PHYSICAL REVIEW VOLUME 130, NUMBER 3 1 MAY 1963

Radiative Corrections to High-Energy Scattering Processes*

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(Received 13 December 1962)

A unified treatment of radiative corrections to a class of scattering experiments is presented. The experiments considered are those in which either (but not both) the scattered or recoil particle is detected. The recoil kinematics are properly treated and the calculation is simplified by retaining only terms of logarithmic order. The general results are applied to specific practical examples in which radiative corrections are likely to be important. Except possibly for the case of Compton scattering with nearly maximum or nearly minimum momentum transfer, the errors are estimated to be less than 2% of the cross section.

I. INTRODUCTION

CALCULATIONS in quantum electrodynamics, while straightforward in principle, are often laborious; and in many cases the results have not been put into a convenient form for application to specific experiments. If one does not insist upon a complete calculation (to a given order of α), it should be possible to pick out the dominant contributions which may then be simpler to calculate. That this is true has been made clear in recent years by work in which the infrared contributions are singled out for special consideration.[1–4] The physical reason that these contributions are the most important at very high energies is well known. They arise from the large-scale distributions of the electromagnetic field, which should be classically describable. At very high energies these fields are strongly Lorentz contracted in the region transverse to the moving particles. They cannot be quickly rearranged when a charged particle is deflected in a scattering process; and, as a result, radiation *must* be emitted (bremsstrahlung) and together with that there must be a strong radiative reaction tending to suppress the elastic part of the scattering cross section. This feature of the radiative corrections has, of course, been well known for many years, but its importance from a practical computational standpoint has perhaps not always been so well appreciated. These general ideas are discussed in more detail in reference 1; it is the purpose of the present paper to exploit them for the calculation of radiative corrections to a specific class of scattering experiments.

In this paper the radiative corrections are separated into two parts, which are called, respectively, the "external radiative corrections" and the "internal radiative corrections." The distinction arises due to the fact that in the scattering process the current density of the interacting system can be split up in a natural way into two parts: The first part is the "external current," which is specified entirely by the momenta and spin states of the initial and final charged particles; the "internal current" is the residue, which depends on the specific details of the scattering interaction. To be more precise, the external radiative contribution is obtained by considering emission and absorption of photons (real or virtual) from external lines. By themselves, these contributions would not correspond to a conserved current because the scattering matrices which they multiply would be shifted off the mass shell due to their dependence upon k, the momentum of the photon. The external radiative correction is by definition the contribution which is obtained when this particular k dependence is neglected. Since these corrections are associated mainly with long-wavelength (infrared) photons, this is a good approximation if the scattering amplitude does not have a strong dependence on k. The residue from this approximation together with the contributions in which a photon terminates on an internal line is then called the internal radiative correction; it clearly depends on the precise details of the scattering process. On the other hand, the external radiative corrections are independent of details. Furthermore, if we are willing to estimate them by considering only terms of logarithmic order, they may be approximated with very little labor. Since the neglected terms of order unity must be multiplied by (α/π) to obtain the fractional error, the error made in this estimate is likely to be only of the order of magnitude of 1 or 2% of the cross section. The throwing away of terms of order unity is, of course, not unique, and we frequently simplify logarithmic terms by making changes of order unity. (Sometimes, terms of order unity are retained in the results if they are well known; for example, those arising from an electron vertex are retained.) An important feature of this estimate is that the result can be factorized; i.e., the correction can be expressed as a factor depending

* This work supported in part by the United States Atomic Energy Commission and in part by the United States Air Force through the European Office, Office of Aerospace Research.
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[1] D. R. Yennie, S. C. Frautschi, and H. Suura, Ann. Phys. (N. Y.) **13**, 379 (1961). Other references to the infrared divergence problem may be found in this paper.
[2] Y. S. Tsai, Phys. Rev. **120**, 269 (1960).
[3] Y. S. Tsai, Phys. Rev. **122**, 1898 (1960).
[4] A. S. Krass, Phys. Rev. **125**, 2172 (1962).

only on the external momenta times the uncorrected cross section. It is, of course, impossible to give a general discussion of the internal radiative corrections; however, in many practical examples one can give arguments that they are not important relative to the dominant external radiative corrections. Of course, in a high-precision scattering experiment (precision of order 1%) it would be necessary to give a complete calculation of the radiative corrections. Even in such a situation, it would probably be of value to split the contributions in the suggested way. The main reason for this is that the external corrections contain all the infrared divergence, which can be evaluated explicitly once and for all. The remaining part of the calculation need then have no artificial infrared cutoff.

In the present paper we present a fairly complete calculation of the external radiative corrections for some typical scattering experiments. The aim is to consider a general situation in which either the incident particle or the target particle is detected; coincidence experiments are not considered. The classic calculation of this type refers to an experiment where the particle is detected at a precisely defined angle but with a spread in possible energies. In current experiments the momentum spectrum of the scattered particles is also of interest. If the kinematics leads to a rapid variation of elastic scattering energy with angle, another type of experiment —precisely defined momentum with spread in angles—is possible.[5] Radiative corrections to these three types of experiment are discussed in a unified way here. Although the experimental conditions envisaged may be somewhat idealized, it is hoped that the principles will be sufficiently well illustrated so that the results may be extended to more realistic experimental situations. We do not wish to specialize to a particular choice of projectile and target; however, in order that the correction be meaningful compared to its error, we impose the restriction that the incident particle be extremely relativistic and suffer a momentum transfer which is large compared with its rest mass.[6] The principal difficulty that makes necessary a new calculation is the fact that recoil effects may become important in the general situation. Thus, additional terms arise dynamically from the fact that the recoiling particle may possess a charge and kinematically from the fact that the phase space is altered. Thus, if the scattered particle has an energy loss ϵ relative to elastic scattering, the energy carried off by an additional unobserved photon will not be ϵ, and it will, in fact, depend upon the direction of its emission. This integration over the phase space of the unobserved photons is the main source of difficulty in making a complete calculation. There is, of course, no difficulty in principle, but if we have the aim of doing the calculation for a completely general situation and presenting the result in a convenient form for applications, the calculation must be carefully arranged to achieve this purpose. We emphasize again that this calculation is made feasible by the fact that we are interested only in obtaining the dominant logarithmic terms associated with the external radiative corrections. A complete calculation would be many times more difficult.

The paper is organized in the following way. In Sec. II are presented the principal features of the calculation, while some of the finer points are relegated to the Appendices. Section III contains some discussion of the errors made in neglecting the internal radiative corrections and considers some special features of particular scattering experiments. In Sec. IV the results are specialized to various experiments in which the radiative corrections are important. Some attempt was made to keep Sec. IV self-contained; but an experimentalist may find it of value to refer also to Sec. IIA, where the "experimental conditions" are defined. Some of the necessary notation is also defined in Sec. II.

II. CALCULATION OF THE EXTERNAL RADIATIVE CORRECTIONS

We want to review and extend some of the considerations of reference 1 concerning the external radiative corrections. Suppose a real or virtual photon of momentum k is emitted from an incoming charged particle of momentum p. For definiteness, assume the charged particle has spin one-half; the corresponding result for zero spin will be obvious by inspection. The matrix element associated with this emission will have the form[7]

$$\cdots\frac{1}{\boldsymbol{p}-\boldsymbol{k}-m}\boldsymbol{e}u_p=\cdots\frac{(2p-k)\cdot e-\frac{1}{2}[\boldsymbol{k},\boldsymbol{e}]}{k^2-2k\cdot p}u_p. \quad (2.1)$$

The dots indicate a basic factor in the matrix element which we need not consider explicitly in computing the external radiative corrections. It is the same factor that would occur in the matrix element without photon emission, except that the momentum argument p is changed to $p-k$. In fact, the rule for calculating the external radiative corrections is to neglect the k dependence of this basic factor; by definition, the correction to this approximation is included in the internal radiative corrections as it depends on the specific details of the interaction. On the right side of (2.1) the factor corresponding to the emission of a photon appears as a sum of two terms; the first is

$$\cdots u_p\frac{(2p-k)\cdot e}{k^2-2k\cdot p}. \quad \text{(convection term)} \quad (2.1a)$$

This term, which is a simple factor times the original matrix element, is independent of the particle's spin.

[5] E. B. Dally, Phys. Rev. 123, 1840 (1961).

[6] Without this restriction the radiative corrections will be very small unless the energy resolution is extremely good.

[7] The following notation is employed: $a\cdot b=a^\mu b_\mu=a_0b_0-\boldsymbol{a}\cdot\boldsymbol{b}$, $a^2=a\cdot a$, $\boldsymbol{a}=\gamma\cdot a$.

