected (L,M) Representation

In this section, we first find a set of "equivalence relations" such that

$$f_\sigma(F_{-1},F_{-5},F_{\pm4})\Phi_{\epsilon K}{}^{\lambda\mu}=g_\sigma(L_\pm,\Lambda_\pm)\Phi_{\epsilon K}{}^{\lambda\mu},\quad\text{(IIIC.1)}$$

where the eight operators f_σ are given in (IIIB.8) and (IIIB.9), and

$$L_+=L_x+iL_y=-\sqrt{2}(F_5+F_{-1}),\quad L_-=L_x-iL_y=-\sqrt{2}(F_{-5}+F_1),\quad\text{(IIIC.2)}$$

$$\Lambda_+=F_4,\quad \Lambda_-=F_{-4}.\quad\text{(IIIC.3)}$$

As shown by Elliott, $\Lambda_\pm$ play the role of step operators with respect to eigenstates of Λ^2 and $\Lambda_0=L_0/2$, just as $L_\pm$ serve as step operators with respect to eigenstates of L^2 and L_0. The nonvanishing matrix elements of $\Lambda_\pm$ are

$$\langle\lambda\mu\epsilon K|\Lambda_+|\lambda\mu\epsilon,K-2\rangle=\langle\lambda\mu\epsilon,K-2|\Lambda_-|\lambda\mu\epsilon K\rangle=[(\Lambda+\tfrac{1}{2}K)(\Lambda-\tfrac{1}{2}K+1)]^{1/2}.\quad\text{(IIIC.4)}$$

We note that $\Lambda_\pm$ step K up or down by two units and leave all other quantum numbers unchanged. These operators are associated with the part of the Hamiltonian that connects different K bands and destroys axial symmetry.

In order to illustrate the techniques involved, we derive the equivalence relation for $F_{-1}F_{-5}$. Since

$$L_+L_-=2(F_5+F_{-1})(F_{-5}+F_1)=2(F_5F_{-5}+F_{-1}F_{-5}+F_5F_1+F_{-1}F_1),$$

we have

$$F_{-1}F_{-5}=\tfrac{1}{2}L_+L_--F_5F_{-5}-F_5F_1-F_{-1}F_1.\quad\text{(IIIC.5)}$$

TABLE IV. Equivalence relations. Operators equivalent to $F_\alpha F_\beta F_\gamma\cdots$ when operating on states $\Phi_{\epsilon K}{}^{\lambda\mu}$.

$F_{-1}F_{-5}$:	$\frac{1}{2}(\mathbf{L}^2-K^2-\epsilon)$
$F_{-1}F_{-1}F_{-4}$:	$\frac{1}{2}L_+L_+\Lambda_- -\frac{1}{4}(\mu+K)(\mu-K+2)$
$F_{-5}F_{-5}F_4$:	$\frac{1}{2}L_-L_-\Lambda_+ -\frac{1}{4}(\mu-K)(\mu+K+2)$
$F_{-1}F_{-1}F_{-5}F_{-5}$:	$\frac{1}{4}\mathbf{L}^4-\frac{1}{2}(K^2+2\epsilon-5)\mathbf{L}^2-\frac{1}{2}(L_+L_+\Lambda_-+L_-L_-\Lambda_+)+\frac{1}{4}[\mu(\mu+2)+K^2(K^2+4\epsilon-10)+2\epsilon(\epsilon-4)]$
$F_{-1}F_{-1}F_{-1}F_{-5}F_{-4}$:	$\frac{1}{4}\mathbf{L}^2L_+L_+\Lambda_- -\frac{3}{8}(\mu+K)(\mu-K+2)\mathbf{L}^2-\frac{1}{4}[K(K-2)+3\epsilon-6]L_+L_+\Lambda_-+\frac{3}{8}(\mu+K)(\mu-K+2)(\epsilon+K^2-2)$
$F_{-5}F_{-5}F_{-5}F_{-1}F_4$:	$\frac{1}{4}\mathbf{L}^2L_-L_-\Lambda_+ -\frac{3}{8}(\mu-K)(\mu+K+2)\mathbf{L}^2-\frac{1}{4}[K(K+2)+3\epsilon-6]L_-L_-\Lambda_++\frac{3}{8}(\mu-K)(\mu+K+2)(\epsilon+K^2-2)$
$F_{-1}F_{-1}F_{-1}F_{-1}F_{-4}F_{-4}$:	$\frac{1}{4}L_+L_+L_+L_+\Lambda_-\Lambda_- -\frac{3}{4}(\mu+K-2)(\mu-K+4)L_+L_+\Lambda_-+\frac{3}{16}(\mu+K)(\mu-K+2)(\mu+K-2)(\mu-K+4)$
$F_{-5}F_{-5}F_{-5}F_{-5}F_4F_4$:	$\frac{1}{4}L_-L_-L_-L_-\Lambda_+\Lambda_+ -\frac{3}{4}(\mu-K-2)(\mu+K+4)L_-L_-\Lambda_++\frac{3}{16}(\mu-K)(\mu+K+2)(\mu-K-2)(\mu+K+4)$

But

$$F_5F_{-5}=[F_5,F_{-5}]+F_{-5}F_5=\tfrac{1}{2}(L_0+Q_0)+F_{-5}F_5, \quad \text{(IIIC.6)}$$

where we have used the commutation rule in Table I. Substituting (IIIC.6) into (IIIC.5) and recalling that F_1, F_5 annihilate $\Phi_{\epsilon K}{}^{\lambda\mu}$ while L_0 and Q_0 are eigenoperators with eigenvalues K and $\epsilon_{\max}$:

$$F_{-1}F_{-5}\Phi_{\epsilon K}{}^{\lambda\mu}=[\tfrac{1}{2}L_+L_- -\tfrac{1}{2}(K+\epsilon)]\Phi_{\epsilon K}{}^{\lambda\mu}. \quad \text{(IIIC.7)}$$

Now

$$L_+L_-=-[-\mathbf{L}^2+L_0{}^2-L_0], \quad \text{(IIIC.8)}$$

so, finally, we obtain

$$F_{-1}F_{-5}\Phi_{\epsilon K}{}^{\lambda\mu}=[\tfrac{1}{2}(\mathbf{L}^2-K^2-\epsilon)\Phi_{\epsilon K}{}^{\lambda\mu}]. \quad \text{(IIIC.9)}$$

Next, consider $F_{-1}F_{-1}F_{-4}$. Starting with

$$L_+L_+\Lambda_-=2(F_5+F_{-1})(F_5+F_{-1})F_{-4}, \quad \text{(IIIC.10)}$$

we commute all F_5's to the right where they annihilate $\Phi_{\epsilon K}{}^{\lambda\mu}$. Then

$$\begin{aligned}L_+L_+\Lambda_-\Phi_{\epsilon K}{}^{\lambda\mu}&=2\{F_5[F_5,F_{-4}]+[F_5,F_{-1}]F_{-4}\\&\quad+F_{-1}[F_5,F_{-4}]+F_4[F_5,F_{-4}]\\&\quad+F_{-1}F_{-1}F_{-4}\}\Phi_{\epsilon K}{}^{\lambda\mu} \quad \text{(IIIC.11)}\\&=2\{F_4F_{-4}+F_{-1}F_{-1}F_{-4}\}\Phi_{\epsilon K}{}^{\lambda\mu}.\end{aligned}$$

The second equality in (IIIC.11) follows from the commutation rules and the fact that F_1 also annihilates the wave function. Substituting the eigenvalues (IIIB.13) of F_4F_{-4} into (IIIC.11), we arrive at the equivalence relation:

$$\begin{aligned}&F_{-1}F_{-1}F_{-4}\Phi_{\epsilon K}{}^{\lambda\mu}\\&=[\tfrac{1}{2}L_+L_+\Lambda_- -\tfrac{1}{4}(\mu+K)(\mu-K+2)]\Phi_{\epsilon K}{}^{\lambda\mu}. \quad \text{(IIIC.12)}\end{aligned}$$

The methods described above can be used to obtain equivalence relations for all the operators (IIIB.8) and (IIIB.9). Inspecting the list of relations in Table IV, we note a symmetry among the operators. For example, the equivalent operator for $F_{-1}F_{-1}F_{-4}$ is obtained from the equivalent operator for $F_{-5}F_{-5}F_{-4}$ by changing L_+ to L_-, Λ_- to Λ_+, and K to $-K$. This symmetry is a consequence of the time-reversal transformations (IIIA.7). Consider the relation:

$$f_\sigma(F_{-1},F_{-5},F_4,F_{-4})\Phi_{\epsilon K}{}^{\lambda\mu}=g_\sigma(L_+,L_-,F_4,F_{-4},K)\Phi_{\epsilon K}{}^{\lambda\mu}, \quad \text{(IIIC.13)}$$

which changes under time reversal to

$$\begin{aligned}&f_\sigma(-F_{-5},\,-F_{-1}F_{-4},\,F_4)\Phi_{\epsilon,-K}{}^{\lambda\mu}\\&\quad=g_\sigma(-L_-,\,-L_+,\,F_{-4},\,F_4,\,-K)\Phi_{\epsilon,-K}{}^{\lambda\mu}.\end{aligned}$$

Since this relation holds for all values of K, we can change the sign of K to obtain

$$\begin{aligned}&f_\sigma(-F_{-5},\,-F_{-1},\,F_{-4},\,F_4)\Phi_{\epsilon K}{}^{\lambda\mu}\\&\quad=g_\sigma(-L_-,\,-L_+,\,F_{-4},\,F_4,\,K)\Phi_{\epsilon K}{}^{\lambda\mu}.\end{aligned}$$

The operators $f_\sigma(F_\beta)$ change ϵ by six or twelve units and therefore involve products of even numbers of F_β's with $\beta=-1,\,-5$. Hence, in general,

$$f_\sigma(-F_{-5},\,-F_{-1},\,F_{-4},\,F_4)=f_\sigma(F_{-5},\,F_{-1},\,F_{-4},\,F_4). \quad \text{(IIIC.14)}$$

Returning to Eq. (IIIB.16), we substitute the equivalent operators $g_\sigma(L_\pm,\Lambda_\pm)$ for the $f_\sigma(F_\beta)$ and obtain:

$$\begin{aligned}\mathcal{H}\Phi_{\epsilon K}{}^{\lambda\mu}=\{&C_\epsilon{}^{\lambda\mu K}+\tfrac{1}{2}a_1{}^{\lambda\mu K}\mathbf{L}^2+\tfrac{1}{2}a_2{}^{\lambda\mu K}L_+L_+\Lambda_-+\tfrac{1}{2}a_3{}^{\lambda\mu K}L_-L_-\Lambda_+\\&-\tfrac{1}{2}a_1{}^{\lambda\mu K}(K^2+\epsilon)-\tfrac{1}{4}a_2{}^{\lambda\mu K}(\mu+K)(\mu-K+2)-\tfrac{1}{4}a_3{}^{\lambda\mu K}(\mu-K)(\mu+K+2)\\&+\tfrac{1}{4}[b_1{}^{\lambda\mu K}\mathbf{L}^4+b_2{}^{\lambda\mu K}\mathbf{L}^2L_+L_+\Lambda_-+b_3{}^{\lambda\mu K}\mathbf{L}^2L_-L_-\Lambda_++b_4{}^{\lambda\mu K}L_+L_+L_+L_+\Lambda_-\Lambda_-+b_5{}^{\lambda\mu K}L_-L_-L_-L_-\Lambda_+\Lambda_+]\\&-[b_1{}^{\lambda\mu K}\tfrac{1}{2}(K^2+2\epsilon-5)+\tfrac{3}{8}b_2{}^{\lambda\mu K}(\mu+K)(\mu-K+2)+\tfrac{3}{8}b_3{}^{\lambda\mu K}(\mu-K)(\mu+K+2)]\mathbf{L}^2\\&-[\tfrac{1}{2}b_1{}^{\lambda\mu K}+\tfrac{1}{4}b_2{}^{\lambda\mu K}(K^2-2K+3\epsilon-6)+\tfrac{3}{4}b_4{}^{\lambda\mu K}(\mu+K-2)(\mu-K+4)]L_+L_+\Lambda_-\\&-[\tfrac{1}{2}b_1{}^{\lambda\mu K}+\tfrac{1}{4}b_3{}^{\lambda\mu K}(K^2+2K+3\epsilon-6)+\tfrac{3}{4}b_5{}^{\lambda\mu K}(\mu-K-2)(\mu+K+4)]L_-L_-\Lambda_+\\&+\tfrac{1}{4}b_1{}^{\lambda\mu K}[\mu(\mu+2)+K^2(K^2+4\epsilon-10)+2\epsilon(\epsilon-4)]+\tfrac{3}{8}b_2{}^{\lambda\mu K}[(\mu+K)(\mu-K+2)(\epsilon+K^2-2)]\\&+\tfrac{3}{8}b_3{}^{\lambda\mu K}[(\mu-K)(\mu+K+2)(\epsilon+K^2-2)]+\tfrac{3}{16}b_4{}^{\lambda\mu K}[(\mu+K)(\mu-K+2)(\mu+K-2)(\mu-K+4)]\\&\quad+\tfrac{3}{16}b_5{}^{\lambda\mu K}[(\mu-K)(\mu+K+2)(\mu-K-2)(\mu+K+4)]\}\Phi_{\epsilon K}{}^{\lambda\mu}+\text{terms with }(\lambda'\mu')\neq(\lambda\mu). \quad \text{(IIIC.15)}\end{aligned}$$

TABLE V. Projection relations derived in Sec. IIIC.

$P_M{}^L L_+L_+\Lambda_-|K\rangle=\frac{1}{2}[(L+K)(L-K+1)(L+K-1)(L-K+2)(\mu+K)(\mu-K+2)]^{1/2}P_M{}^L|K-2\rangle$

$P_M{}^L L_-L_-\Lambda_+|K\rangle=\frac{1}{2}[(L-K)(L+K+1)(L-K-1)(L+K+2)(\mu-K)(\mu+K+2)]^{1/2}P_M{}^L|K+2\rangle$

$P_M{}^L L_+L_+L_+L_+\Lambda_-\Lambda_-|K\rangle$
$=\frac{1}{4}[(L+K)(L-K+1)(L+K-1)(L-K+2)(L+K-2)(L-K+3)(L+K-3)(L-K+4)$
$\times(\mu+K-2)(\mu-K+4)(\mu+K)(\mu-K+2)]^{1/2}P_M{}^L|K-4\rangle$

$P_M{}^L L_-L_-L_-L_-\Lambda_-\Lambda_-|K\rangle$
$=\frac{1}{4}[(L-K)(L+K+1)(L-K-1)(L+K+2)(L-K-2)(L+K+3)(L-K-3)(L+K+4)$
$\times(\mu-K-2)(\mu+K+4)(\mu-K)(\mu+K+2)]^{1/2}P_M{}^L|K+4\rangle$

$P_M{}^L|-K\rangle=(-1)^{L+\lambda+\mu}P_M{}^L|K\rangle$

If transitions with $\Delta\epsilon=18$ are neglected, the terms with $(\lambda'\mu')\neq(\lambda\mu)$ are obtained from the sum in brackets simply by substituting λ', μ', for λ, μ.

We now apply the projection operator to (IIIC.15). It is clear that

$$P_M{}^L\mathcal{H}|\lambda\mu\epsilon K\rangle=\mathcal{H}P_M{}^L|\lambda\mu\epsilon K\rangle, \tag{IIIC.16}$$

$$P_M{}^L\mathbf{L}^2|\lambda\mu\epsilon K\rangle=L(L+1)P_M{}^L|\lambda\mu\epsilon K\rangle, \tag{IIIC.17}$$

since $\mathcal{H}$ and L^2 both commute with $P_M{}^L$. In order to evaluate operators of the form $P_M{}^Lg_\sigma(L_\pm,\Lambda_\pm)$, we require the matrix elements of $\Lambda_\pm$ and $L_\pm$:

$$\langle\lambda\mu\epsilon, K+2|\Lambda_+|\lambda\mu\epsilon K\rangle=\langle\lambda\mu\epsilon K|\Lambda_-|\lambda\mu\epsilon, K+2\rangle=\tfrac{1}{2}[(\mu+K+2)(\mu-K)]^{1/2}, \tag{IIIC.18}$$

$$\langle L, M+1|L_+|L, M\rangle=\langle L, M|L_-|L, M+1\rangle=[(L+M+1)(L-M)]^{1/2}. \tag{IIIC.19}$$

In Eq. (IIIC.18) the symbol $|\lambda\mu\epsilon K\rangle$ designates an intrinsic state $\phi_{\epsilon K}{}^{\lambda\mu}$. In Eq. (IIIC.19) the symbol $|L,M\rangle$ designates the projected state $\Psi_M{}^L=P_M{}^L\Phi_{\epsilon K}{}^{\lambda\mu}$, which is an eigenstate of L^2 and L_z with eigenvalues $L(L+1)$ and M, respectively. Comparing (IIIC.18) with (IIIC.19), we see that $\mu/2$ plays the role of L and $K/2$ plays the role of M. This is a consequence of the fact that Λ transforms like the angular momentum operator and has eigenvalue $\mu/2$ for $\epsilon=\epsilon_{\max}$. We note that the matrix elements of $L_\pm$ cannot be easily evaluated in the $\Phi_{\epsilon K}{}^{\lambda\mu}$ representation since L is not a good quantum number. Operators such as $P_M{}^LL_\pm$ can nevertheless be dealt with by writing

$$P_M{}^L=P_M{}^LP_K{}^L, \tag{IIIC.20}$$

$$P_M{}^LL_\pm|\lambda\mu\epsilon K'\rangle=P_M{}^LL_\pm P_{K'}{}^L|\lambda\mu\epsilon K'\rangle. \tag{IIIC.21}$$

Equation (IIIC.20) follows directly from the definition that $P_M{}^L$ projects out L and changes the eigenvalue of L_0 to M. This operation is performed whether one applies $P_M{}^LP_K{}^L$ or $P_M{}^L$ alone. In Eq. (IIIC.21), $P_{M'}{}^L$ is a polynomial in L^2 that projects out L without changing K'. Hence it commutes with $L_\pm$, and (IIIC.21) is simply a special case of (IIIC.20).

The evaluation of $P_M{}^LL_+L_+\Lambda_-|\lambda\mu\epsilon K\rangle$ is now carried out with the aid of (IIIC.18)–(IIIC.21):

$$P_M{}^LL_+L_+\Lambda_-|\lambda\mu\epsilon K\rangle=\tfrac{1}{2}[(\mu+K)(\mu-K+2)]^{1/2}P_M{}^LL_+L_+|\lambda\mu\epsilon, K-2\rangle, \tag{IIIC.22}$$

$$\begin{aligned}P_M{}^LL_+L_+|\lambda\mu\epsilon, K-2\rangle&=P_M{}^LL_+L_+P_{K-2}{}^L|\lambda\mu\epsilon, K-2\rangle=P_M{}^LL_+P_{K-1}{}^L\langle L, K-1|L_+|L, K-2\rangle|\lambda\mu\epsilon, K-2\rangle\\&=P_M{}^LP_K{}^L\langle L, K|L_+|L, K-1\rangle\langle L, K-1|L_+|L, K-2\rangle|\lambda\mu\epsilon, K-2\rangle\\&=[(L+K)(L-K+1)(L+K-1)(L-K+2)]^{1/2}P_M{}^L|\lambda\mu\epsilon, K-2\rangle.\end{aligned} \tag{IIIC.23}$$

Combining (IIIC.22) and (IIIC.23), we have

$$P_M{}^LL_+L_+\Lambda_-|\lambda\mu\epsilon K\rangle=\tfrac{1}{2}[(L+K)(L-K+1)(L+K-1)(L-K+2)(\mu+K)(\mu-K+2)]^{1/2}P_M{}^L|\lambda\mu\epsilon, K-2)\rangle. \tag{IIIC.24}$$

Similarly,

$$P_M{}^LL_-L_-\Lambda_+|\lambda\mu\epsilon K\rangle=\tfrac{1}{2}[(L-K)(L+K+1)(L-K-1)(L+K+2)(\mu-K)(\mu+K+2)]^{1/2}P_M{}^L|\lambda\mu\epsilon, K+2\rangle. \tag{IIIC.25}$$

We note the appearance of $P_M{}^L|\lambda\mu\epsilon, K-2\rangle$ in Eq. (IIIC.22). The projected functions $P_M{}^L|\lambda\mu\epsilon K\rangle$ are defined only for $K\geqslant0$. Negative values of K are dealt with by means of the relation given by Elliott:

$$P_M{}^L|\lambda\mu\epsilon, -K\rangle=(-)^{L+\lambda+\mu}P_M{}^L|\lambda\mu\epsilon K\rangle. \tag{IIIC.26}$$

The relation between the states $P_M{}^L g_\sigma(L_\pm,\Lambda_\pm)|\lambda\mu\epsilon K\rangle$ and the states $\Psi_M{}^L = P_M{}^L|\lambda\mu\epsilon K\rangle$ for any combination of $L_\pm$, $\Lambda_\pm$, is immediately apparent from (IIIC.24) and (IIIC.25). For example,

$$P_M{}^L L_+L_+L_+L_+\Lambda_-\Lambda_-|\lambda\mu\epsilon K\rangle$$
$$=\{\langle L,K|L_+|L,K-1\rangle\langle L,K-1|L_+|L,K-2\rangle\langle L,K-2|L_+|L,K-3\rangle\langle L,K-3|L_+|L,K-4\rangle$$
$$\times\langle\lambda\mu\epsilon,K-4|\Lambda_-|\lambda\mu\epsilon,K-2\rangle\langle\lambda\mu\epsilon,K-2|\Lambda_-|\lambda\mu\epsilon,K\rangle\}P_M{}^L|\lambda\mu\epsilon,K-4\rangle.$$

We are now in a position to investigate the mixing of different K bands by the operators $\Lambda_\pm$. Applying the projection operator to (IIIC.15) and considering only terms with $\Delta\epsilon=0$, 6, we have

$$\mathcal{H}P_M{}^L\phi_{\epsilon K}{}^{\lambda\mu}$$
$$=[\tfrac{1}{2}a_1{}^{\lambda\mu K}L(L+1)+C_\epsilon{}^{\lambda\mu K}-\tfrac{1}{2}a_1{}^{\lambda\mu}(K^2+\epsilon)-\tfrac{1}{4}a_2{}^{\lambda\mu}(\mu+K)(\mu-K+2)-\tfrac{1}{4}a_3{}^{\lambda\mu}(\mu-K)(\mu+K+2)]P_M{}^L\Phi_{\epsilon,K}{}^{\lambda\mu}$$
$$+\tfrac{1}{4}a_2{}^{\lambda\mu K}[(L+K)(L-K+1)(L+K-1)(L-K+2)(\mu+K)(\mu-K+2)]^{1/2}P_M{}^L\Phi_{\epsilon,K-2}{}^{\lambda\mu}$$
$$+\tfrac{1}{4}a_3{}^{\lambda\mu K}[(L-K)(L+K+1)(L-K-1)(L+K+2)(\mu-K)(\mu+K+2)]^{1/2}P_M{}^L\Phi_{\epsilon,K+2}{}^{\lambda\mu}. \quad \text{(IIIC.27)}$$

We compare (IIIC.27) with the matrix elements of an asymmetric rotator[13]:

$$\langle LMK|H_{\text{a.r.}}|LMK\rangle=\tfrac{1}{4}(1/I_A+1/I_B)[L(L+L)-K^2]+K^2/2I_C,$$
$$\langle LM,K+2|H_{\text{a.r.}}|LMK\rangle=\tfrac{1}{8}(1/I_A-1/I_B)[(L-K)(L-K-1)(L+K+1)(L+K+2)]^{1/2}, \quad \text{(IIIC.28)}$$

where the eigenstates $|LMK\rangle$ are the standard top wave functions and

$$H_{\text{a.r.}}=\tfrac{1}{2}\{(L_x')^2/I_A+(L_y')^2/I_B+(L_z')^2/I_C\}.$$

In the limit of large values of μ/K and with $a_2{}^{\lambda\mu K}=a_3{}^{\lambda\mu K}$, we can identify corresponding terms in (IIIC.27) and (IIIC.28):

$$\tfrac{1}{4}a_3{}^{\lambda\mu K}=\tfrac{1}{8}(1/I_A-1/I_B),$$
$$\tfrac{1}{2}a_1{}^{\lambda\mu K}=\tfrac{1}{4}(1/I_A+1/I_B),$$
$$\tfrac{1}{4}(a_2{}^{\lambda\mu K}+a_3{}^{\lambda\mu K})=1/2I_C.$$

If the term in (IIIC.27) are evaluated for a typical force such as that considered in Sec. IVA, we find that $a_2{}^{\lambda\mu}$ and $a_3{}^{\lambda\mu}$ are much smaller than $a_1{}^{\lambda\mu}$. The $\Delta\epsilon=12$ terms also are very small. It thus appears that K is approximately a good quantum number and asymmetry effects are small in the *sd* oscillator level. For the forces considered, however, the $a_i{}^{\lambda\mu K}$ are quite K-dependent and consequently an exact analogy cannot be drawn between these constants and the parameters of an asymmetric rotator.

Finally, the development in Secs. IIIB and IIIC has led to an expression for the matrix elements of $\mathcal{H}$ in the projected (L,M) representation:

$$\mathcal{H}P_M{}^L\Phi_{\epsilon K}{}^{\lambda\mu}=\sum_{(\lambda'\mu')K'}[(\lambda\mu)K|L|(\lambda'\mu')K']\times P_M{}^L\Phi_{\epsilon'K'}{}^{\lambda'\mu'}. \quad \text{(IIIC.29)}$$

The coefficients $[(\lambda\mu)K|L|(\lambda'\mu')K']$, which, in general, are not symmetric, incorporate the ${}^{\lambda\mu}a_i{}^{\lambda'\mu'K}$, ${}^{\lambda\mu}b_j{}^{\lambda'\mu'K}$ of Sec. IIIC together with matrix elements of $L_\pm$, $\Lambda_\pm$, introduced by projection relations such as (IIIC.24) and (IIIC.25). We note that these coefficients have been expressed in a form such that only the inhomogeneous matrix elements of $\mathcal{H}$ in the intrinsic (K,ϵ,Λ) representation remain to be determined.