The infrared divergent contributions, as well as some ultraviolet divergences, arise from the convection terms. The other term in the emission factor is

$$\cdots\frac{-\tfrac{1}{2}[\boldsymbol{k},\boldsymbol{e}]}{k^2-2k\cdot p}u_p. \quad \text{(spin term)} \tag{2.1b}$$

This depends explicitly on the Dirac matrices and, hence, it cannot be written as a simple factor times the original matrix element. However, it will be seen later that the largest (i.e., logarithmic in E/m) contributions which arise from the spin term can also be reduced to a simple factor. If the photon is absorbed rather than emitted, k must be replaced by $-k$ in these expressions. For absorption of a photon on an outgoing charged particle, the corresponding terms are

$$\frac{(2p'-k)\cdot e}{k^2-2k\cdot p'}\bar{u}_{p'}\cdots, \quad \text{(convection term)} \tag{2.2a}$$

$$\bar{u}_{p'}\frac{-\tfrac{1}{2}[\boldsymbol{e},\boldsymbol{k}]}{k^2-2k\cdot p'}\cdots. \quad \text{(spin term)} \tag{2.2b}$$

For purposes of calculation it will prove convenient to catalog the various contributions to the external radiative corrections according to whether the photons are emitted or absorbed by the convection or spin part of the current. The major correction arises from the *convection contribution*, which contains all the infrared divergence. For virtual photons the convection contribution corresponds to both emission and reabsorption by a convection term; for real photons it refers to the contribution obtained by squaring the convection part of the emission matrix element. An important correction also arises from the cross term between convection and spin terms (this is called the *spin-convection contribution*). It is interesting to note that in the case of electron scattering from an external potential treated in Born approximation, all of the ultraviolet divergence is associated with the convection contribution. This is in spite of the extra powers of k in the spin terms; the divergent part of the spin contributions actually turns out to be zero as a result of the properties of the γ matrices.

The external radiative corrections due to virtual photons are now obtained by summing the contributions from all Feynman diagrams in which a photon is emitted from one external line and absorbed by another, together with the wave-function renormalizations. For the convection terms the derivation is given in reference 1, and only the notation and result will be quoted here. Consider an arbitrary process containing a number of charged incoming and outgoing particles. The ith external line represents a particle of charge eZ_i and momentum p_i; a number θ_i distinguishes incoming ($\theta_i=-1$) and outgoing ($\theta_i=+1$) particles. If the original matrix element for the process is M_0, the virtual photon convection contribution to this matrix element is simply

$$\alpha B M_0, \tag{2.3a}$$

where

$$B=\sum_{\text{pairs}}\frac{-iZ_i\theta_iZ_j\theta_j}{8\pi^3}\times\int\frac{d^4k}{k^2-\lambda^2}\left[\frac{(2p_i\theta_i-k)_\mu}{k^2-2k\cdot p_i\theta_i}+\frac{(2p_j\theta_j+k)_\mu}{k^2+2k\cdot p_j\theta_j}\right]^2. \tag{2.3b}$$

The sum extends over each pair of external lines. The infrared divergence is cut off by the introduction of a small photon mass λ; this makes the real and virtual photon contributions separately convergent before the final cancellation of the infrared divergence.

In reference 1 the probability for emitting an unobserved soft photon is calculated under the assumption that recoil effects are small. Roughly speaking, this means that the requirements of energy-momentum conservation are taken into account in computing the phase space available to the emitted photon; but changes in the cross section due to the dependence of the momentum of the recoil particle on that of the photon are neglected. This is a valid approximation if the experimental conditions are such as to assure that only very soft photons are emitted, and it leads to a demonstration of the canceling of the infrared divergence to all orders of approximation. However, for our present considerations such an approximation is not justified; and as we shall see, important corrections can arise when the kinematics are treated correctly. Nevertheless, since it will provide a convenient way for handling the canceling of the infrared divergence, we give here the probability for emitting an unobserved soft photon when recoil is neglected:

$$2\alpha\tilde{B}\sigma_0, \tag{2.4a}$$

where

$$\tilde{B}=\sum_{\text{pairs}}\frac{Z_i\theta_iZ_j\theta_j}{8\pi^2}\int_0^{K_m}\frac{d^3k}{(\mathbf{k}^2+\lambda^2)^{1/2}}\left[\frac{p_{i\mu}}{k\cdot p_i}-\frac{p_{j\mu}}{k\cdot p_j}\right]^2, \tag{2.4b}$$

and σ_0 is the uncorrected cross section proportional to $|M_0|^2$. The upper limit K_m generally is a function of the direction of the photon, depending on the details of the experimental arrangement. In determining K_m as a function of direction, it is, of course, important *not* to ignore k in the over-all conservation laws. If, as is the case in the problems under investigation here, K_m is independent of direction in some Lorentz frame, the integral in (2.4b) may be carried out explicitly. When the result is combined with (2.3), the net contribution to the radiative correction is

$$2\alpha(\mathrm{Re}B+\tilde{B})\sigma_0, \tag{2.5a}$$

where

$$\mathrm{Re}B+\tilde{B}=\sum_{\text{pairs}}\frac{Z_i\theta_iZ_j\theta_j}{2\pi}\left\{\ln\frac{K_m^2}{\tilde{E}_i\tilde{E}_j}-\tfrac{1}{2}p_i\cdot p_j\int_{-1}^{1}\ln\frac{K_m^2}{\tilde{E}_x^2}\frac{dx}{p_x^2}-\frac{1}{4}\int_{-1}^{1}\ln\frac{p_x^2}{m_im_j}dx\right\},\quad(2.5\text{b})$$

and $2p_x=(1+x)p_i+(1-x)p_j$. Some unimportant contributions of order unity have been neglected. The energies $\tilde{E}_i$, $\tilde{E}_j$, and $\tilde{E}_x$ appearing here have to be evaluated in the Lorentz frame in which K_m is isotropic. In case particle i is extremely relativistic relative to particle j (i.e., $p_i\cdot p_j\gg m_im_j$), the leading logarithmic contributions to the summand of (2.5b) may easily be evaluated; the result is

$$\{\quad\}_{ij}\cong-\left\{\left(\ln\frac{2p_i\cdot p_j}{m_im_j}-1\right)\ln\frac{K_m^2}{\tilde{E}_i\tilde{E}_j}+\ln\frac{m_j}{m_i}\ln\frac{\tilde{E}_j}{\tilde{E}_i}-\tfrac{1}{2}\ln^2\frac{\tilde{E}_j}{\tilde{E}_i}-\tfrac{1}{2}\ln^2\frac{2p_i\cdot p_j}{m_j^2}\theta(m_j^2-2p_i\cdot p_j)-\tfrac{1}{2}\ln^2\frac{2p_i\cdot p_j}{m_i^2}\theta(m_i^2-2p_i\cdot p_j)+\tfrac{1}{2}\ln\frac{2p_i\cdot p_j}{m_im_j}\right\}$$
$$(\text{when}\quad p_i\cdot p_j\gg m_im_j),\quad(2.6)$$

where $\theta(\alpha)=1$ or 0 for $\alpha>0$ or $\alpha<0$. The only contributions which have been neglected are those of order unity (i.e., terms which remain bounded or tend to zero as the various energy ratios become large). It is also of some practical significance to note that (2.6) contains no Spence functions. In fact, the calculation has been arranged in such a manner that all the Spence functions which occur have argument less than one; they are therefore of order unity and can be ignored. Of course, in a complete calculation these terms would have to be recovered; this, however, would be one of the least difficulties in doing a complete calculation. The terms multiplying the θ function can occur only when $m_j\gg m_i$ or $m_i\gg m_j$. Some additional remarks should be made about this result. The last term in both (2.5b) and (2.6) is related to the ultraviolet divergent part of the convection contribution. As discussed in the Introduction, the approximation of neglecting k inside the residual matrix element may, therefore, not be terribly well justified for this term. In particular applications it is then necessary to make a detailed study to verify whether it is justified to retain this term in comparison with other neglected contributions. Another term of similar order of magnitude is the vacuum polarization and it should be put in explicitly whenever it occurs. The only other important logarithmic contributions that are known are those associated with the spin-convection contribution; they will be discussed below. We now turn to a more detailed discussion of the kinematical problem and the computation of the external radiative corrections with recoil properly treated.

A. Kinematical Considerations

We would like to derive the external radiative corrections to a scattering process in which either (but not both) of the particles is detected. While we do not wish to specialize the calculation to any particular physical system, rather idealized experimental conditions will be assumed. One of these is that the incident beam is perfectly defined; in practice our result would have to be folded into the energy spectrum of the incident beam. It is also assumed that the detector spans a well-defined angular range ($\theta_{\max}>\theta>\theta_{\min}$) and momentum range ($p_{\max}>p>p_{\min}$) and that the probability for detecting a particle is uniform in this range. Three special cases will be considered: (a) Angular resolution is sharp and the momentum resolution includes elastic scattering; the result then depends on Δp, the maximum momentum the particle can lose below its elastic scattering value. (b) The energy spectrum of particles scattered in a fixed small solid angle. (c) Sharp momentum resolution and the angular resolution includes elastic scattering; the result depends on $\Delta\theta$, the difference between the elastic scattering angle (θ_{el}) and the minimum detection angle. Case (c) can arise when the elastic-scattering momentum has a rapid angular dependence. As will be evident later, the results for Cases (a) and (c) can be determined by a single calculation. Case (b) is simply determined from Case (a) by differentiation.