The coefficients $[(\lambda\mu)K|L|(\lambda'\mu')K']$ are here defined as the matrix elements of $\mathcal{H}$. Since the representation spanned by the $P_M{}^L\Phi_{\epsilon K}{}^{\lambda\mu}$ is not orthogonal, the matrix elements so defined are not, in general, equal to the expression:

$$\langle P_M{}^L\Phi_{\epsilon K}{}^{\lambda\mu}|\mathcal{H}|P_M{}^L\Phi_{\epsilon'K'}{}^{\lambda'\mu'}\rangle.$$

We can, nevertheless, write an eigenvector of $\mathcal{H}$ as

$$\Psi_E{}^{L,M}=\sum_{\lambda'\mu'K'}A^L(\lambda'\mu',K')P_M{}^L\Phi_{\epsilon'K'}{}^{\lambda'\mu'}, \quad \text{(IIIC.30)}$$

where the $P_M{}^L\Phi_{\epsilon'K'}{}^{\lambda'\mu'}$ form a complete set. From (IIIC.29) and (IIIC.30), the amplitude $A^L(\lambda'\mu',K')$ must satisfy the equation:

$$\sum_{\lambda'\mu'K'}A^L(\lambda'\mu',K')[(\lambda'\mu')K'|L|(\lambda\mu)K]=EA^L(\lambda,\mu,K). \quad \text{(IIIC.31)}$$

Hence, the secular equation for the eigenvalues E assumes the usual form

$$\text{Det}\{[(\lambda'\mu')K'|L|(\lambda\mu)K]-E\delta_{\lambda\lambda'}\delta_{\mu\mu'}\delta_{KK'}\}=0.$$

IV. INHOMOGENEOUS MATRIX ELEMENTS NEGLECTING SPIN

In this section, we express a generalized Hamiltonian for the *sd* oscillator level without spin as a sum of ten scalar operators X^σ. We then tabulate the matrix elements of the X^σ in the $|l_1,l_2,L\rangle$ representation as well as the inhomogeneous matrix elements

$$\langle\lambda\mu\epsilon K|f^+(F_\beta)X^\sigma|\lambda\mu\epsilon K\rangle$$

for each of the eight operators f_i, f_j defined in Sec. III.

A. The Operators X^σ

Let $|l_1,l_2,L\rangle_x$ be the wave function of two particles in the *sd* oscillator shell with angular momenta l_1 and

[13] L. D. Landau and E. M. Lifschitz, *Quantum Mechanics Nonrelativistic Theory* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts), p. 373 ff.

TABLE VI. Matrix elements M_ρ, as defined in (IVA.1) of operators $X^{(i)}$, as defined in (IVA.8).[a]

ρ \ $X_{ij}^{(\sigma)}$	$X^{(1)}$	$X^{(2)}$	$X^{(3)}$	$X^{(5)}$	$X^{(6)}$	$X^{(7)}$	$X^{(8)}$	$X^{(9)}$	$X^{(10)}$	$X^{(11)}$
1	20			1	1					
2	4									112
3	18		12	1	1		144/5	144	−168/5	25
4	24	6	6	1	1	36		36		112
5	2									20
6	14	6	12	1	1	36	288/35	144	96/5	−75/14
7	28	20	12	1	1	400	8/35	144	12/5	50/7
8	16	2	12	−1	1	4	−96/5	144		25/2
9	16	6	6	−1	1	36		36		−112
10	16	12	12	−1	1	144	−72/35	144	−12	−100/7

[a] Blank spaces denote vanishing matrix elements.

l_2 coupled to a total angular momentum L. If $l_1=l_2$, the wave function is symmetric under space exchange for even values of L and antisymmetric for odd L. If $l_1\neq l_2$, both symmetric and antisymmetric combinations are possible and these are distinguished by setting the subscript x equal to s or a. A two-body scalar symmetric Hamiltonian in the space of the sd shell without spin is then defined by its ten matrix elements M_ρ:

$$\begin{aligned} &M_1\equiv\langle s^2,0|\mathcal{H}|s^2,0\rangle, && M_6\equiv\langle d^2,2|\mathcal{H}|d^2,2\rangle,\\ &M_2\equiv(1/5^{1/2})\langle s^2,0|\mathcal{H}|d^2,0\rangle_s, && M_7\equiv\langle d^2,4|\mathcal{H}|d^2,4\rangle,\\ &M_3\equiv\langle d^2,0|\mathcal{H}|d^2,0\rangle, && M_8\equiv\langle d^2,1|\mathcal{H}|d^2,1\rangle,\\ &M_4\equiv\langle sd,2|\mathcal{H}|sd,2\rangle_s, && M_9\equiv\langle sd,2|\mathcal{H}|sd,2\rangle_a,\\ &M_5\equiv[1/(14)^{1/2}]\langle sd,2|\mathcal{H}|d^2,2\rangle_s, && M_{10}\equiv\langle d^2,3|\mathcal{H}|d^2,3\rangle. \end{aligned} \tag{IVA.1}$$

Alternatively, we may express an arbitrary two-body Hamiltonian for the sd shell without spin in terms of ten linearly independent scalar two-body operators $X_{ij}^{(\sigma)}$, such that

$$M_\rho=\sum_\sigma g_\sigma X_\rho^{(\sigma)}, \tag{IVA.2}$$

where the $X_\rho^{(\sigma)}$ are matrix elements defined by substituting the operators $X_{ij}^{(\sigma)}$ for $\mathcal{H}$ in (IVA.1). The coefficients g_σ in (IVA.2) depend linearly on the M_ρ:

$$g_\sigma=\sum_\rho S_\rho^\sigma M_\rho. \tag{IVA.3}$$

Comparing (IVA.2) and (IVA.3), we see that

$$\sum_\rho S_\rho^\sigma X_\rho^{(\sigma')}=\delta_{\sigma\sigma'}. \tag{IVA.4}$$

It is convenient to construct the $X_{ij}^{(\sigma)}$ from the operators L_μ and Q_μ. For this purpose we need the relations:

$$(Q\times Q)_q^{(K)}\equiv\sum_{\mu,\nu} Q_\mu Q_\nu\begin{bmatrix}2 & 2 & K\\ \mu & \nu & q\end{bmatrix}, \tag{IVA.5}$$

$$(Q\times L)_q^{(K)}=\sum_{\mu,\nu} Q_\mu L_\nu\begin{bmatrix}2 & 1 & K\\ \mu & \nu & q\end{bmatrix}, \tag{IVA.6}$$

$$\sum_q(-1)^q T_q^{(K)}A_{-q}^{(K)}\equiv T^{(K)}\cdot A^{(K)}, \tag{IVA.7}$$

where the symbol in brackets is the vector coupling coefficient and $T^{(K)}$, $A^{(K)}$, are irreducible tensor operator of rank K. The ten scalar two-body operators $X_{ij}^{(\sigma)}$ are defined as follows:

$X_{ij}^{(1)}\equiv 9C_{ij}$; nine times the Casimir operator of particles i and j.

$X_{ij}^{(2)}\equiv(L_{ij})^2=(l_i+l_j)^2$; the square of the total angular momentum of particles i and j.

$X_{ij}^{(3)}\equiv l_i^2+l_j^2$; the sum of the squares of the single-particle angular momenta.

$X_{ij}^{(5)}\equiv p_{ij}^x$; the space exchange operator of particles i and j.

$X_{ij}^{(6)}\equiv 1$; unity.

$X_{ij}^{(7)}\equiv(L_{ij})^4$; the square of $X_{ij}^{(2)}$.

$X_{ij}^{(8)}\equiv\frac{1}{9}[(Q\times Q)_i^{(4)}\cdot(Q\times Q)_j^{(4)}]$; one-ninth of the scalar product of the rank-4 tensors $(Q\times Q)$ for the ith and jth particles.

$X_{ij}^{(9)}\equiv(l_i^2+l_j^2)^2$; the square of $X_{ij}^{(3)}$. (IVA.8)

$X_{ij}^{(10)}\equiv(Q\times L)_i^{(3)}\cdot(Q\times L)_j^{(3)}$.

$X_{ij}^{(11)}\equiv(Q\times Q)_i^{(2)}\cdot(Q\times Q)_j^{(2)}$.

(The number 4 is reserved for the one-body spin-orbit operator discussed in Sec. V.)

By means of standard Racah techniques, all matrix elements $X_\rho^{(\sigma)}$ involving the operators L_μ and Q_μ can be expressed in terms of the reduced single-particle matrix elements $(l\|L\|l')$, $(l\|Q\|l')$. For a harmonic oscillator, the reduced matrix elements are:

$$\begin{aligned} &(d\|Q\|d)=-(70)^{1/2}, && (d\|L\|d)=(30)^{1/2},\\ &(d\|Q\|s)=(s\|Q\|d)=-2(10)^{1/2}, && (s\|L\|s)=0. \end{aligned} \tag{IVA.9}$$

With the aid of (IVA.9) and the standard Racah formulas, we evaluate the matrix elements $X_\rho^{(\sigma)}$ to obtain the array given in Table VI. The S_ρ^σ are obtained by inverting this and they are given in Table VII.

As an example of the expansion of a Hamiltonian in the operators $X_{ij}^{(\sigma)}$, we consider the Yukawa potential with a Serber exchange:

$$V=V_0\frac{e^{-r/a}}{r/a}(1+P^x). \tag{IVA.10}$$

TABLE VII. S_ρ^σ, as defined in (IVA.3). The superscript labels the columns and the subscript labels the rows.

$\rho \backslash S^\sigma$	S^1	S^2	S^3	S^5	S^6	S^7	S^8	S^9	S^{10}	S^{11}
1			$\frac{-1}{4}$		1			$\frac{1}{72}$		
2	$\frac{-5}{36}$	$\frac{-977}{3360}$	$\frac{341}{3360}$	-1	$\frac{34}{9}$	$\frac{13}{672}$	$\frac{1}{16}$	$\frac{-157}{10080}$	$\frac{1}{24}$	$\frac{1}{72}$
3		$\frac{-34}{525}$	$\frac{149}{2100}$			$\frac{1}{420}$	$\frac{1}{80}$	$\frac{-43}{12600}$	$\frac{-1}{120}$	
4		$\frac{47}{420}$	$\frac{69}{420}$	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{-1}{168}$	$\frac{-1}{32}$	$\frac{-31}{2520}$	$\frac{-1}{48}$	
5	$\frac{7}{9}$	$\frac{241}{336}$	$\frac{-317}{336}$		$\frac{-140}{9}$	$\frac{-19}{336}$		$\frac{95}{1008}$		$\frac{-1}{36}$
6		$\frac{33}{980}$	$\frac{-178}{2940}$			$\frac{-1}{392}$	$\frac{1}{56}$	$\frac{31}{4410}$	$\frac{1}{42}$	
7		$\frac{-99}{1225}$	$\frac{369}{4900}$			$\frac{3}{490}$	$\frac{1}{1120}$	$\frac{-51}{9800}$	$\frac{3}{560}$	
8		$\frac{-3}{28}$	$\frac{234}{2100}$			$\frac{3}{840}$	$\frac{-1}{40}$	$\frac{-3}{700}$		
9		$\frac{-47}{420}$	$\frac{71}{420}$	$\frac{-1}{2}$	$\frac{1}{2}$	$\frac{1}{168}$	$\frac{1}{32}$	$\frac{-13}{840}$	$\frac{1}{48}$	
10		$\frac{23}{105}$	$\frac{-589}{2100}$			$\frac{-1}{105}$	$\frac{-1}{160}$	$\frac{83}{4200}$	$\frac{-1}{48}$	

Following Elliott and Flowers, we chose $V_0=-45$ MeV, $a=1.37\times10^{-13}$ cm, and the harmonic oscillator length parameter $b=1.64\times10^{-13}$ cm for the *sd* shell. The matrix elements M_ρ defined by (IVA.1) can then be calculated and the coefficients g_σ determined with the aid of the results in Table VII. We find the results given in Table VIII. The coefficient g_1 is sufficiently large that eigenstates of the Casimir operator comprise a good zero-order scheme. That this is true for various potentials used in intermediate shell-model calculations can easily be verified and will be demonstrated in a subsequent paper reporting numerical results for the *sd* shell. In general, the usefulness of the Elliott scheme for the *sd* shell is a consequence of the fact that eigenstates of $\mathcal{H}$ are approximate eigenstates of C.

TABLE VIII. Matrix elements M_ρ, as defined in (IVA.1), g_σ, as defined in (IVA.2) and $X^{(\sigma)}$, as defined in (IVA.8) for the potential given by (IVA.9).

M_ρ (MeV)	g_σ (MeV)	$X_{ij}^{(\sigma)}$
$M_1=-4.24035$	$g_1=-0.21121$	$9C_{ij}$
$M_2=-0.74420$	$g_2=0.17730$	$(l_i+l_j)^2$
$M_3=-7.09785$	$g_3=0.18560$	$(l_i^2+l_j^2)$
$M_4=-3.59415$	$g_5=-1.05288$	p_{ij}^x
$M_5=-0.40455$	$g_6=1.03674$	1
$M_6=-3.18780$	$g_7=-0.0016949$	$(L_{ij})^4$
$M_7=-3.72330$	$g_8=-0.083167$	$(Q\times Q)_i^4\cdot(Q\times Q)_j^4/9$
$M_8=0$	$g_9=-0.0200182$	$(l_i^2+l_j^2)^2$
$M_9=0$	$g_{10}=0.0071742$	$(Q\times L)_i^3\cdot(Q\times L)_j^3$
$M_{10}=0$	$g_{11}=0.000899$	$(Q\times Q)_i^2\cdot(Q\times Q)_j^2$

We must now analyze the structure of the operators

$$\sum_{j=1}^{N}\sum_{i<j} X_{ij}^{(\sigma)}$$

for a system of N particles in the *sd* shell. Considering first $X^{(1)}$, from the definition (IIB.9) of the Casimir operator, we have

$$X_{ij}^{(1)}\equiv 9C_{ij} = \tfrac{1}{4}[3(l_i+l_j)^2+(Q_i+Q_j)\cdot(Q_i+Q_j)]. \quad \text{(IVA.11)}$$

Hence,

$$\sum_{i<j}^{N} X_{ij}^{(1)}$$
$$=\tfrac{1}{4}\sum_{i<j}^{N}[3l_i\cdot l_i+6l_i\cdot l_j+3l_j\cdot l_j+Q_i\cdot Q_i+2Q_i\cdot Q_j+Q_j\cdot Q_j]$$
$$=\tfrac{1}{4}\sum_{i<j}[6l_i\cdot l_j+2Q_i\cdot Q_j]+\frac{(N-1)}{4}\sum_{i=1}^{N}(3l_i\cdot l_i+Q_i\cdot Q_i)$$
$$=\tfrac{1}{4}\sum_{i,j=1}^{N}(3l_i\cdot l_j+Q_i\cdot Q_j)+\frac{(N-2)}{4}\sum_{i=1}^{N}(3l_i\cdot l_i+Q_i\cdot Q_i). \quad \text{(IVA.12)}$$

But nine times the Casimir operator of N particles is

$$9C=\tfrac{1}{4}\sum_{i,j}^{N}(3l_i\cdot l_j+Q_i\cdot Q_j)=\frac{3\mathbf{L}\cdot\mathbf{L}+Q\cdot Q}{4}, \quad \text{(IVA.13)}$$

where

$$\mathbf{L}=\sum_i l_i, \quad Q=\sum_i Q_i.$$

Combining (IVA.12) and (IVA.13), we have

$$\sum_{i<j}^{N} X_{ij}^{(1)}=9C+\frac{(N-2)}{4}\sum_{i=1}^{N}(3l_i\cdot l_i+Q_i\cdot Q_i). \quad \text{(IVA.14)}$$

The wave function of a single particle in the sd shell belongs to the irreducible representation $(\lambda,\mu)=(2,0)$. Hence, the single-particle Casimir operator,

$$C_i\equiv\frac{1}{36}(3l_i\cdot l_i+Q_i\cdot Q_i),$$

can be equated to its eigenvalue of 10/9, in accordance with Eq. (IIB.11). So

$$\sum_{i<j}^{N} X_{ij}^{(1)}=9C+10N(N-2). \quad \text{(IVA.15)}$$

Next,

$$\sum_{i<j} X_{ij}^{(2)}=\sum_{i<j}(l_i+l_j)^2=\mathbf{L}\cdot\mathbf{L}+(N-2)\sum_{i=1}^{N} l_i^2, \quad \text{(IVA.16)}$$

$$\sum_{i<j} X_{ij}^{(3)}=\sum_{i<j}(l_i^2+l_j^2)=(N-1)\sum_{i=1}^{N} l_i^2, \quad \text{(IVA.17)}$$

$$\sum_{i<j} X_{ij}^{(6)}=\frac{N(N-1)}{2}, \quad \text{(IVA.18)}$$

$$\sum_{i<j} X_{ij}^{(7)}=\sum_{i<j}(l_i+l_j)^4. \quad \text{(IVA.19)}$$

The operator (IVA.19) can be manipulated into a factored form which is much more convenient for computation. Define l_μ^i as the μth spherical component of the orbital angular momentum operator for the ith particle. Then

$$(l_i\cdot l_i)^2=\sum_{\mu,\nu} l_\mu^i l_{-\mu}^j l_\nu^i l_{-\nu}^j(-)^{\mu+\nu}. \quad \text{(IVA.20)}$$

The various tensor products $(l\times l)$ of rank K are given by

$$(l^i\times l^i)_g^{(K)}\equiv\sum_{\sigma\rho}\begin{bmatrix}1 & 1 & K\\ \rho & \sigma & g\end{bmatrix} l_\rho^i l_\sigma^i. \quad \text{(IVA.21)}$$

It follows from (IVA.21) and the unitary property of the vector coupling coefficients that

$$l_\mu^i l_\nu^i=\sum_\lambda\begin{bmatrix}1 & 1 & \lambda\\ \mu & \nu & \mu+\nu\end{bmatrix}(l^i\times l^i)_{\mu+\nu}^{\lambda}. \quad \text{(IVA.22)}$$

Substituting (IVA.22) into (IVA.20) and noting that

$$(l\times l)_0^0=-l^2/\sqrt{3}; \quad (l\times l)_\mu^1=-l_\mu/\sqrt{2}, \quad \text{(IVA.23)}$$

we obtain the factored form:

$$(l_i\cdot l_j)^2=\tfrac{1}{3}l_i^2l_j^2-\tfrac{1}{2}(l_i\cdot l_j)+[(l^i\times l^i)^{(2)}\cdot(l^j\times l^j)^{(2)}]+\delta_{ij}l_il_j. \quad \text{(IVA.24)}$$

Returning to

$$(l_i+l_j)^4=l_i^4+l_j^4+4(l_i\cdot l_j)^2+4(l_i\cdot l_j)(l_i^2+l_j^2)+2l_i^2l_j^2, \quad i\neq j \quad \text{(IVA.25)}$$

we note that the operator l_i^4 is equivalent to $6l_i^2$ for the sd shell since only s and d orbitals are present. Similarly, the expression $(l_i\cdot l_j)(l_i^2+l_j^2)$ is equivalent to $12(l_i\cdot l_j)$. So, in the sd shell,

$$(l_i+l_j)^4=6(l_i^2+l_j^2)+4(l_i\cdot l_j)^2+48(l_i\cdot l_j)+2l_i^2l_j^2. \quad \text{(IVA.26)}$$

Write

$$(l_i\cdot l_j)=\tfrac{1}{2}(l_i+l_j)^2-l_i^2-l_j^2]. \quad \text{(IVA.27)}$$

Substituting (IVA.27) into (IVA.26) we obtain

$$(l_i+l_j)^4=-18(l_i^2+l_j^2)+24(l_i+l_j)^2+4(l_i\cdot l_j)^2+2l_i^2l_j^2. \quad \text{(IVA.28)}$$

For the sd shell, we have

$$\sum_{i<j} l_i^2l_j^2=(\sum_i l_i^2)^2-6\sum_i l_i^2. \quad \text{(IVA.29)}$$

Summing (IVA.28) over $i<j$ and using (IVA.24), (IVA.29), and (IVA.16), we obtain

$$\sum_{i<j}^{N}(l_i+l_j)^4$$
$$=\sum_{i<j} X_{ij}^{(7)}=23\mathbf{L}\cdot\mathbf{L}+(6N-46)\sum_i l_i^2+(5/3)(\sum_i l_i^2)^2$$
$$+2\sum_{i,j}[(l^i\times l^i)^{(2)}\cdot(l^j\times l^j)^{(2)}]. \quad \text{(IVA.30)}$$

We now have a factored form for $\sum_{i<j} X_{ij}^{(7)}$ consisting of sums and products of symmetric one-body operators. The new operator which appears is $\sum_i(l^i\times l^i)_\mu^{(2)}$.

The operators $X^{(8)}$, $X^{(10)}$, and $X^{(11)}$ are already factored. For $X^{(9)}$, we note that

$$\sum_{i<j}^{N} X_{ij}^{(9)}=\sum_{i<j}^{N}(l_i^2+l_j^2)^2$$
$$=(\sum_i^N l_i^2)^2+6(N-2)\sum_i^N l_i^2. \quad \text{(IVA.31)}$$

The total Hamiltonian for N particles in the sd shell,

$$\mathcal{H}=\sum_{i<j}^{N}\sum_{\sigma} g_\sigma X_{ij}^{(\sigma)},$$

can now be written as

$$\mathcal{H}=9g_1C+(g_2+23g_7)\mathbf{L}\cdot\mathbf{L}+[(g_2+6g_9)(N-2)+g_3(N-1)]\sum_i l_i^2+g_9(\sum_i l_i^2)^2+g_7[\sum_{i<j} X_{ij}^7-23\mathbf{L}\cdot\mathbf{L}]$$
$$+\frac{g_8}{9}\sum_{i<j}[(Q\times Q)_i^{(4)}\cdot(Q\times Q)_j^{(4)}]+g_{10}\sum_{i<j}[(Q\times L)_i^{(3)}\cdot(Q\times L)_j^{(3)}]+g_{11}\sum_{i<j}[(Q\times Q)_i^{(2)}\cdot(Q\times Q)_j^{(2)}]$$
$$+g_5\sum_{i<j} p_{ij}^\times+\left[10g_1N(N-2)+g_6\frac{N(N-1)}{2}\right]1. \quad \text{(IVA.32)}$$

B. Inhomogeneous Forms

In Sec. III, we expressed the matrix elements of $\mathcal{H}$ in terms of homogeneous matrix elements which could be calculated immediately, and inhomogeneous matrix elements which explicitly involve the Hamiltonian. These inhomogeneous forms can now be dealt with by means of the expansion (IVA.32). Consider first the operator $\sum_{i=i}^N l_i^2$. The matrix of l_i^2 in the ϕ_k representation is:

$$\begin{array}{c|cccccc} & \phi_0 & \phi_1 & \phi_{-1} & \phi_2 & \phi_{0'} & \phi_{-2} \\ \hline \phi_0 & 4 & 0 & 0 & 0 & 2\sqrt{2} & 0 \\ \phi_1 & 0 & 6 & 0 & 0 & 0 & 0 \\ \phi_{-1} & 0 & 0 & 6 & 0 & 0 & 0 \\ \phi_2 & 0 & 0 & 0 & 6 & 0 & 0 \\ \phi_{0'} & 2\sqrt{2} & 0 & 0 & 0 & 2 & 0 \\ \phi_{-2} & 0 & 0 & 0 & 0 & 0 & 6 \end{array} \quad \text{(IVB.1)}$$

Hence, in terms of the operators $\chi_\mu^{\dagger(i)}$, $\chi_\nu^{(i)}$, we have

$$l_i^2=2[2n_0^{(i)}+3n_1^{(i)}+3n_{-1}^{(i)}+n_{0'}^{(i)}+3n_2^{(i)}+3n_{-2}^{(i)}$$
$$+\sqrt{2}\chi_0^{\dagger(i)}\chi_{0'}^{(i)}+\sqrt{2}\chi_{0'}^{\dagger(i)}\chi_0^{(i)}], \quad \text{(IVB.2)}$$

where $n_\alpha^{(i)}\equiv\chi_\alpha^{\dagger(i)}\chi_\alpha^{(i)}$. Referring to the diagram in Fig. 3 of the ϕ_K states, we see that $\chi_0^{\dagger(i)}\chi_{0'}^{(i)}$ increases ϵ by six units whereas $\chi_{0'}^{\dagger(i)}\chi_0^{(i)}$ decreases ϵ by six units. By definition, the operators $n_\alpha^{(i)}$ leave ϵ unchanged. The nonvanishing inhomogeneous matrix elements involving $\sum_i l_i^2$ for the case $(\lambda'\mu')=(\lambda\mu)$ are, therefore,

$$2[\langle\lambda\mu\epsilon K|2n_0+3n_1+3n_{-1}+n_{0'}$$
$$+3n_2+3n_{-2}|\lambda\mu\epsilon K\rangle], \quad \text{(IVB.3)}$$

$$2\sqrt{2}[\lambda\mu\epsilon K|F_5F_1\textstyle\sum_i \chi_{0'}^{\dagger(i)}\chi_0^{(i)}|\lambda\mu\epsilon K\rangle],$$
$$2\sqrt{2}[\langle\lambda\mu\epsilon K|F_4F_1F_1\textstyle\sum_i \chi_{0'}^{\dagger(i)}\chi_0^{(i)}|\lambda\mu\epsilon K\rangle], \quad \text{(IVB.4)}$$
$$2\sqrt{2}[\langle\lambda\mu\epsilon K|F_{-4}F_5F_5\textstyle\sum_i \chi_{0'}^{\dagger(i)}\chi_0^{(i)}|\lambda\mu\epsilon K\rangle].$$

Again, we have denoted the N-particle state

$$\Phi([f](\lambda,\mu)K,\epsilon_{\max})$$

simply as $|\lambda\mu\epsilon K\rangle$, and we define the many-particle operator $n_\alpha\equiv\sum_i n_\alpha^{(i)}$.