We shall try to evaluate all integrals for arbitrary values of mass, energy, and momentum transfer; the results may then be specialized later to given choices of projectile and target. The only restrictions will be that the incident particle be extremely relativistic and that the momentum transfer be large compared with the mass of the incident particle. To avoid an awkward nomenclature, we shall often refer to the incident particle as an electron and the target particle as a proton; in fact, this particular scattering process is one of the major applications of our result. However, by setting the masses equal, the result will apply equally to electron-electron or electron-positron scattering. By setting the mass and charge of the projectile equal to zero, we shall obtain the radiative corrections to Compton scattering. The latter process has not previously been evaluated for actual experimental conditions. We do it here at the expense of omitting some terms of order unity; those terms could of course be recovered by comparing the present calculation with that of Brown and Feynman.[8] See also the remarks in Sec. IIIA.

For elastic scattering the electron's initial and final momenta are, respectively, p_1 and p_3, while those of the proton are p_2 and p_4. The angle of the elastically

[8] L. M. Brown and R. P. Feynman, Phys. Rev. 85, 231 (1952).

scattered electron is θ_3 and that of the recoil proton is θ_4; both of these are measured from the direction of the incident beam. Furthermore,

$$p_1^2=p_3^2=m_1^2,\quad p_2^2=p_4^2=m_2^2. \tag{2.7}$$

For scattering with bremsstrahlung, the final momenta are primed. Energy and momentum conservation in the two cases are expressed by

$$p_1+p_2=p_3+p_4, \tag{2.8a}$$

$$p_1+p_2=p_3'+p_4'+k. \tag{2.8b}$$

Experiments in which the incident or target particle is detected will be labeled, respectively, I or II, with a subscript a, b, or c to denote the type of detection. For example, Experiment I_a means the electron is detected at an angle θ_3 with a momentum loss smaller than Δp_3. To keep the discussion general, the charge of the electron is called Z_1e and that of the proton Z_2e. Unless specifically indicated, energies and momenta are given in the laboratory system.

Some important kinematical relationships will now be derived and listed. The first of these are the energy and momenta of the final particles as a function of their direction for elastic scattering. We always assume conditions such that the incident and scattered particle is extremely relativistic ($E_1\gg m_1$ and $E_3\gg m_1$); then we easily find

$$E_3\cong p_3\cong E_1/\eta \quad\text{with}\quad \eta=1+(E_1/m_2)(1-\cos\theta_3) \tag{2.9}$$

and

$$p_4\cong\frac{2E_1m_2(m_2+E_1)\cos\theta_4}{(m_2+E_1)^2-E_1^2\cos^2\theta_4}, \tag{2.10a}$$

$$E_4\cong m_2\frac{(m_2+E_1)^2+E_1^2\cos^2\theta_4}{(m_2+E_1)^2-E_1^2\cos^2\theta_4}. \tag{2.10b}$$

For each given momentum loss of the particle being detected, there exists a Lorentz frame in which the energy of the photon is isotropic. This frame is the center-of-momentum frame of the photon and the unobserved particle. Suppose the four-momentum of the electron is p_3', while the corresponding elastic scattering value is p_3. Then the energy of the photon in the special frame may be determined from

$$(p_4'+k)^2-m_2^2=2k\cdot p_4'+\lambda^2\cong 2\delta p_3\cdot(p_1+p_2),$$

where

$$\delta p_3=p_3-p_3'.$$

For Experiments I_a and I_b, $\mathbf{p}_3'$ is parallel to $\mathbf{p}_3$, hence,

$$\gamma_1\equiv\delta p_3\cdot(p_1+p_2) =m_2\eta(|\mathbf{p}_3|-|\mathbf{p}_3'|),\quad (\mathrm{I}_a\text{ and }\mathrm{I}_b) \tag{2.11a}$$

and for Experiment I_c

$$\gamma_1=p_1p_3\sin\theta_3(\theta_3-\theta_3').\quad(\mathrm{I}_c) \tag{2.11b}$$

Then

$$k\cdot p_4'=\tilde{\omega}(\tilde{E}_4'+\tilde{\omega})=\gamma_1, \tag{2.11c}$$

where $\tilde{\omega}=(\tilde{k}^2+\lambda^2)^{1/2}$ and $\tilde{E}_4'=(\tilde{k}^2+m_2^2)^{1/2}$. Solving for $\tilde{\omega}$, we find

$$\tilde{\omega}=\gamma_1/(m_2^2+2\gamma_1)^{1/2}. \tag{2.11d}$$

Also let Γ_1 be the maximum value of γ_1 for either Experiment I_a or I_c. Then it is interesting to notice the behavior for two situations. If $m_2^2\gg\Gamma_1$ (for example, if $m_2>E_1$), we have simply $\tilde{\omega}=\gamma_1/m_2$ and the recoil proton is never relativistic in the special frame. On the other hand, if $\Gamma_1\gg m_2^2$, the recoil proton has a nonrelativistic velocity in the special frame for small γ_1 and a relativistic velocity for large γ_1. It is just this dependence of $\tilde{E}_4'$ on $\tilde{k}$ which was neglected in the calculation of $\tilde{B}$. For experiment II, the corresponding expressions are

$$k\cdot p_3'=\tilde{\omega}(\tilde{E}_3'+\tilde{\omega})=\gamma_2, \tag{2.12a}$$

$$\gamma_2=\frac{p_4(E_1+m_2)m_2}{E_4(E_4+m_2)}(|\mathbf{p}_4|-|\mathbf{p}_4'|),\quad(\mathrm{II}_a\text{ and }\mathrm{II}_b) \tag{2.12b}$$

$$\gamma_2=p_1p_4\sin\theta_4(\theta_4-\theta_4'),\quad(\mathrm{II}_c) \tag{2.12c}$$

$$\tilde{\omega}=\gamma_2/(m_1^2+2\gamma_2)^{1/2}, \tag{2.12d}$$

where

$$\tilde{E}_3'=(m_1^2+\tilde{k}^2)^{1/2}.$$

Again, Γ_2 is the maximum value of γ_2.

Consider next the integrals over the final phase space of the particles. For the elastic scattering part this is of the form

$$\int\cdots\int_R\frac{d^3p_3}{E_3}\frac{d^3p_4}{E_4}\delta(p_1+p_2-p_3-p_4)|M_0|^2,$$

where R denotes the region of phase space permitted by the detection arrangement. This expression is invariantly defined, and hence, the following analysis can be performed equally well in the laboratory or the center-of-mass coordinate system; however, most present experiments of the type under consideration correspond to the laboratory system which will be employed here. For the various experiments this reduces to

$$\frac{p_3^2}{m_2E_1}d\Omega_3\ |M_0|^2,\quad(\mathrm{I}_a\text{ and }\mathrm{I}_b) \tag{2.13a}$$

$$\frac{p_3}{p_1E_3}dp_3d\phi_3\ |M_0|^2,\quad(\mathrm{I}_c) \tag{2.13b}$$

$$\frac{p_4(E_4+m_2)}{m_2(E_1+m_2)}d\Omega_4\ |M_0|^2,\quad(\mathrm{II}_a\text{ and }\mathrm{II}_b) \tag{2.13c}$$

$$\frac{p_4}{p_1E_4}dp_4d\phi_4\ |M_0|^2. \tag{2.13d}$$

The inelastic scattering contribution to the observed cross section takes the form

$$\int_R \cdots \int \frac{d^3p_3'}{E_3'}\frac{d^3p_4'}{E_4'}\frac{d^3k}{\omega}\delta(p_1+p_2-p_3'-p_4'-k)(\cdots).$$

Consider Experiment I. Since the integrand is an invariant, the integration over k and p_4' may be carried out in any reference frame. For each fixed value of p_3', it is convenient to use that frame in which the photon energy is isotropic; it is specified by the vanishing of the space part of $(p_1+p_2-p_3')$. With the aid of the δ function, the integral may be reduced to

$$\int_R \cdots \int \frac{d^3p_3'}{\gamma_1 E_3'}\int \tilde{\omega}\tilde{k}d\tilde{\Omega}(\cdots),$$

where $\tilde{\omega}$ and $\tilde{E}_4'$ are defined in (2.11). The remaining factor in the integrand is to be evaluated at the appropriate values specified by the δ function; $d\tilde{\Omega}$ indicates an integration over angles in the special frame.

For experiments I_a and I_b, we have

$$\frac{d^3p_3'}{E_3'}=\frac{d\Omega_3 p_3'^2 dp_3'}{E_3'}$$

$$\cong\frac{d\Omega_3 p_3^2}{m_2E_1}d\gamma_1,$$

while for Experiment I_c, we find

$$\frac{d^3p_3'}{E_3'}=\frac{p_3^2dp_3d\phi_3}{E_3}\sin\theta_3'd\theta_3'$$

$$\cong\frac{p_3dp_3d\phi_3}{p_1E_3}d\gamma_1.$$

Note that the kinematical factors are, respectively, the same as in (2.13a) and (2.13b); a similar result is true for Experiment II. Since the incident flux factors are the same for elastic and inelastic scattering, the fractional corrections from inelastic scattering take the form

$$\int_0^{\Gamma_i}\frac{d\gamma_i}{\gamma_i}\int \tilde{\omega}\tilde{k}d\tilde{\Omega}(\cdots)/|M_0|^2,\quad (\mathrm{I}_a, \mathrm{I}_c, \mathrm{II}_a, \text{or } \mathrm{II}_c)\quad (2.14)$$

where $i=1$ or 2.