We now restrict the analysis to states in which ϕ_2, ϕ_{-2}, $\phi_{0'}$, are not occupied; that is,

$$\chi_\alpha^{(i)}|\lambda\mu\epsilon K\rangle=0;\quad \alpha=\pm2,0'. \quad \text{(IVB.5)}$$

Equation (IVB.5) holds for most of the bands of interest in the first half of the sd shell. If other bands must be considered, they are generated from the bands with $Q_{\pm2}$, $Q_{0'}$ unoccupied by applying suitable creation and annihilation operators. Then

$$N=\sum_{i,\alpha} n_\alpha^{(i)}=n_0+n_1+n_{-1}, \quad \text{(IVB.6)}$$

$$Q_0=\sum_i Q_0^{(i)}=4n_0+n_1+n_{-1}, \quad \text{(IVB.7)}$$

$$L_0=\sum_i L_0^{(i)}=n_1-n_{-1}. \quad \text{(IVB.8)}$$

Equations (IVB.7) and (IVB.8) follow from (IIIA.4) with $n_{0'}=n_2=n_{-2}=0$. Hence,

$$n_0=\tfrac{1}{3}(Q_0-N), \quad \text{(IVB.9)}$$

$$n_1=\tfrac{1}{2}(\tfrac{4}{3}N-\tfrac{1}{3}Q_0+L_0), \quad \text{(IVB.10)}$$

$$n_{-1}=\tfrac{1}{2}(\tfrac{4}{3}N-\tfrac{1}{3}Q_0-L_0). \quad \text{(IVB.11)}$$

Since $|\lambda\mu\epsilon K\rangle$ is an eigenvector of L_0, Q_0, and N, with eigenvalues K, $2\lambda+\mu$, and N, it is also an eigenvector of the operators n_0, n_1, n_{-1}. We shall designate the eigenvalues of the operators n_α also as n_α. The matrix elements (IVB.3) can then be evaluated immediately:

$$2[\langle\lambda\mu\epsilon K|\textstyle\sum_i l_i^2|\lambda\mu\epsilon K\rangle]=4n_0+6(n_1+n_{-1}). \quad \text{(IVB.12)}$$

The matrix elements (IVB.4) are evaluated with the aid of (IIIA.4) and the commutation rules (IIIA.2). For example,

$$2\sqrt{2}\langle\lambda\mu\epsilon K|F_5F_1\textstyle\sum_i \chi_{0'}^{\dagger(i)}\chi_0^{(i)}|\lambda\mu\epsilon K\rangle$$
$$=2\sqrt{2}\langle\lambda\mu\epsilon K|F_5[F_1,\textstyle\sum_i \chi_{0'}^{\dagger(i)}\chi_0^{(i)}]|\lambda\mu\epsilon K\rangle$$
$$+2\sqrt{2}\langle\lambda\mu\epsilon K|F_5\textstyle\sum_i \chi_{0'}^{\dagger(i)}\chi_0^{(i)}F_1|\lambda\mu\epsilon K\rangle. \quad \text{(IVB.13)}$$

The second term on the right vanishes since F_1 annihilates $|\lambda\mu\epsilon K\rangle$. Substituting for F_1 the first of Eqs. (IIIA.4), we evaluate the commutator

$$[F_1, \textstyle\sum_i \chi_{0'}^{\dagger(i)}\chi_0^{(i)}] = -\textstyle\sum_i(-\sqrt{2}\chi_{0'}^{\dagger(i)}\chi_1^{(i)}+\chi_{-1}^{\dagger(i)}\chi_0^{(i)}). \quad \text{(IVB.14)}$$

Since F_5 also annihilates $|\lambda\mu\epsilon K\rangle$, we use the same technique to obtain

$$\begin{aligned} F_5(\textstyle\sum_i -\sqrt{2}\chi_{0'}^{\dagger(i)}\chi_1^{(i)}+\chi_{-1}^{\dagger(i)}\chi_0^{(i)}|\lambda\mu\epsilon K\rangle &= -\sqrt{2}(\textstyle\sum_i \chi_0^{\dagger(i)}\chi_0^{(i)}-\chi_{-1}^{\dagger(i)}\chi_{-1}^{(i)} \\ &\quad -\chi_1^{\dagger(i)}\chi_1^{(i)}+\chi_{0'}^{\dagger(i)}\chi_{0'}^{(i)})|\lambda\mu\epsilon K\rangle \\ &= -\sqrt{2}(n_0-n_{-1}-n_1)|\lambda\mu\epsilon K\rangle. \end{aligned} \quad \text{(IVB.15)}$$

Combining (IVB.15), (IVB.14), and (IVB.13), we have

$$\begin{aligned} \langle\lambda\mu\epsilon K|F_5F_1\textstyle\sum_i l_i^2|\lambda\mu\epsilon K\rangle &= 12\sqrt{2}\lambda\mu\epsilon K|F_5F_1\textstyle\sum_i\chi_{0'}^{\dagger(i)}\chi_0^{(i)}|\lambda\mu\epsilon K\rangle \\ &= 4(n_0-n_1-n_{-1}). \end{aligned} \quad \text{(IVB.16)}$$

As an example of the treatment of a two-body operator, consider

$$\begin{aligned} \sum_{i<j}(Q\times Q)_i^{(4)}\cdot(Q\times Q)_j^{(4)} &= \tfrac{1}{2}[\sum_i (Q\times Q)_i^{(4)}][\sum_j (Q\times Q)_j^{(4)}] \\ &\quad -\tfrac{1}{2}\sum_i (Q\times Q)_i^{(4)}\cdot(Q\times Q)_i^{(4)}. \end{aligned} \quad \text{(IVB.17)}$$

The one-body operator $(Q\times Q)_i^{(4)}$ can be expanded in terms of the generators Q_μ:

$$(Q\times Q)_{\mu i}^{(4)} = \sum_{\rho\sigma}\begin{bmatrix} 2 & 2 & 4 \\ \rho & \sigma & \mu \end{bmatrix} Q_\rho^i Q_\sigma^i. \quad \text{(IVB.18)}$$

For $\mu=0$, we have

$$\begin{aligned} (Q\times Q)_{(0)i}^{(4)} &= (18/35)^{1/2}Q_0^iQ_0^i+[6/(70)^{1/2}](F_4^iF_{-4}^i+F_{-4}^iF_4^i) \\ &\quad -\frac{3\times 8^{1/2}}{(35)^{1/2}}[(F_1^i-F_{-5}^i)(F_{-1}^i-F_5^i) \\ &\quad +(F_{-1}^i-F_5^i)(F_1^i-F_{-5}^i)], \end{aligned} \quad \text{(IVB.19)}$$

TABLE IX. Inhomogeneous forms involving the operators in (IVA.35) for the case $(\lambda'\mu')=(\lambda\mu)$.

	$\sum_{i>j}(l\times Q)_i^{(3)}\cdot(l\times Q)_j^{(3)}$	$\sum_{i>j}(Q\times Q)_i^{(2)}\cdot(Q\times Q)_j^{(2)}$	$\sum_{i>j}(Q\times Q)_i^{(4)}\cdot(Q\times Q)_j^{(4)}$
$\langle H\rangle$	$\frac{24}{5}K^2-8(n_1+n_{-1})$	$\frac{1}{28}(44n_0+5n_1+5n_{-1})(44n_0+5n_1+5n_{-1}-44)$ $-\frac{1407}{28}(n_1+n_{-1})+\frac{75}{28}[\mu(\mu+2)-K^2]$	$\frac{576}{35}(\xi^2-n_0)+\frac{144}{7}[\mu(\mu+2)-K^2]$ $-\frac{3456}{35}(n_1+n_{-1})$
$\langle F_{15}H\rangle$	$\frac{32}{5}[\mu(\mu+2)-K^2]-\frac{72}{5}K^2$ $+\frac{32}{5}(\xi^2-n_0)-\frac{8}{5}(n_1+n_{-1})$	$-\frac{3}{56}(44n_0+5n_1+5n_{-1}+16\xi)(44n_0+5n_1+5n_{-1}-44)$ $-\frac{2139}{28}(n_1+n_{-1})+\frac{645}{56}\mu(\mu+2)+\frac{1878}{56}K^2$	$-\frac{1728}{35}(\xi^2-n_0)-\frac{36}{7}\mu(\mu+2)$ $+\frac{180}{7}K^2+\frac{2232}{35}(n_1+n_{-1})$
$\frac{\langle F_{-455}H\rangle}{\langle F_{-44}\rangle}$	$\frac{4}{5}(\xi+12K-8)$	$-\frac{3}{14}(452n_0+185n_1+185n_{-1}-1220+25K)$	$-\frac{288}{35}(12\xi+5K+8)$
$\frac{\langle F_{411}H\rangle}{\langle F_{4-4}\rangle}$	$\frac{4}{5}(\xi-12K-8)$	$-\frac{3}{14}(452n_0+185n_1+185n_{-1}-1220-25K)$	$-\frac{288}{35}(12\xi-5K+8)$
$\langle F_{1155}H\rangle$	$-32(\xi^2-n_0+\mu[\mu+2]-2K^2)$	$\frac{576}{7}(\xi^2-n_0)+72\mu(\mu+2)-\frac{936}{7}K^2$	$\frac{576}{35}[9(\xi^2-n_0)+7\mu(\mu+2)-12K^2]$
$\frac{\langle F_{45111}H\rangle}{\langle F_{4-4}\rangle}$	$-96(\xi+1-K)$	$\frac{432}{7}(4\xi+3-3K)$	$\frac{1728}{35}(9\xi+5-5K)$
$\frac{\langle F_{-41555}H\rangle}{\langle F_{-44}\rangle}$	$-96(\xi+1+K)$	$\frac{432}{7}(4\xi+3+3K)$	$\frac{1728}{35}(9\xi+5+5K)$
$\frac{\langle F_{441111}H\rangle}{\langle F_{44-4-4}\rangle}$	0	$\frac{864}{7}$	$\frac{13\,824}{35}$
$\frac{\langle F_{-4-45555}H\rangle}{\langle F_{-4-444}\rangle}$	0	$\frac{864}{7}$	$\frac{13{,}824}{35}$

TABLE IX (*continued*)

	$\sum_{i>j} \mathbf{L}_{ij}^4 - 23\mathbf{L}\cdot\mathbf{L}$	$\sum_i \mathbf{l}_i^2$	$(\sum_i \mathbf{l}_i^2)^2$
$\langle H\rangle$	$56n_0^2+148n_0(n_1+n_{-1})-148n_0+99(n_1+n_{-1})^2$ $-259(n_1+n_{-1})+9[\mu(\mu+2)-K^2]$	$4n_0+6(n_1+n_{-1})$	$(6N-2n_0)^2+8n_0$
$\langle F_{15}H\rangle$	$80n_0^2-\frac{233}{2}(n_1+n_{-1})^2+12n_0(n_1+n_{-1})+169(n_1+n_{-1})$ $-\frac{57}{2}\mu(\mu+2)+41K^2-172n_0$	4ξ	$4\xi[8n_0+12(n_1+n_{-1})-2]$
$\frac{\langle F_{-455}H\rangle}{\langle F_{-44}\rangle}$	$2[52n_0+127(n_1+n_{-1})-9K-208]$	8	$8[8n_0+12(n_1+n_{-1})-2]$
$\frac{\langle F_{411}H\rangle}{\langle F_{4-4}\rangle}$	$2[52n_0+127(n_1+n_{-1})+9K-208]$	8	$8[8n_0+12(n_1+n_{-1})-2]$
$\langle F_{1155}H\rangle$	$192(\xi^2-n_0)+128\mu(\mu+2)-192K^2$	0	$32[2\xi^2-2n_0+\mu(\mu+2)-K^2]$
$\frac{\langle F_{45111}H\rangle}{\langle F_{4-4}\rangle}$	$192(3\xi-K+1)$	0	192ξ
$\frac{\langle F_{-41555}H\rangle}{\langle F_{-44}\rangle}$	$192(3\xi+K+1)$	0	192ξ
$\frac{\langle F_{441111}H\rangle}{\langle F_{44-4-4}\rangle}$	768	0	384
$\frac{\langle F_{-4-45555}H\rangle}{\langle F_{-4-444}\rangle}$	768	0	384
	$N=n_0+n_1+n_{-1}$	$\xi=n_0-n_1-n_{-1}$	

where the Q_μ's are expressed in terms of the F_β according to (IIB.1). A product of two one-body operators which operate on the same particle can be reduced by writing out the expression in terms of the χ's. For example,

$$F_4{}^iF_{-4}{}^i|\lambda\mu\epsilon K\rangle = (\chi_1{}^{\dagger(i)}\chi_{-1}{}^{(i)}+\sqrt{2}\chi_{0'}{}^{\dagger(i)}\chi_{-2}{}^{(i)}+\sqrt{2}\chi_2{}^{\dagger(i)}\chi_{0'}{}^{(i)}) \times(\chi_{-1}{}^{\dagger(i)}\chi_1{}^{(i)}+\sqrt{2}\chi_{-2}{}^{\dagger(i)}\chi_{0'}{}^{(i)} +\sqrt{2}\chi_{0'}{}^{\dagger(i)}\chi_2{}^{(i)})|\lambda\mu\epsilon K\rangle. \quad \text{(IVB.20)}$$