In summary, the calculation is to be carried out in the following manner: for each fixed value of γ_i, the integration over photon angles is to be carried out in the special Lorentz frame in which the photon energy is isotropic. The result is then to be integrated with respect to γ_i in order to obtain the desired radiative correction.

B. Details of the Convection Contributions

It is convenient to rearrange (2.3) into the sum of direct terms for each of the particles and an interference contribution

$$B=B(1)+B(2)+B(12),\quad (2.15)$$

where $B(1)$ is the $i=1$, $j=3$ contribution, $B(2)$ is the $i=2$, $j=4$ contribution, and all other terms are combined into the single expression

$$B(12)=-\frac{iZ_1Z_2}{4\pi^3}\int\frac{d^4k}{k^2-\lambda^2}\left[\frac{(2p_1-k)_\mu}{k^2-2k\cdot p_1}+\frac{(2p_3+k)_\mu}{k^2+2k\cdot p_3}\right]\times\left[\frac{(2p_2+k)^\mu}{k^2+2k\cdot p_2}+\frac{(2p_4-k)^\mu}{k^2-2k\cdot p_4}\right].\quad (2.16)$$

If the two particles are identical, B is symmetric under the interchange of the two initial or the two final momenta, but the separate terms in the decomposition do not have this property.

Now (2.4) must be generalized for the case where recoil is important. To make the external radiative correction approximation, we neglect all k dependence in the integrands of (2.14) except that appearing in the convection factors. It should be remarked that this approximation involves neglecting k not only in the Dirac operator, but also in the final-state Dirac spinors which are held fixed at their elastic scattering values. In place of (2.4), we then have for the real photon contribution to the observable cross section

$$2\alpha\tilde{B}'\sigma_0,\quad (2.17a)$$

where

$$2\alpha\tilde{B}'=\int_0^{\Gamma_i}\frac{d\gamma_i}{\gamma_i}\int\tilde{k}\tilde{\omega}d\tilde{\Omega}\,\tilde{S}'.\quad (2.17b)$$

We shall *always* use the index i for the detected particle and j for the undetected one. Thus, we set $i=3$, $j=4$ for Experiment I, and the reverse is true for Experiment II. $\tilde{S}'$ is defined by

$$\tilde{S}'=\tilde{S}'(1)+\tilde{S}'(2)+\tilde{S}'(12),\quad (2.18a)$$

where

$$\tilde{S}'(1)=-\frac{\alpha Z_1^2}{4\pi^2}\left(\frac{p_{1\mu}}{k\cdot p_1}-\frac{p_{3\mu}'}{k\cdot p_3'}\right)^2,\quad (2.18b)$$

$$\tilde{S}'(2)=-\frac{\alpha Z_2^2}{4\pi^2}\left(\frac{p_{2\mu}}{k\cdot p_2}-\frac{p_{4\mu}'}{k\cdot p_4}\right)^2,\quad (2.18c)$$

$$\tilde{S}'(12)=-\frac{\alpha Z_1Z_2}{2\pi^2}\left(\frac{p_{1\mu}}{k\cdot p_1}-\frac{p_{3\mu}'}{k\cdot p_3'}\right)\left(\frac{p_2^{\mu}}{k\cdot p_2}-\frac{p_4'^{\mu}}{k\cdot p_4'}\right).\quad (2.18d)$$

The difference between $\tilde{B}$ and $\tilde{B}'$ is that in the expression for $\tilde{B}$ all the momenta p_l' are approximated by their elastic scattering values p_l; in $\tilde{B}'$ the p_l' are functions of δp_i through the conservation laws.

In order to make use of the infrared cancellation which has already been included in (2.5), we rewrite $\tilde{B}'$ in the form

$$\tilde{B}'=\tilde{B}+\delta\tilde{B}, \qquad (2.19a)$$

where

$$2\alpha\tilde{B}=\int_0^{\Gamma_i}\frac{d\gamma_i}{\gamma_i}\int d\tilde{\Omega}\,[\tilde{k}\tilde{\omega}\tilde{S}], \qquad (2.19b)$$

$$2\alpha\delta\tilde{B}=\int_0^{\Gamma_i}\frac{d\gamma_i}{\gamma_i}\int d\tilde{\Omega}\,\{[\tilde{k}^2\tilde{S}']-[\tilde{k}^2\tilde{S}]\}. \qquad (2.19c)$$

Again, i refers to the detected particle in either experiment. In the last of these equations it is safe to set the photon mass equal to zero because $\tilde{S}'-\tilde{S}$ vanishes for $\gamma_i=0$. In order to agree with the definition of $\tilde{B}$ given in (2.4), it is necessary for the photon energy $\tilde{\omega}$ to be defined differently when it is associated with $\tilde{S}$. It is given simply by γ_i/m_j; the square brackets in (2.19c) are to emphasize that $\tilde{k}$ is to be calculated differently in the two terms. If the experimental situation is such that the undetected particle is nonrelativistic in the special Lorentz frame for all $\tilde{k}$ (i.e., if $m_j^2>\Gamma_i$), $\delta\tilde{B}$ may be neglected and the convection contribution reduces to

$$\begin{aligned}\tilde{B}+\mathrm{Re}B&\cong\frac{Z_1^2}{2\pi}\left\{\rho_1\ln\left(\frac{\Gamma_1^2}{m_2^2E_1E_3}\right)+\rho_1'-\tfrac{1}{2}\ln^2\eta\right\}\\&+\frac{Z_2^2}{2\pi}\left\{\rho_2\ln\left(\frac{\Gamma_1^2}{m_2^2a_2}\right)+\rho_2'-\tfrac{1}{2}\ln^2\left(\frac{a_2}{m_2^2}\right)\right\}\\&+\frac{Z_1Z_2}{2\pi}\left\{-\ln\eta\ln\left(\frac{\Gamma_1^4}{m_2^2E_1E_3a_2^2}\right)-\ln\eta+\beta(2E_1/m_2)\right.\\&\left.-\beta(2E_3/m_2)\right\},\quad (\mathrm{I}_a\text{ and }\mathrm{I}_c)\end{aligned}\qquad (2.20a)$$

$$\begin{aligned}\tilde{B}+\mathrm{Re}B&\cong\frac{Z_1^2}{2\pi}\left\{\rho_1\ln\left(\frac{\Gamma_2^2}{m_1^2a_1}\right)+\rho_1'-\tfrac{1}{2}\ln^2\left(\frac{a_1}{m_1^2}\right)\right\}\\&+\frac{Z_2^2}{2\pi}\left\{\rho_2\ln\left(\frac{\Gamma_2^2}{m_2^2E_1E_3}\right)+\rho_2'-\tfrac{1}{2}\ln^2\eta\right\}\\&+\frac{Z_1Z_2}{2\pi}\left\{-\ln\eta\ln\left(\frac{\Gamma_2^4}{m_2^2E_1E_3a_1^2}\right)-\ln\eta+\beta(2E_1/m_2)\right.\\&\left.-\beta(2E_3/m_2)\right\},\quad (\mathrm{II}_a\text{ and }\mathrm{II}_c)\end{aligned}\qquad (2.20b)$$

where the following notation has been introduced:

$$\begin{aligned}a_1&=p_1\cdot p_3\cong E_1E_3(1-\cos\theta_3)\\&\cong m_2(E_4-m_2),\\a_2&=p_2\cdot p_4=m_2E_4,\\\rho_l&=\frac{a_l}{(a_l^2-m_l^4)^{1/2}}\ln\left(\frac{a_l+(a_l^2-m_l^4)^{1/2}}{m_l^2}\right)-1\\&\cong\ln\frac{2a_l}{m_l^2}-1\qquad\text{for}\quad a_l\gg m_l^2,\\&\cong\frac{2}{3}\frac{(a_l-m_l^2)}{m_l^2}\qquad\text{for}\quad a_l\cong m_l^2,\\\rho_l'&=\frac{1}{2}\left(\frac{a_l+m_l^2}{a_l-m_l^2}\right)^{1/2}\ln\left(\frac{a_l+(a_l^2-m_l^4)^{1/2}}{m_l^2}\right)-1\\&\cong\frac{1}{2}\ln\frac{2a_l}{m_l^2}-1,\\\beta(\lambda)&=(ln^2\lambda)\theta(1-\lambda).\end{aligned}\qquad (2.21)$$

The approximate form taken for ρ_l' is actually valid only for $a_l\gg m_l^2$. In the nonrelativistic region, ρ_l' tends to zero; however, the error made in using the approximation for all a_l is only of order unity. The function ρ_l is not similarly treated since it multiplies a logarithm of Γ_l. In evaluating (2.5) and (2.6), we have used the fact that the energies $\bar{E}_l$ are the energies of the particles as seen in the rest frame of the recoiling particle when the scattering is elastic. Thus, for Experiment I: $\bar{E}_l=p_l\cdot p_4/m_2$; while for Experiment II: $\bar{E}_l=p_l\cdot p_3/m_1$.