With the aid of (IVB.5), the right side of (IVB.20) reduces to

$$(\chi_1{}^{\dagger(i)}\chi_{-1}{}^{(i)}\chi_{-1}{}^{\dagger(i)}\chi_1{}^{(i)}+\sqrt{2}\chi_{0'}{}^{\dagger(i)}\chi_{-2}{}^{(i)}\chi_{-1}{}^{\dagger(i)}\chi_1{}^{(i)} +\sqrt{2}\chi_2{}^{\dagger(i)}\chi_{0'}{}^{(i)}\chi_{-1}{}^{\dagger(i)}\chi_1{}^{(i)})|\lambda\mu\epsilon K\rangle. \quad \text{(IVB.21)}$$

Now, $\chi_\alpha{}^{(i)}\chi_\beta{}^{(i)}|\lambda\mu\epsilon K\rangle=0$ because particle i cannot be in two states at the same time. Commuting the second operator in each term of (IVB.21) to the right, we then obtain:

$$F_4{}^iF_{-4}{}^i|\lambda\mu\epsilon K\rangle =\{\chi_1{}^{\dagger(i)}[\chi_{-1}{}^{(i)},\chi_{-1}{}^{\dagger(i)}]\chi_1{}^{(i)} +\sqrt{2}\chi_{0'}{}^{\dagger(i)}[\chi_{-2}{}^{(i)},\chi_{-1}{}^{\dagger(i)}]\chi_1{}^{(i)} +\sqrt{2}\chi_2{}^{\dagger(i)}[\chi_{0'}{}^{(i)},\chi_{-1}{}^{\dagger(i)}]\chi_1{}^{(i)}\}|\lambda\mu\epsilon K\rangle =\chi_1{}^{\dagger(i)}\chi_1{}^{(i)}|\lambda\mu\epsilon K\rangle=n_1{}^{(i)}|\lambda\mu\epsilon K\rangle. \quad \text{(IVB.22)}$$

Equivalently, one could simply multiply the matrices for $F_4{}^i$ and $F_{-4}{}^i$:

$$\begin{array}{c}\\0\\1\\-1\\2\\0'\\-2\end{array}\begin{bmatrix}0&1&-1&2&0'&-2\\ &&&&&\\ &&1&&&\\ &&&&&\\ &&&&\sqrt{2}&\\ &&&&&\sqrt{2}\\ &&&&&\end{bmatrix}\times\begin{array}{c}\\0\\1\\-1\\2\\0'\\-2\end{array}\begin{bmatrix}0&1&-1&2&0'&-2\\ &&&&&\\ &&&&&\\ &1&&&&\\ &&&&&\\ &&&\sqrt{2}&&\\ &&&&\sqrt{2}&\end{bmatrix}=\begin{array}{c}\\0\\1\\-1\\2\\0'\\-2\end{array}\begin{bmatrix}0&1&-1&2&0'&-2\\ &&&&&\\ &1&&&&\\ &&&&&\\ &&&2&&\\ &&&&2&\\ &&&&&\end{bmatrix}$$

$$=(n_1+2n_2+2n_{0'})=n_1,$$

where the last equality again follows from (IVB.5).

The techniques illustrated above can be used to evaluate all inhomogeneous forms involving the operators in (IVA.35) for the case $(\lambda',\mu)=(\lambda,\mu)$, $n_2=n_{-2}=n_{0'}=0$. These forms are given in Table IX. Terms with $(\lambda'\mu')\neq(\lambda\mu)$ or with $\epsilon\neq\epsilon_{\max}$ must be handled by means of special techniques which are developed in a later paper as various cases arise.

V. SPIN

In the preceding section, we have been dealing with states with given space symmetry $[f]$. There exist in general several such states, corresponding to the different members of the irreducible representation labeled by $[f]$. If we take certain linear combinations of these functions multiplied by appropriate spin functions, we can form totally antisymmetric wave functions. The new wave functions are characterized by the above quantum numbers and in addition by S, the spin quantum number, and by σ, the z component of S. We now consider the totally antisymmetric functions:

$$\Phi([f](\lambda\mu)K\epsilon\Lambda:S\sigma),$$

and the new representation

$$\Psi_M{}^{J,L,S}\equiv P_M{}^J\Phi([f](\lambda\mu)K\epsilon_{\max}:S\sigma),$$

where $P_M{}^J$ now projects out the part of Φ with total angular momentum J and changes the eigenvalue of J_z to M.

A. The (JLS,M) Representation

In analogy with the notation of Secs. III and IV, we designate the state $\Phi([f](\lambda\mu)K\epsilon_{\max}:S\sigma)$ as $|\lambda\mu\epsilon K\sigma\rangle$. We first show that a complete set can be formed if one takes all states $P_M{}^J|\lambda\mu\epsilon K\sigma\rangle$ with $K+\sigma\geqslant 0$.

Consider a product of an orbital function $\psi_K{}^L$ with fixed L and K, and spin function $\chi_\sigma{}^S$ with fixed S and σ. We wish to evaluate

$$P^J\psi_K{}^L\chi_\sigma{}^S. \tag{VA.1}$$

Clearly, we can construct eigenstates of J:

$$\Psi_\mu{}^{JLS}\equiv\sum_{K'\sigma'}\begin{bmatrix}L & S & J\\ K' & \sigma' & \mu\end{bmatrix}\psi_{K'}{}^L\chi_{\sigma'}{}^S. \tag{VA.2}$$

Then

$$\psi_K{}^L\chi_\sigma{}^S=\sum_{\mu,J}\begin{bmatrix}L & S & J\\ K & \sigma & \mu\end{bmatrix}\Psi_\mu{}^{JLS}. \tag{VA.3}$$

Applying the projection operators P^J and $P_M{}^J$ to (VA.3) yields

$$\begin{aligned}P^J\psi_K{}^L\chi_\sigma{}^S&=\begin{bmatrix}L & S & J\\ K & \sigma & K+\sigma\end{bmatrix}\Psi_{K+\sigma}{}^{JLS},\\ P_M{}^J\psi_K{}^L\chi_\sigma{}^S&=\begin{bmatrix}L & S & J\\ K & \sigma & K+\sigma\end{bmatrix}\Psi_M{}^{JLS},\end{aligned} \tag{VA.4}$$

or, equivalently,

$$P_M{}^J\psi_K{}^L\chi_\sigma{}^S=\begin{bmatrix}L & S & J\\ K & \sigma & K+\sigma\end{bmatrix}\sum_{K'\sigma'}\begin{bmatrix}L & S & J\\ K' & \sigma' & M\end{bmatrix}\times P_{K'}{}^LP_{\sigma'}{}^S\psi_{K'}{}^L\chi_{\sigma'}{}^S, \tag{VA.5}$$

where we have substituted (VA.2) into (VA.4). In fact, if a sum is carried out over L,S as well as over K',σ', we have an operator which projects J out of any linear combination

$$\sum_{LS}a_{LS}\psi_K{}^L\chi_\sigma{}^S. \tag{VA.6}$$

But any arbitrary function of space and spin with K,σ, good quantum numbers can be written in the form (VA.6). Hence, we conclude that

$$P_M{}^J|\lambda\mu\epsilon,-K,-\sigma\rangle=\sum_{LM_LM_S}\begin{bmatrix}L & S & J\\ -K & -\sigma & -K-\sigma\end{bmatrix}\times P_{M_L}{}^LP_{M_S}{}^S\begin{bmatrix}L & S & J\\ M_L & M_S & M\end{bmatrix}\times|\lambda\mu\epsilon,-K,-\sigma\rangle. \tag{VA.7}$$

Considering the right side of (VA.7), we note that

$$P_{M_S}{}^S|\lambda\mu\epsilon,-K,-\sigma\rangle=P_{M_S}{}^S|\lambda\mu\epsilon,-K,\sigma\rangle, \tag{VA.8}$$

since S is a good quantum number for the state $|\lambda\mu\epsilon,-K,-\sigma\rangle$. With the aid of Elliott's relation (IIIC.26), we then have

$$P_{M_L}{}^LP_{M_S}{}^S|\lambda\mu\epsilon,-K,-\sigma\rangle=(-1)^{L+\lambda+\mu}P_{M_L}{}^LP_{M_S}{}^S|\lambda\mu\epsilon K\sigma\rangle. \tag{VA.9}$$

Now

$$\begin{bmatrix}L & S & J\\ -K & -\sigma & -K-\sigma\end{bmatrix}=(-)^{-L-S+J}\begin{bmatrix}L & S & J\\ K & \sigma & K+\sigma\end{bmatrix}. \tag{VA.10}$$

Substuting (VA.9) and (VA.10) on the right side of (VA.7), we see that

$$\begin{aligned}P_M{}^J|\lambda\mu\epsilon,-K,-\sigma\rangle&=(-1)^{J-S+\lambda+\mu}\sum_{LM_LM_S}\begin{bmatrix}L & S & J\\ K & \sigma & K+\sigma\end{bmatrix}\times P_{M_L}{}^LP_{M_S}{}^S\begin{bmatrix}L & S & J\\ M_L & M_S & M\end{bmatrix}|\lambda\mu\epsilon K\sigma\rangle\\ &=(-1)^{J-S+\lambda+\mu}P_M{}^J|\lambda\mu\epsilon K\sigma\rangle.\end{aligned} \tag{VA.11}$$

It is, thus, clear that no new states are formed by changing the sign of $K+\sigma$. From (VA.11), we have the

TABLE X. Allowed values of K and J for given $S=1$, $(\lambda\mu)=(8,4)$.

$K=0$	L	0	2	4	6	8				
	J	1	1, 2, 3	3, 4, 5	5, 6, 7	7, 8, 9				
$K=2$	L	2	3	4	5	6	7	8	9	10
	J	1, 2, 3	2, 3, 4	3, 4, 5	4, 5, 6	5, 6, 7	6, 7, 8	7, 8, 9	8, 9, 10	9, 10, 11
$K=4$	L	4	5	6	7	8	9	10	11	12
	J	3, 4, 5	4, 5, 6	5, 6, 7	6, 7, 8	7, 8, 9	8, 9, 10	9, 10, 11	10, 11, 12	11, 12, 13

relation:

$$P_M{}^J|\lambda\mu\epsilon,0,0\rangle=(-1)^{J-S+\lambda+\mu}P_M{}^J|\lambda\mu\epsilon,0,0\rangle. \quad \text{(VA.12)}$$

Hence, $|\lambda\mu\epsilon,0,0\rangle$ contains only even or only odd angular momenta, depending on whether $(-1)^{\lambda+\mu-S}$ is even or odd.

In our new representation, then, $K+\sigma\leqslant\tau$ is the "band" quantum number corresponding to the projection of J on the body-fixed axis. The quantum number, τ, assumes all values consistent with $K+\sigma\geqslant 0$ where

$$\pm K=\min(\lambda,\mu),\ \min(\lambda,\mu)-2,\ \cdots,\ 0 \text{ or } 1,$$

$$\pm\sigma=S,\ S-1,\ \cdots,\ 0 \text{ or } 1/2.$$

As an example of the (JLS,M) representation, consider a triplet band with $(\lambda,\mu)=(8,4)$, $S=1$. The possible values of $\mathbf{J}=\mathbf{L}+\mathbf{S}$ for each of the allowed K's are listed in Table X. From Table X, we see that there are 67 states with J ranging from 1 to 13. The states are grouped into bands characterized by the quantum number τ, as shown in Table XI. The number of states of given J is shown in the last line of Table X. If we assume $J=\tau,\tau+1,\cdots,\tau+\max(\lambda,\mu)$, with the restriction that J must be odd or even if $(-1)^{\lambda+\mu-S}$ equals $-$ or $+$ for $K=\sigma=0$, we arrive at the correct number of states. That these states are linearly independent can be shown using methods similar to those employed by Elliott to demonstrate the linear independence of the states $P_M{}^L\Phi([f](\lambda\mu)K\epsilon_{\max})$.

TABLE XI. States listed in Table X regrouped into bands labeled by K, σ, and τ.

K	σ	τ	J												
0	0	0	1		3		5		7						
0	1	1	1	2	3	4	5	6	7	8	9				
2	−1	1	1	2	3	4	5	6	7	8	9				
2	0	2		2	3	4	5	6	7	8	9	10			
2	1	3			3	4	5	6	7	8	9	10	11		
4	−1	3			3	4	5	6	7	8	9	10	11		
4	0	4				4	5	6	7	8	9	10	11	12	
4	1	5					5	6	7	8	9	10	11	12	13
	Total		3	3	6	6	8	7	8	7	7	5	4	2	1

The projection relations for the functions $|\lambda\mu\epsilon,K,\sigma\rangle$ are constructed in the same manner as those for the spin-independent case. We shall need the matrix elements of $S_\pm$:

$$\begin{aligned}\langle\lambda\mu\epsilon K,\ \sigma+1|S_+|\lambda\mu\epsilon K,\ \sigma\rangle &=\langle\lambda\mu\epsilon K,\ \sigma|S_-|\lambda\mu\epsilon K,\ \sigma+1\rangle\\ &=[(S+\sigma+1)(S-\sigma)]^{1/2}. \quad \text{(VA.13)}\end{aligned}$$

Then, for $\mathbf{L}^2$, we have

$$\begin{aligned}&P_M{}^J\mathbf{L}^2|\lambda\mu\epsilon K,\sigma\rangle\\ &=P_M{}^J(\mathbf{J}^2-2\mathbf{J}\cdot\mathbf{S}+\mathbf{S}^2)|\lambda\mu\epsilon K,\sigma\rangle=[J(J+1)+S(S+1)]P_M{}^J|\lambda\mu\epsilon K,\sigma\rangle-2P_M{}^J[J_0S_0+J_+S_-/2+J_-S_+/2]|\lambda\mu\epsilon K,\sigma\rangle\\ &=[J(J+1)+S(S+1)-2(K+\sigma)\sigma]P_M{}^J|\lambda\mu\epsilon K,\sigma\rangle\\ &\quad-[(S+\sigma)(S-\sigma+1)(J-K-\sigma+1)(J+K+\sigma)]^{1/2}P_M{}^J|\lambda\mu\epsilon K,\ \sigma-1\rangle\\ &\quad\quad-[(S-\sigma)(S+\sigma+1)(J+K+\sigma+1)(J-K-\sigma)]^{1/2}P_M{}^J|\lambda\mu\epsilon K,\ \sigma+1\rangle. \quad \text{(VA.14)}\end{aligned}$$

Similarly,

$$\begin{aligned}&P_M{}^JL_+L_+\Lambda_-|\lambda\mu\epsilon K,\sigma\rangle=P_M{}^J(J_+-S_+)(J_+-S_+)\Lambda_-|\lambda\mu\epsilon K,\sigma\rangle\\ &=\tfrac{1}{2}[(\mu+K)(\mu-K+2)]^{1/2}\{[(J-K-\sigma+2)(J+K+\sigma-1)(J-K-\sigma+1)(J+K+\sigma)]^{1/2}P_M{}^J|\lambda\mu\epsilon,\ K-2,\ \sigma\rangle\\ &\quad-2[(S-\sigma)(S+\sigma+1)(J-K-\sigma+1)(J+K+\sigma)]^{1/2}P_M{}^J|\lambda\mu\epsilon,\ K-2,\ \sigma+1\rangle\\ &\quad\quad+[(S-\sigma)(S+\sigma+1)(S-\sigma-1)(S+\sigma+2)]^{1/2}P_M{}^J|\lambda\mu\epsilon,\ K-2,\ \sigma+2\rangle\}, \quad \text{(VA.15)}\end{aligned}$$

$$\begin{aligned}&P_M{}^JL_-L_-\Lambda_+|\lambda\mu\epsilon K,\sigma\rangle=P_M{}^J(J_--S_-)(J_--S_-)\Lambda_+|\lambda\mu\epsilon K,\sigma\rangle\\ &=\tfrac{1}{2}[(\mu-K)(\mu+K+2)]^{1/2}\{[(J+K+\sigma+2)(J-K-\sigma-1)(J+K+\sigma+1)(J-K-\sigma)]^{1/2}P_M{}^L|\lambda\mu\epsilon,\ K+2,\ \sigma\rangle\\ &\quad-2[(S+\sigma)(S-\sigma+1)(J+K+\sigma+1)(J-K-\sigma)]^{1/2}P_M{}^L|\lambda\mu\epsilon,\ K+2,\ \sigma-1\rangle\\ &\quad\quad+[(S+\sigma)(S-\sigma+1)(S+\sigma-1)(S-\sigma)]^{1/2}P_M{}^J|\lambda\mu\epsilon,\ K+2,\ \sigma-2\rangle\}. \quad \text{(VA.16)}\end{aligned}$$

The projection relations for the spin-dependent case are given in Table XII. Again the time-reversal symmetry is apparent. We note that $\mathbf{S}\rightarrow-\mathbf{S}$ and $|\lambda\mu\epsilon,K\sigma\rangle\rightarrow e^{i\beta}|\lambda\mu\epsilon,\ -K-\sigma\rangle$ under time reversal, where β is a real phase depending on S.

TABLE XII. Projection relations including spin.

$$P_M{}^J\mathbf{L}^4|K,\sigma\rangle$$
$$=\{[J(J+1)+S(S+1)-2(K+\sigma)\sigma]^2+2(K+\sigma)\sigma+2[J(J+1)-(K+\sigma)^2][S(S+1)-\sigma^2]\}P_M{}^J|K,\sigma\rangle$$
$$+2[(K+\sigma)(\sigma-1)+\sigma(K+\sigma-1)+1-J(J+1)-S(S+1)][(S+\sigma)(S-\sigma+1)(J+K+\sigma)(J-K-\sigma+1)]^{1/2}P_M{}^J|K,\sigma-1\rangle$$
$$+2[(K+\sigma)(\sigma+1)+\sigma(K+\sigma+1)+1-J(J+1)-S(S+1)][(S-\sigma)(S+\sigma+1)(J-K-\sigma)(J+K+\sigma+1)]^{1/2}P_M{}^J|K,\sigma+1\rangle$$
$$+[(J+K+\sigma)(J-K-\sigma+1)(J+K+\sigma-1)(J-K-\sigma+2)(S+\sigma-1)(S-\sigma+2)(S+\sigma)(S-\sigma+1)]^{1/2}P_M{}^J|K,\sigma-2\rangle$$
$$+[(J-K-\sigma)(J+K+\sigma+1)(J-K-\sigma-1)(J+K+\sigma+2)(S-\sigma-1)(S+\sigma+2)(S-\sigma)(S+\sigma+1)]^{1/2}P_M{}^J|K,\sigma+2\rangle$$

$$P_M{}^J\mathbf{L}^2L_+L_+\Lambda_-|K\sigma\rangle=\tfrac{1}{2}[(\mu+K)(\mu-K+2)]^{1/2}P_M{}^J\mathbf{L}^2L_+L_+|K-2,\sigma\rangle$$

$$P_M{}^J\mathbf{L}^2L_+L_+|K-2,\sigma\rangle$$
$$=-[(S+\sigma)(S-\sigma+1)]^{1/2}[(J+K+\sigma)(J-K-\sigma+1)(J+K+\sigma-1)(J-K-\sigma+2)(J+K+\sigma-2)(J-K-\sigma+3)]^{1/2}P_M{}^J|K-2,\sigma-1\rangle$$
$$-[(S+\sigma+3)(S-\sigma-2)(S+\sigma+2)(S-\sigma-1)(S+\sigma+1)(S-\sigma)(J+K+\sigma+1)(J-K-\sigma)]^{1/2}P_M{}^J|K-2,\sigma+3\rangle$$
$$+[J(J+1)+3S(S+1)-2\sigma(\sigma+1)-2\sigma(K+\sigma-2)][(J+K+\sigma)(J-K-\sigma+1)(J+K+\sigma-1)(J-K-\sigma+2)]^{1/2}P_M{}^J|K-2,\sigma\rangle$$
$$+[(S+\sigma+1)(S-\sigma)(J+K+\sigma)(J-K-\sigma+1)]^{1/2}[-3J(J+1)-3S(S+1)+(K+\sigma)(5\sigma+K+1)+\sigma(\sigma-1)]P_M{}^J|K-2,\sigma+1\rangle$$
$$+[(S+\sigma+2)(S-\sigma-1)(S+\sigma+1)(S-\sigma)]^{1/2}[3J(J+1)+S(S+1)-2(K+\sigma)(2\sigma+K+1)]P_M{}^J|K-2,\sigma+2\rangle$$

$$P_M{}^J\mathbf{L}^2L_-L_-\Lambda_+|K\sigma\rangle=\tfrac{1}{2}[(\mu-K)(\mu+K+2)]^{1/2}P_M{}^J\mathbf{L}^2L_-L_-|K+2,\sigma\rangle$$

$$P_M{}^J\mathbf{L}^2L_-L_-|K+2,\sigma\rangle$$
$$=-[(S-\sigma)(S+\sigma+1)(J-K-\sigma)(J+K+\sigma+1)(J-K-\sigma-1)(J+K+\sigma+2)(J-K-\sigma-2)(J+K+\sigma+3]^{1/2}P_M{}^J|K+2,\sigma+1\rangle$$
$$-[(S-\sigma+3)(S+\sigma-2)(S-\sigma+2)(S+\sigma-1)(S-\sigma+1)(S+\sigma)(J-K-\sigma+1)(J+K+\sigma)]^{1/2}P_M{}^J|K+2,\sigma-3\rangle$$
$$+[(J(J+1)+3S(S+1)-2\sigma(\sigma-1)-2\sigma(K+\sigma+2)][(J-K-\sigma)(J+K+\sigma+1)(J-K-\sigma+1)(J+K+\sigma+2)]^{1/2}P_M{}^J|K+2,\sigma\rangle$$
$$+[(S-\sigma+1)(S+\sigma)(J-K-\sigma)(J+K+\sigma+1)]^{1/2}[-3J(J+1)-3S(5+1)+(K+\sigma)(5\sigma+K-1)+\sigma(\sigma+1)]P_M{}^J|K+2,\sigma-1\rangle$$
$$+[(S-\sigma+2)(S+\sigma-1)(S-\sigma+1)(S+\sigma)]^{1/2}[SJ(J+1)+S(J+1)-2(K+\sigma)(2\sigma+K-1)]P_M{}^J|K+2,\sigma-2\rangle$$

$$P_M{}^JL_+{}^4\Lambda_-\Lambda_-|K,\sigma\rangle=\tfrac{1}{4}[(\mu+K-2)(\mu-K+4)(\mu+K)(\mu-K+2)]^{1/2}P_M{}^JL_+{}^4|K-4,\sigma\rangle$$

$$P_M{}^JL_+{}^4|K-4,\sigma\rangle$$
$$=[(J+K+\sigma)(J-K-\sigma+1)(J+K+\sigma-1)(J-K-\sigma+2)(J+K+\sigma-2)(J-K-\sigma+3)(J+K+\sigma-3)(J-K-\sigma+4)]^{1/2}P_M{}^J|K-4,\sigma\rangle$$
$$-4[(S+\sigma+1)(S-\sigma)(J+K+\sigma)(J-K-\sigma+1)(J+K+\sigma-1)(J-K-\sigma+2)(J+K+\sigma-2)(J-K-\sigma+3)]^{1/2}P_M{}^J|K-4,\sigma+1\rangle$$
$$+6[(S+\sigma+2)(S-\sigma-1)(S+\sigma+1)(S-\sigma)(J+K+\sigma)(J-K-\sigma+1)(J+K+\sigma-1)(J-K-\sigma+2)]^{1/2}P_M{}^J|K-4,\sigma+2\rangle$$
$$-4[(S+\sigma+3)(S-\sigma-2)(S+\sigma+2)(S-\sigma-1)(S+\sigma+1)(S-\sigma)(J+K+\sigma)(J-K-\sigma+1)]^{1/2}P_M{}^J|K-4,\sigma+3\rangle$$
$$+[(S+\sigma+4)(S-\sigma-3)(S+\sigma+3)(S-\sigma-2)(S+\sigma+2)(S-\sigma-1)(S+\sigma+1)(S-\sigma)]^{1/2}P_M{}^J|K-4,\sigma+4\rangle$$

$$P_M{}^JL_-{}^4\Lambda_+\Lambda_+|K,\sigma\rangle=\tfrac{1}{4}[(\mu-K-2)(\mu+K+4)(\mu-K)(\mu+K+2)]^{1/2}P_M{}^JL_-{}^4|K+4,\sigma\rangle$$

$$P_M{}^JL_-{}^4|K+4,\sigma\rangle$$
$$=[(J-K-\sigma)(J+K+\sigma+1)(J-K-\sigma-1)(J+K+\sigma+2)(J-K-\sigma-2)(J+K+\sigma+3)(J-K-\sigma-3)(J+K+\sigma+4)]^{1/2}P_M{}^J|K+4,\sigma\rangle$$
$$-4[(S-\sigma+1)(S+\sigma)(J-K-\sigma)(J+K+\sigma+1)(J-K-\sigma-1)(J+K+\sigma+2)(J-K-\sigma-2)(J+K+\sigma+3)]^{1/2}P_M{}^J|K+4,\sigma-1\rangle$$
$$+6[(S-\sigma+2)(S+\sigma-1)(S-\sigma+1)(S+\sigma)(J-K-\sigma)(J+K+\sigma+1)(J-K-\sigma-1)(J+K+\sigma+2)]^{1/2}P_M{}^J|K+4,\sigma-2\rangle$$
$$-4[(S-\sigma+3)(S+\sigma-2)(S-\sigma+2)(S+\sigma-1)(S-\sigma+1)(S+\sigma)(J-K-\sigma)(J+K+\sigma+1)]^{1/2}P_M{}^J|K+4,\sigma-3\rangle$$
$$+[(S-\sigma+4)(S+\sigma-3)(S-\sigma+3)(S+\sigma-2)(S-\sigma+2)(S+\sigma-1)(S-\sigma+1)(S+\sigma)]^{1/2}P_M{}^J|K+4,\sigma-4\rangle$$