When the energy of the recoil particle can be relativistic in the special Lorentz frame, $\delta\tilde{B}$ can make an important contribution. The details of this calculation are relegated to the Appendix; however, some of the general features will be discussed here briefly. We recall that each value of $\tilde{k}$ corresponds to a definite choice for the special Lorentz frame. We see from (2.17) and (2.18), that the angular integration in the special Lorentz frame involves terms of the form

$$\int d\tilde{\Omega}\left(\frac{\tilde{k}^2p_l'\cdot p_k'}{k\cdot p_l'k\cdot p_k'}-\frac{\tilde{k}^2p_l\cdot p_k}{k\cdot p_lk\cdot p_k}\right). \qquad (2.22)$$

It is necessary to state carefully what this expression means because two different Lorentz frames are involved. The angular integration in the second term is carried out in the rest frame of the unobserved recoil particle when the scattering is elastic; the factors of $\tilde{k}$, of course, cancel out for this term. The angular integration in the first term is carried out in the special Lorentz frame; for uniformity of notation, we have set $p_1=p_1'$ and $p_2=p_2'$. If neither l nor k corresponds to the unobserved particle, p_l' and p_k' are both independent of angle and the integral may be easily evaluated; the result is

$$2\pi\int_{-1}^{1}\left(\frac{p_l'\cdot p_k'}{p_x'^2(lk)}-\frac{p_l\cdot p_k}{p_x^2(lk)}\right)dx, \tag{2.23}$$

where $2p_x(lk)=(1+x)p_l+(1-x)p_k$, with a similar definition for p_x'. If l and k are any combination of 1 and 2, p_x equals p_x' and this integral is identically zero; the same is true for $l=k=i$. Under the assumption that the momentum resolution is good ($\Delta p_3/p_3\ll 1$ or $\Delta p_4/p_4\ll 1$), it can be shown that the remaining terms of this form ($l, k\neq j$) are also unimportant; the details are in the Appendix.

In case k or l corresponds to the unobserved particle, the corresponding momentum will depend upon the angular variables in the integration. For example, in Experiment I we have

$$k\cdot p_4'=\gamma_1,$$

and

$$\begin{aligned}p_l'\cdot p_4'&=p_l'\cdot(p_4+\delta p_3)-p_l'\cdot k\\&\cong p_4\cdot p_l-k\cdot p_l'.\end{aligned} \tag{2.24}$$

In evaluating the integrals, we are interested only in keeping contributions which can be large under foreseeable experimental conditions. If terms of nonlogarithmic order are neglected, the calculation is relatively easy; the details are given in the Appendix and the results are contained in the following formula:

$$\delta\tilde{B}=\frac{Z_j^2}{2\pi}\left\{-\tfrac{1}{2}\ln^2\left(1+\frac{2\Gamma_i}{m_j^2}\right)\right\}. \tag{2.25}$$

The energy spectrum may now be obtained from (2.20) and (2.25) by differentiation:

$$\frac{d\sigma}{dp_3}=\frac{\alpha\sigma_0}{\pi\delta p_3}\left\{2Z_1^2\rho_1+Z_2^2\left[2\rho_2-\ln\left(1+\frac{2\Gamma_1}{m_2^2}\right)\right]-4Z_1Z_2\ln\eta\right\} \quad \text{(exp. I)}, \tag{2.26a}$$

$$\frac{d\sigma}{dp_4}=\frac{\alpha\sigma_0}{\pi\delta p_4}\left\{2Z_2^2\rho_2+Z_1^2\left[2\rho_1-\ln\left(1+\frac{2\Gamma_2}{m_1^2}\right)\right]-4Z_1Z_2\ln\eta\right\} \quad \text{(exp. II)}. \tag{2.26b}$$

In the terms arising from $\delta\tilde{B}$, the denominators should be $\delta p_i[1+(m_j^2/2\Gamma_i)]$ rather than δp_i. However, these terms are important only if $\Gamma_i\gg m_j^2$, and in that case the given approximation is valid.

C. Spin-Convection Contributions

The convection contributions discussed in the previous subsection are independent of the spin of the charged particles. If the particles have spin, additional terms, such as (2.1b), will appear in the factors for emission and absorption of photons. These terms depend on the details of the current distribution at somewhat smaller distances than the convection terms. This is evidenced by the extra powers of k they contain, which tend to emphasize the harder photon contributions. However, as will be seen, the interference between the spin and convection terms has a part which is large (i.e., logarithmic in a large energy ratio) and is independent of the specific details of the scattering interaction. In contrast to the infrared part of the convection terms, which is characterized by an integral of the form $\int dk/k$, the spin-convection contribution is characterized by $\int dk/E$, with an upper cutoff of order E for virtual photons and ΔE for real ones. Thus, the approximation of neglecting k inside the residual factor in the matrix element is not likely to be as good in the latter case. However, there seems to be no indication that the correction to this approximation contains logarithms of large energy ratios; this of course does not prevent it from having a large numerical value.

In view of these remarks, the significance of the spin-convection contributions is somewhat uncertain in the general scattering situation. However, they may then give us some information about the order of magnitude of the errors in the straight convection approximation. In any case, there are numerous important applications where the basic scattering is given quite well by the Born approximation; the approximations required can then be studied in detail and they are generally found to be quite adequate. The following analysis will be for Dirac particles only, with no anomalous moment included. The contributions of the anomalous moment of the proton in electron-proton scattering will be discussed explicitly in Sec. III.

Consider the virtual photons first. If the incident particle has spin one-half, it contributes the following spin-convection term:

$$\delta M(1)=\frac{iZ_1^2\alpha}{4\pi^3}\int\frac{d^4k}{k^2}\frac{\bar{u}(p_3)\{\Gamma(p_3-k,\,p_1-k)[k,p_3]+[p_1,k]\Gamma(p_3-k,\,p_1-k)\}u(p_1)}{(k^2-2k\cdot p_3)(k^2-2k\cdot p_1)}, \tag{2.27}$$

where Γ is the γ-matrix operator appearing in the basic scattering matrix element. In order not to make use of the detailed properties of Γ, we wish to arrange the calculation in such a way that γ matrices need not be shifted through it. In fact, if we restrict our attention to logarithmic terms, we will find that it is possible to eliminate the extra γ matrices and thus express $\delta M(1)$ in terms of M_0. If the k dependence of Γ is neglected, the integration with respect to k is easily carried out and it leads to the result

$$\delta M(1)=\frac{Z_1^2\alpha}{16\pi}\int_{-1}^{1}\frac{dx}{p_x^2}\{(1+x)\bar{u}(p_3)\Gamma[p_1,p_3]u(p_1)+(1-x)\bar{u}(p_3)[p_1,p_3]\Gamma u(p_1)\},$$

with $2p_x=(1+x)p_1+(1-x)p_3$. Now if terms of order m_1 are neglected (ultimately in the cross section they would be of order m_1^2/E_1^2), the commutators can be replaced by the invariant scalar product $2p_i\cdot p_2$ and we find

$$\delta M(1)=\alpha C(1)M_0, \tag{2.28a}$$

with

$$C(1)=\frac{Z_1^2}{4\pi}p_1\cdot p_3\int_{-1}^{1}\frac{dx}{p_x^2}\cong\frac{Z_1^2}{2\pi}\ln\frac{2p_1\cdot p_3}{m_1^2}. \tag{2.28b}$$

By the same arguments (we see in the Appendix that the neglect of terms proportional to m_2 gives no appreciable error), we find for the interference of spin and convection currents of the target particle

$$C(2)=\frac{Z_2^2}{2\pi}\frac{a_2}{(a_2^2-m_2^4)^{1/2}}\ln\left(\frac{a_2+(a_2^2-m_2^4)^{1/2}}{m_2^2}\right), \tag{2.29a}$$

which reduces to

$$C(2)=\frac{Z_2^2}{2\pi}\ln\frac{2p_2\cdot p_4}{m_2^2}\quad\text{when}\quad p_2\cdot p_4\gg m_2^2. \tag{2.29b}$$

The latter approximation will be used for all values of $p_2\cdot p_4$ although it yields a small (order unity) error for small $p_2\cdot p_4$.

For the cross terms between particle 1 and particle 2, we have to distinguish the contribution from the interference of the spin current of particle 1 with the convection current of particle 2 called $C(12)$ with the corresponding contribution $C(21)$. The result, whose derivation is presented in the Appendix, is

$$C(12)=C(21)=\frac{Z_1Z_2}{2\pi}\ln\frac{p_2\cdot p_3}{p_1\cdot p_2}=-\frac{Z_1Z_2}{2\pi}\ln\eta. \tag{2.30}$$

Thus, the contribution from the spin of particle 1 to radiative corrections is given by

$$2\alpha[C(1)+C(12)],$$

and similarly the contribution from the spin of particle 2 is

$$2\alpha[C(2)+C(21)].$$

At first sight, the real photon spin-convection terms involve integrals of order $\alpha\int dk/E$ and they should accordingly be of relative order $\alpha\Delta E/E$. However, if the undetected particle is extremely relativistic in the laboratory, a photon emitted parallel to it can carry off considerable energy and a much larger contribution might be obtained. When this situation attains, it is no longer profitable to attempt a general analysis since other features, such as variation of the traces through the dependence of the final momentum on k, will be of comparable importance. Accordingly, we do not include these terms among what we choose to define as the external radiative corrections. These contributions are discussed in greater detail in the following section.

III. REFINEMENTS AND LIMITATIONS

The preceding section contains most of what can be said in a general way about the radiative corrections without a detailed consideration of the basic processes. Before turning to some of the refinements which are possible for specific processes, let us review qualitatively the origin of the logarithmic terms. The doubly logarithmic terms are associated principally with the infrared divergent integrals; roughly speaking, one logarithm comes from the strongly peaked angular integration and the other from the dk/k integration.[9] In the case of the virtual photons, the upper limit of the dk/k integration is effectively determined by the external momenta. In making the external radiative correction approximation, the dependence of the basic factor on k was neglected. If this variation with k is, in fact, not too violent, the doubly logarithmic terms should be well estimated.[10] The effect of the variation of the basic factor on k may perhaps be estimated by expanding it in a power series in k. The linear term in k would no longer contain an infrared divergence, but it could yield a single logarithm from the angular integration. This procedure will be used in one of the estimates that follows.

In the noninfrared parts of the external radiative corrections (occurring in both the convection and spin-convection contributions), some single-logarithmic terms are associated with the strongly peaked angular integrations times a nonlogarithmic dk/E integral. Others are residues of the spurious ultraviolet divergence, which is logarithmic. Clearly if there is any important variation of the basic factor, these terms have not been reliably estimated. In that situation there is no justification in retaining them if the corrections mentioned in the preceding paragraph are ignored. In the most general situation, we therefore regard only the dominant doubly logarithmic terms as having been reliably estimated.

Fortunately, in most of the contemporary or possible experiments in which radiative corrections are likely to be an important consideration, a more detailed study is

[9] More precisely, the form of the doubly logarithmic terms occurring in B and $\tilde{B}$ separately depend on the type of infrared cutoff employed. However, the ambiguous terms cancel in the sum $B+\tilde{B}$.

[10] In case the basic factor already contains infrared divergences, its variation with k is important. This problem of overlapping infrared divergences is discussed in Appendix A of reference 1, where it is shown that the neglect of this k dependence is compensated by the neglect of photon emission and absorption from internal lines.

possible. Some of these refinements on the general discussion will now be presented.

A. The Effect of the k Dependence of the Basic Process

Suppose the basic scattering is due to the exchange of a single photon. Consider two-photon exchange; in obtaining the convection contribution for the soft photon, we have made the following approximation relative to the hard photon:

$$1/(q-k)^2 \to 1/q^2. \tag{3.1}$$

The correction to this approximation corresponds to inserting an extra factor $(2k\cdot q/q^2)$ into the definition of $B(12)$, Eq. (2.16). The resulting correction is easily found to be

$$\delta B(12)=\frac{3Z_1Z_2}{2\pi}\ln\eta, \tag{3.2}$$

which is important enough to be included in our final formulas.

Suppose the basic interaction is more complicated, but still may be expressed by a function $G(q^2)$. Then (3.2) must be multiplied by a factor

$$-q^2G'(q^2)/G(q^2),$$

where the prime denotes differentiation with respect to the argument. If the scattering is due to electromagnetic interaction, but involves finite structure, $G(q^2)$ takes the form $F(q^2)/q^2$, where F is the form factor, and the correction factor which should multiply (3.2) becomes

$$1-q^2F'(q^2)/F(q^2). \tag{3.3}$$

The second term, which might become more important than the first in some circumstances, has been omitted from the tabulated formulas presented in the following section.

The preceding argument is valid only when the basic factor does not vary much within the range of values of k which are important in the integral. Some examples where this variation must be considered more completely will now be cited. The first of these is electron-electron scattering at small center-of-mass angles. Then q^2 is small and the magnitude of $(2k\cdot q/q^2)$ might become large. To see how important these effects might be, we may compare the contributions obtained by the present methods with the exact two-photon exchange contributions.[2] The surprising result is that our methods yield quite accurate answers for this example. Another example is Compton scattering near 180° in the center-of-mass system. The dominant diagram is the one in which the incoming electron emits the final photon before absorbing the initial one. The intermediate electron propagator then yields the small denominator $2p_3\cdot p_2$. In this case if we compare our result with the exact one,[8] we find a difference (exact minus approximate) in B of

$$\Delta B\cong+\frac{1}{4\pi}\ln^2\left(\frac{p_1\cdot p_4}{p_1\cdot p_3}\right). \tag{3.3}$$

An exact calculation of $\tilde{B}$ for large energy loss has not been done, so we cannot determine the corresponding error $\Delta\tilde{B}$ in the calculation of real photon emission. To the extent that ΔB and $\Delta\tilde{B}$ are associated with infrared photons, they may tend to cancel like the doubly logarithmic terms which depend on the type of infrared cutoff. The term (3.3) is not included in the tabulated result of Sec. IV; the results for Compton scattering are clearly less reliable than those for the other processes tabulated.

We conjecture that the difference between electron-electron scattering and Compton scattering arises as follows. The effective range of integration over k which yields the major contribution is determined by the external charged lines. For small q in electron-electron scattering, the range is proportional to q. Thus, as q decreases, $k\cdot q/q^2$ does not increase in importance. On the other hand, for Compton scattering the important range of k is probably determined from $(p_2-p_4)^2$, which is large relative to $2p_2\cdot p_3$ in the situation under consideration. Important corrections result.

B. The Effect of the Anomalous Magnetic Moment of the Proton

In electron-proton scattering, suppose that the extra soft photon exchanged between the two particles interacts with the anomalous moment of the proton rather than with its convection or spin current. Corresponding to the fact that the photon is assumed to be soft, we consider the terms with the least number of powers of k in the numerator and we neglect the dependence of the basic interaction on k. It is then easy to give an argument why these contributions vanish to logarithmic order. For example, suppose the photons are exchanged between the incident particles; we then have to study the structure

$$(p_2+m_2)\int\frac{[\boldsymbol{k},\boldsymbol{p}_1]}{(k^2-2k\cdot p_2)(k^2+2k\cdot p_1)}\frac{d^4k}{k^2}u(p_2).$$

But the result of the k integration can only replace the $\boldsymbol{k}$ in the commutator by $\boldsymbol{p}_2$; it is then trivial to see that the remaining Dirac operators acting on the proton spinor give zero.

The point of this demonstration is that no logarithms arise from the interaction of soft photons with the proton's anomalous moment. However, if we take into account the variation of the basic interaction with k, or the contributions from electron spin interacting with the proton moment, a nonvanishing contribution could occur. These contributions come mainly from the region of very large k and they are not easily included within the framework of our present discussion.

C. Small Virtual Electron Four-Momentum

A situation in which the integration over virtual photon momentum might have a large contribution due to several denominators becoming small simultaneously occurs as follows. The virtual photon emitted by the incoming electron takes nearly all the energy and momentum of the electron. It then scatters from the proton as a nearly real photon and is reabsorbed by the electron. We might view this qualitatively as a Compton scattering of the Lorentz contracted proper field of the electron. Letting p be the momentum of the virtual electron, we consider the contribution arising from the region of small p:

$$\bar{u}(p_3)\int d^4p\,\frac{1}{(p^2-2p_1\cdot p+m_1^2)}\,\frac{1}{(p^2-2p_3\cdot p+m_1^2)}\times\gamma_\lambda\frac{(p+m_1)}{p^2-m_1^2}\gamma_\mu u(p_1).$$

As in the case of the spin convection and infrared terms, a logarithmic factor arises from the angular integration. More important, the factor p is replaced (in form) by p_1+p_3; we then find

$$(\gamma_\lambda p_{1\mu}+p_{3\lambda}\gamma_\mu)\times\log.$$

When this is combined with the factor associated with the scattering of the photon by the proton, which we denote simply by $F_{\mu\lambda}(p_1,p_3,p_2,p_4)$ (p is here neglected), the result is zero by gauge invariance:

$$p_1{}^\mu F_{\mu\lambda}=p_3{}^\lambda F_{\mu\lambda}=0.$$

Thus, no large contribution arises from the situation in which the virtual electron is "soft."