B. Matrix Elements of the Spin-Orbit Potential

The spin-orbit operator can be written as

$$\mathbf{l}^i\cdot\mathbf{S}^i=\frac{l_+{}^iS_-{}^i}{2}+\frac{l_-{}^iS_+{}^i}{2}+l_0{}^iS_0{}^i$$
$$=-\frac{1}{\sqrt{2}}[(F_{-1}{}^i+F_5{}^i)S_-{}^i+(F_{-5}{}^i+F_1{}^i)S_+{}^i]+l_0{}^iS_0{}^i. \quad \text{(VB.1)}$$

We proceed as in Sec. III and consider first

$$\sum_i\mathbf{l}^i\cdot\mathbf{S}^i\Phi([f](\lambda\mu)K\epsilon_{\max}:S\sigma)$$
$$=[a(K,\sigma)+b(K,\sigma)F_{-5}S_++C(K,\sigma)F_{-1}S_-+d(K,\sigma)F_{-5}F_4S_-+e(K,\sigma)F_{-1}F_{-4}S_+]$$
$$\times\Phi([f](\lambda\mu)K\epsilon_{\max}:S\sigma)+\chi, \quad \text{(VB.2)}$$

where

$$F_\alpha=\sum_i F_\alpha{}^i,\quad \mathbf{S}=\sum_i S^i. \quad \text{(VB.3)}$$

TABLE XIII. Homogeneous matrix elements for treating the spin-orbit force.

	$F_{-5}S_+$	$F_{-1}S_-$	$F_{-5}F_4S_-$	$F_{-1}F_{-4}S_+$
S_-F_5	$\tfrac{1}{2}(K+\epsilon)\times[S(S+1)-\sigma(\sigma+1)]$	0	0	$\tfrac{1}{2}[\mu(\mu+2)-K(K-2)]\times[S(S+1)-\sigma(\sigma+1)]$
S_+F_1	0	$\tfrac{1}{2}(-K+\epsilon)\times[S(S+1)-\sigma(\sigma-1)]$	$\tfrac{1}{4}[\mu(\mu+2)-K(K+2)]\times[S(S+1)-\sigma(\sigma-1)]$	0
$S_+F_{-4}F_5$	0	$\tfrac{1}{4}[\mu(\mu+2)-K(K+2)]\times[S(S+1)-\sigma(\sigma-1)]$	$(K+\epsilon+2)\times\tfrac{1}{8}[\mu(\mu+2)-K(K+2)]\times[S(S+1)-\sigma(\sigma-1)]$	0
$S_-F_4F_1$	$\tfrac{1}{4}[\mu(\mu+2)-K(K-2)]\times[S(S+1)-\sigma(\sigma+1)]$	0	0	$(-K+\epsilon+2)\times\tfrac{1}{8}[\mu(\mu+2)-K(K-2)]\times[S(S+1)-\sigma(\sigma+1)]$

The symbol χ denotes a sum over states

$$\Phi([f'](\lambda'\mu')K\epsilon_{\max}:S'\sigma),$$

with $[f']\neq[f]$, $(\lambda'\mu')\neq(\lambda\mu)$, or $S'\neq S$. We note that $\sum_i \mathbf{l}^i\cdot\mathbf{S}^i$ operating on $\Phi([f](\lambda\mu)K\epsilon_{\max}:S\sigma)$ changes ϵ by 0 or 3 units and leaves $\tau=K+\sigma$ unchanged. The most general operator meeting these two conditions and leaving $[f]$, $(\lambda\mu)$, and S unchanged is the sum in brackets on the right of (VB.2).

The coefficients $a(K,\sigma)$, $b(K,\sigma)$, $\cdots$, $e(K,\sigma)$ are evaluated in the same manner as the corresponding coefficients for the spin-independent case. For the homogeneous forms, the relations

$$S_-S_+=\mathbf{S}^2-S_0{}^2-S_0,\qquad S_+S_-=\mathbf{S}^2-S_0{}^2+S_0, \tag{VB.4}$$

eliminate the S operator from the problem. The homogeneous matrix elements are given in Table XIII.

Finally, we arrive at two sets of simultaneous equations, one for $b(K,\sigma)$ and $e(K,\sigma)$, the other for $c(K,\sigma)$ and $d(K,\sigma)$:

$$\langle\lambda\mu\epsilon K\sigma|S_-F_5\sum_i \mathbf{l}^i\mathbf{S}^i|\lambda\mu K\sigma\rangle=\left\{b(K,\sigma)\left(\frac{K+\epsilon}{2}\right)+e(K,\sigma)\left[\frac{\mu(\mu+2)-K(K-2)}{4}\right]\right\}[S(S+1)-\sigma(\sigma+1)],$$
$$\langle\lambda\mu\epsilon K\sigma|S_-F_4F_1\sum_i \mathbf{l}^i\cdot\mathbf{S}^i|\lambda\mu\epsilon K\sigma\rangle=\left\{b(K,\sigma)+e(K,\sigma)\frac{(-K+\epsilon+2)}{2}\right\}\left[\frac{\mu(\mu+2)-K(K-2)}{4}\right][S(S+1)-\sigma(\sigma+1)]; \tag{VB.5}$$

$$\langle\lambda\mu\epsilon K\sigma|S_+F_1\sum_i \mathbf{l}^i\cdot\mathbf{S}^i|\lambda\mu\epsilon K\sigma\rangle=\left\{C(K,\sigma)\left(\frac{-K+\epsilon}{2}\right)+d(K,\sigma)\left[\frac{\mu(\mu+2)-K(K+2)}{4}\right]\right\}[S(S+1)-\sigma(\sigma-1)],$$
$$\langle\lambda\mu\epsilon K\sigma|S_+F_{-4}F_5\sum_i \mathbf{l}^i\cdot\mathbf{S}^i|\lambda\mu\epsilon K\sigma\rangle=\left\{C(K,\sigma)+d(K,\sigma)\frac{(K+\epsilon+2)}{2}\left[\frac{\mu(\mu+2)-K(K+2)}{4}\right]\right\}[S(S+1)-\sigma(\sigma-1)]. \tag{VB.6}$$

In fact, only one set of equations must be solved since

$$a(K,\sigma)=a(-K,\,-\sigma);\quad b(-K,\,-\sigma)=c(K,\sigma);\quad d(-K,\,-\sigma)=e(K,\sigma). \tag{VB.7}$$

To see this, apply the time-reversal transformation to Eq. (VB.2) and note that the $a(K,\sigma)$, $\cdots$, $e(K,\sigma)$ are real. Relations (VB.7) are also evident from inspection of equations (VB.5) and (VB.6) when it is recognized that

$$\langle\lambda\mu\epsilon K\sigma|S_-F_5\textstyle\sum_i \mathbf{l}^i\cdot\mathbf{S}^i|\lambda\mu\epsilon K\sigma\rangle=\langle\lambda\mu\epsilon,\,-K,\,-\sigma|S_+F_1\sum_i \mathbf{l}^i\cdot\mathbf{S}^i|\lambda\mu\epsilon,\,-K,\,-\sigma\rangle,$$
$$\langle\lambda\mu\epsilon K\sigma|S_-F_4F_1\textstyle\sum_i \mathbf{l}^i\cdot\mathbf{S}^i|\lambda\mu\epsilon K\sigma\rangle=\langle\lambda\mu\epsilon,\,-K,\,-\sigma|S_+F_{-4}F_5\sum_i \mathbf{l}^i\cdot\mathbf{S}^i|\lambda\mu\epsilon,\,-K,\,-\sigma\rangle. \tag{VB.8}$$

That is, the matrix elements of the time-reversed operators in the time-reversed representation are equal to the complex conjugates of the original matrix elements.

The equivalence relations for the operators in (VB.2) are quite simple:

$$F_{-5}S_+|\lambda\mu\epsilon K\sigma\rangle=-\frac{1}{2\sqrt{2}}(J_--S_-)S_+|\lambda\mu\epsilon K\sigma\rangle,$$
$$F_{-1}S_-|\lambda\mu\epsilon K\sigma\rangle=-\frac{1}{2\sqrt{2}}(J_+-S_+)S_-|\lambda\mu\epsilon K\sigma\rangle,$$
$$F_{-5}F_4S_-|\lambda\mu\epsilon K\sigma\rangle=-\frac{1}{\sqrt{2}}(J_--S_-)S_-\Lambda_+|\lambda\mu\epsilon K\sigma\rangle,$$
$$F_{-1}F_{-4}S_+|\lambda\mu\epsilon K\sigma\rangle=-\frac{1}{\sqrt{2}}(J_+-S_+)S_+\Lambda_-|\lambda\mu\epsilon K\sigma\rangle. \tag{VB.9}$$

From (VB.9) it follows that

$$P_M{}^J F_{-5} S_+ |\lambda\mu\epsilon K\sigma\rangle$$
$$= -\frac{1}{\sqrt{2}}\{[(S+\sigma+1)(S-\sigma)(J+K+\sigma+1)(J-K-\sigma)]^{1/2} P_M{}^J |\lambda\mu\epsilon K, \sigma+1\rangle - (\mathbf{S}^2-\sigma^2-\sigma) P_M{}^J |\lambda\mu\epsilon K\sigma\rangle\},$$

$$P_M{}^J F_{-1} S_- |\lambda\mu\epsilon K\sigma\rangle$$
$$= -\frac{1}{\sqrt{2}}\{[(S-\sigma+1)(S+\sigma)(J-K-\sigma+1)(J+K+\sigma)]^{1/2} P_M{}^J |\lambda\mu\epsilon K, \sigma-1\rangle - (\mathbf{S}^2-\sigma^2+\sigma) P_M{}^J |\lambda\mu\epsilon K\sigma\rangle\},$$

$$P_M{}^J F_{-5} F_4 S_- |\lambda\mu\epsilon K\sigma\rangle = -\frac{1}{2\sqrt{2}}[(\mu+K+2)(\mu-K)]^{1/2}$$
$$\times\{[(S+\sigma)(S-\sigma+1)(J+K+\sigma+1)(J-K-\sigma)]^{1/2} P_M{}^J |\lambda\mu\epsilon, K+2, \sigma-1\rangle$$
$$-[(S+\sigma-1)(S-\sigma+2)(S+\sigma)(S-\sigma+1)]^{1/2} P_M{}^J |\lambda\mu\epsilon, K+2, \sigma-2\rangle\},$$

$$P_M{}^J F_{-1} F_{-4} S_+ |\lambda\mu\epsilon K\sigma\rangle = -\frac{1}{2\sqrt{2}}[(\mu-K+2)(\mu+K)]^{1/2}$$
$$\times\{[(S-\sigma)(S+\sigma+1)(J-K-\sigma+1)(J+K+\sigma)]^{1/2} P_M{}^J |\lambda\mu\epsilon, K-2, \sigma+1\rangle$$
$$-[(S-\sigma-1)(S+\sigma+2)(S-\sigma)(S+\sigma+1)]^{1/2} P_M{}^J |\lambda\mu\epsilon, K-2, \sigma+2\rangle\}.$$

Using the expression (VB.1) for $\sum_i \mathbf{l}^i \cdot \mathbf{S}^i$, the inhomogeneous matrix elements on the left of (VB.5) and (VB.6) are easily evaluated by the methods discussed in Sec. IV. Again, for matrix elements between states of different $(\lambda\mu)$ symmetry, special operators must be developed which generate excited configurations from the ground state. Examples are given in the next paper.

C. Spin-Dependent Two-Body Force

The remaining type of operator of interest has the form $(\sigma_1 \cdot \sigma_2) V(r_{12})$, where $V(r_{12})$ may contain a space-exchange component. In order to handle such operators, we need to consider tensors which combine space and spin; for example:

$$(S\times Q)\binom{K}{q}, \quad (S\times L)\binom{K}{q}.$$

A total of 20 matrix elements is required to define the spin-dependent two-body Hamiltonian for the *sd* shell. We might think of these as comprising 10 singlet and 10 triplet terms. The detailed tables for the spin-dependent matrix elements are given elsewhere.

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