In this subsection and the preceding one, the two-photon exchange terms have been studied from the point of view of radiative corrections. Other studies[11] have placed the emphasis on the off-the-mass-shell Compton scattering by a physical nucleon. While a critical study has not been made of the extent to which the two methods overlap and the extent to which they are supplementary, it seems significant that they agree that the specific two-photon terms are unimportant at energies below 1 BeV. Our analysis shows that, as a consequence of gauge invariance, an unusually large nucleon Compton scattering need not result in a large two-photon contribution. There is no theoretical indication that these terms will become important at higher energies, but neither is there proof that they do not. The situation is also subject to experimental study by comparison of electron-proton and positron-proton scattering, the difference in cross sections being due to the interference between the one- and two-photon terms. Present experiments[12] give no indication of a significant two-photon term at incident energies of 200 and 300 MeV. Two-photon contributions could also introduce terms in the cross section which would make it impossible to fit the experimental data with the Rosenbluth formula.[13] There is no experimental evidence for such a "breakdown" of the Rosenbluth formula.[14]

D. Radiative Corrections to Electromagnetic Scattering of Spin-Zero Particles

Radiative corrections to scattering of spin-zero particles have already been partially included in the so-called convection contributions. However, in case the basic interaction is electromagnetic, there are certain additional refinements which we would like to describe briefly. These refinements are actually of no practical importance, because in actual physical situations the basic interaction is nonelectromagnetic.

The first of these refinements is that two photon lines may terminate at the same vertex because of the $A^2\phi^\dagger\phi$ term in the Lagrangian. Thus, it is possible for a virtual photon to have one end terminate on the external boson line and the other terminate at the same vertex as the exchanged photon. It is not difficult to show that this gives purely a contribution to the "spurious charge renormalization," and is, hence, not of interest. The second refinement comes about because the photon emission operator depends on the momentum of the charged particle. Thus, in the radiative correction in which the boson emits a virtual photon, exchanges a photon, and then reabsorbs the virtual photon, the emission operator for the exchanged photon has the factor $(p_1+p_3-2k)_\mu$ in place of the factor $(p_1+p_3)_\mu$ in the basic matrix element. Using standard methods, the $-2k_\mu$ results in a contribution to be added to B:

$$(3\alpha/2\pi)\ln(2p_1\cdot p_3/m_1^2).$$

E. Other Refinements Involving Virtual Photons

Some other refinements are relegated to the Appendix; these are necessary for the justification of some of the approximations which have been used, but they are not in themselves of any great intrinsic interest. They will be described very briefly here. One of these is electron spin-proton convection contribution arising from an additional exchanged photon. In Sec. IIB, this

[11] S. Drell and M. Ruderman, Phys. Rev. **106**, 561 (1957); S. Drell and S. Fubini, *ibid.* **113**, 741 (1959).

[12] D. Yount and J. Pine, Phys. Rev. **128**, 1842 (1962).

[13] M. N. Rosenbluth, Phys. Rev. **79**, 615 (1950); the possible form of such anomalous terms has been studied by D. Flamm and W. Kammer (to be published).

[14] K. Berkelman, R. M. Littauer, and G. Rouse, in *Proceedings of the 1962 International Conference on High-Energy Physics at CERN*, edited by J. Prentki (CERN, Geneva, Switzerland, 1962), p. 194.

term was treated by neglecting powers of the proton mass in the numerator in comparison with its energy. On its face, this is not a good approximation unless the electron energy is very much greater than the proton mass; nevertheless, the error is shown to be unimportant.

F. Refinements in the Calculation of the Real Photon Contribution

In some situations, the kinematics permit the unobserved photon to have an energy comparable to that of the unobserved particle. For example, if the proton is detected in high-energy electron-proton scattering, an unobserved photon emitted parallel to the electron can carry away most of the unobserved energy. The approximation of neglecting powers of k in the numerator is then no longer valid, and the calculation must be reconsidered carefully. In addition to the explicit k dependence of the integrand, there is an implicit one due to the dependence of the final electron projection operator on k:

$$\frac{p_3'+m_1}{2m_1}=(p_3-k+\delta p_4+m_1)/2m_1.$$

In the calculation of Sec. IIB, this projection operator was approximated by $(p_3+m_1)/2m_1$.

A direct calculation of the additional contributions arising from k dependence would be possible, but somewhat lengthy. Fortunately, as we are interested only in the logarithmic terms, it is possible to give a rather detailed discussion without explicit evaluation of the traces. As usual, the logarithmic terms turn out to be a simple multiple of the original traces. For definiteness, the discussion will be given for Experiment II. Recall that the product of the traces can be reduced ultimately to a polynomial of invariant products of the momenta p_1, p_2, p_3, p_4, δp_4, p_3', and k. Because of the conservation equations (2.8), there are various relations between the invariant products. Clearly, since δp_4 is small, it can be neglected everywhere in the trace. Also, p_3' may be eliminated by:

$$\begin{aligned} p_3'&=p_3-k+\delta p_4,\\ &\cong p_3-k. \end{aligned}$$

The product of traces then depends on invariant products of p_1, p_2, p_3, p_4, and k. Its value for $k=0$ is just the trace occurring in elastic scattering. We have then to consider integrals in the special frame with polynomials in $k\cdot p_l$ in the numerator. The detailed considerations are given in Appendix D; and the results, which are simple, are the following. If the integrand has the form

$$m^2/(k\cdot p_3')^2 \text{ or } 1/(k\cdot p_l)(k\cdot p_k);\quad l,\, k\neq 3,$$

extra powers of k in the numerator may be neglected. In integrals containing a factor $(1/k\cdot p_3')$, the other k's in numerator and denominator may be replaced according to the substitution

$$k\to \tfrac{1}{2}p_3.$$

For example,

$$\frac{k\cdot p_l}{k\cdot p_3' k\cdot p_k}\to\frac{p_3\cdot p_l}{k\cdot p_3' p_3\cdot p_k}\quad l,\, k\neq 3$$

and

$$\frac{k\cdot p_l k\cdot p_k}{k\cdot p_3' k\cdot p_m}\to\frac{1}{2}\frac{p_3\cdot p_l p_3\cdot p_k}{k\cdot p_3' p_3\cdot p_m}\quad l,\, k,\, m\neq 3.$$

If there are higher than two powers of k in the numerator, an additional numerical factor is required. However, in all these cases, it will turn out that cancellations will give a result which is identically zero.

Because of these simple results for the integrals, we can neglect all terms in the trace which do not involve photon emission from external line 3; wherever possible, factors of m_1 are neglected. Let us consider first the interference terms; the pertinent factor in the traces is

$$\cdots p_3'\frac{(2p_{3\mu}'+\gamma_\mu k)}{2k\cdot p_3'}\cdots$$
$$=\cdots(p_3 2p_{3\mu}'-2kp_{3\mu}+p_3\gamma_\mu k)/2k\cdot p_3'\cdots.$$

The first term is the one already included in $\tilde{B}'$. With the substitution $k\to\frac{1}{2}p_3$ and the approximation $p_3^2=m_1^2\cong 0$, the second and third terms cancel. Thus, the interference terms may be ignored; this applies also to interference terms in which the other factor corresponds to emission from an internal line. Finally, the term involving emission only from line 3 involves

$$\cdots\frac{(2p_{3\mu}'+k\gamma_\mu)}{2k\cdot p_3'}p_3'\frac{(2p_{3\mu}'+\gamma_\mu k)}{2k\cdot p_3'}\cdots$$
$$\cong\cdots\left\{\frac{m^2}{(k\cdot p_3')^2}p_3-\frac{1}{2k\cdot p_3'}p_3\right\}\cdots.$$

The first term is already incorporated in $\tilde{B}'$. The second term is proportional to the original trace and yields the radiative correction

$$\frac{\alpha Z_j^2}{4\pi}\ln\left(1+\frac{2\Gamma_i}{m_j^2}\right), \tag{3.4}$$

where i refers to the observed and j to the unobserved particle.

IV. SUMMARY OF RESULTS FOR VARIOUS EXPERIMENTS

The aim of this section is to assemble the various contributions derived in Secs. II and III into convenient formulas for various possible experiments. Since terms of order unity have already been neglected in approximating the various integrals, we omit terms from the general formula which will be small in any foreseeable practical energy range. An exception to this remark is that we keep certain terms of order one when they are associated with the electron vertex function or vacuum polarization. There is no particular justification for this, except that the numbers are simple and well known. It would be feasible to calculate these terms of order unity for pure electrodynamic processes—in fact, they are partly contained in some earlier exact calculations—but present experimental accuracies for this type of experiment do not seem to warrant the effort at the present time. To get an estimate of the error involved in neglecting these contributions of order unity, we note that they are to be multiplied by (α/π) to yield a relative correction to the cross section. An educated guess is that errors as large as 1% are likely, but errors larger than 2% are not likely. Although some new notation is introduced here most of the quantities are defined in Secs. IIA and IIB. Particularly to be noted are (2.9), (2.10), and (2.21).

A. Electron Scattering from a Proton with the Electron Detected

To conform with the notation of Tsai, we set $m_1=m$, $m_2=M$, $Z_1=1$, $Z_2=-Z$, where Z is 1 for electron-proton scattering and -1 for positron-proton scattering; ΔE_3 is the energy resolution of the electron detector as discussed in Sec. IIA. According to (2.10) and (3.2) the fractional correction is then given by

$$
\begin{aligned}
\delta &\cong 2\alpha(\mathrm{Re}B+\tilde{B}),\\
&\cong \frac{\alpha}{\pi}\left\{\left[\ln\left(\frac{2p_1\cdot p_3}{m^2}\right)-1\right]\ln\left[\eta\left(\frac{\Delta E_3}{E_3}\right)^2\right]+\frac{13}{6}\ln\left(\frac{2p_1\cdot p_3}{m^2}\right)-\tfrac{1}{2}\ln^2\eta-\frac{28}{9}\right\}\\
&\quad+\frac{Z\alpha}{\pi}\left\{\ln\eta\,\ln\left[\eta\left(\frac{E_1}{E_4}\right)^2\left(\frac{\Delta E_3}{E_3}\right)^4\right]-\beta(2E_1/M)+\beta(2E_3/M)\right\}\\
&\quad+\frac{Z^2\alpha}{\pi}\left\{\left[\frac{E_4}{p_4}\ln\left(\frac{E_4+p_4}{M}\right)-1\right]\ln\left[\frac{E_1{}^2}{ME_4}\left(\frac{\Delta E_3}{E_3}\right)^2\right]+\tfrac{3}{2}\ln\left(\frac{2E_4}{M}\right)-\tfrac{1}{2}\ln^2\left(\frac{E_4}{M}\right)\right\}. \quad (4.1)
\end{aligned}
$$

It has not been necessary to include $\delta\tilde{B}$ in this expression since it is negligible for all feasible energies. By differentiation, we find for the spectrum of scattered electrons

$$
\frac{d\sigma}{dp_3}\cong\frac{\sigma_0}{\delta p_3}\frac{\alpha}{\pi}\left\{2\left[\ln\left(\frac{2p_1\cdot p_3}{m^2}\right)-1\right]+4Z\ln\eta+2Z^2\left[\frac{E_4}{p_4}\ln\left(\frac{E_4+p_4}{M}\right)-1\right]\right\}. \quad (4.2)
$$

TABLE I. The table contains the fractional radiative corrections for electron-proton ($e^- - p$) and positron-proton ($e^+ - p$) scattering. A and B are experiments in which the electron (or positron) is detected and A' and B' are experiments in which the proton is detected. The results of this paper are given by δ, and those of reference 3 by δ^*. The experimental conditions for the various experiments are:

$A(A')$: $E_1=900$, $E_3=327$, $E_4=1511$, $\Delta E_3=13.1$ $(\Delta p_4=10)$;
$B(B')$: $E_1=5000$, $E_3=500$, $E_4=5438$, $\Delta E_3=10$ $(\Delta p_4=110)$;

where the energies and momenta are in MeV.

	Coefficient of			$e^- - p$		$e^+ - p$	
Case	α/π	$Z\alpha/\pi$	$Z^2\alpha/\pi$	δ	δ^*	δ	δ^*
A	−47.7	−13.0	−0.8	−0.142	−0.150	−0.082	−0.086
B	−58.2	−31.2	−7.2	−0.225	−0.210	−0.080	−0.099
A'	−53.5	−18.4	−1.9	−0.171		−0.086	
B'	−44.1	−30.1	−7.1	−0.191		−0.051	

Table I contains two numerical examples of the application of (4.1); they are the same examples given by Tsai, whose results are labeled δ^*. We may note the following differences between the present calculation and Tsai's: (i) The terms retained by Tsai are expressed in terms of Spence functions; in effect, our calculation is arranged so that all Spence functions are of order unity and they are neglected. (ii) In the Z and Z^2 contributions we retain spin-convection and noninfrared convection terms which Tsai neglects; these terms have a single power of a logarithm of energy ratios, but they are not numerically very important in the cases considered. The difference between these two approximations is not unreasonably large; and as far as accuracy is concerned, there is no great basis for preferring one over the other.

B. Electron Scattering from a Proton with Proton Detected

The principal difference between this and the preceding example is that $\delta\tilde{B}$ is quite important, corresponding to the fact that a photon can carry off a large amount of energy if it is emitted parallel to the final electron, even if the energy loss of the proton is relatively small. Another complication of lesser importance is that it is necessary to treat the electron trace in its entirety, including spin contributions, as discussed in Sec. IIIF. The fractional correction is

$$\delta\cong\frac{\alpha}{\pi}\left\{\left[\ln\left(\frac{2p_1\cdot p_3}{m^2}\right)-1\right]\ln\Lambda+\frac{17}{12}\ln\left(\frac{2p_1\cdot p_3}{m^2}\right)-\frac{28}{9}-\tfrac{1}{2}\ln^2\Lambda-\tfrac{3}{4}\ln\Lambda\right\}$$
$$+\frac{Z\alpha}{\pi}\left\{\ln\eta\ln\left[\frac{(p_1\cdot p_3)^2}{M^2E_1E_3}\Lambda^4\right]-\beta(2E_1/M)+\beta(2E_3/M)\right\}$$
$$+\frac{Z^2\alpha}{\pi}\left\{\left[\frac{E_4}{p_4}\ln\left(\frac{E_4+p_4}{M}\right)-1\right]\ln\left[\frac{(p_1\cdot p_3)^2}{M^2E_1E_3}\Lambda^2\right]+\tfrac{3}{2}\ln\left(\frac{2E_4}{M}\right)-\tfrac{1}{2}\ln^2\eta\right\},\quad(4.3)$$

where $\Lambda=[(E_1+M)/E_4](\Delta p_4/p_4)$. The energy distribution of the recoil protons is

$$\frac{d\sigma}{dp_4}\cong\frac{\sigma_0}{\delta p_4}\frac{\alpha}{\pi}\left\{\left[\ln\left(\frac{2p_1\cdot p_3}{m^2}\right)-1-\ln\Lambda-\tfrac{3}{4}\right]+4Z\ln\eta+2Z^2\left[\frac{E_4}{p_4}\ln\left(\frac{E_4+p_4}{M}\right)-1\right]\right\}.\quad(4.4)$$

It should be noted that the "radiative tail" is considerably smaller in proportion to the cross section than for the case where the electron is detected; it also decreases somewhat more rapidly with increasing energy loss. Table I contains numerical examples for the same experimental parameters worked out previously for the electron detection experiment. It should be noted that (4.3) disagrees with the result obtained by Krass.[4] The disagreement can be traced to an error in the hard photon calculation in reference 4. In particular, the equation giving $k\cdot p_3$ after (3.18) should be replaced by

$$k\cdot p_3\cong(p_3^{\text{el}}/p_3)(k\cdot p_3^{\text{el}}).$$

With this change, his result can be reconciled with ours.[15]

C. Electron-Electron Scattering

There is now no physical distinction between Experiments I and II; the radiative corrections may be calculated in either way and both methods give the same results. Adhering to the restrictions imposed earlier, we assume that both final electrons are extremely relativistic in the final state (laboratory system) when the scattering is elastic. A slight complication is introduced because exchange gives rise to two terms in the scattering amplitude. In the doubly logarithmic corrections, both terms are corrected by the same factor. However, for the singly logarithmic corrections, these factors are different and the scattering amplitude is not altered by a common factor. This difficulty may be overcome by the following observation. When the two terms are comparable, the factors are the same. When they are not comparable, we may simply use the factor associated with the biggest term, with negligible error. To see this, consider vacuum polarization which modifies the photon propagator occurring in elastic scattering

$$1/(p_1-p')^2$$

by the factor

$$\left[1+\frac{\alpha}{3\pi}\ln\left(\frac{2p_1\cdot p'}{m^2}\right)\right].$$

Here p' refers to either p_3 or p_4, and a linear combination of both photon propagators occurs in the elastic scattering matrix element, corresponding to the direct and exchange contributions. If $p_1\cdot p_3$ and $p_1\cdot p_4$ are comparable (i.e., the same within a factor of 3 or 4), we can use either as the argument of the logarithm with an error of order unity. If they differ by a large factor, the photon propagators will also differ by a large factor; and the term with the smallest values of $p_1\cdot p'$ dominates. Accordingly, we can take as our rule that the minimum value of $p_1\cdot p'$ be used as the argument of the logarithm; this is simply mE_m, where E_m is the smaller of E_3 and E_4. A similar argument may be used with respect to the two-photon terms. Recall that the single logarithmic contributions arising from convection and spin-convection were fortuitously cancelled by the contribution of Sec. IIIA. As a consequence, only the single logarithmic terms from the vertex parts survive. These may also be expressed in terms of E_m. The fractional radiative correction is then

$$\delta\cong\frac{\alpha}{\pi}\left\{\left[\ln\frac{2E_3E_4}{mE_1}-1\right]\ln\left(\frac{E_1^2}{E_3E_4}r^3\right)-\tfrac{1}{2}\ln^2\left(\frac{E_1^2}{E_3E_4}r\right)+\frac{11}{3}\ln\left(\frac{2E_m}{m}\right)-\tfrac{3}{4}\ln\left(\frac{E_1}{m}r\right)\right\},\quad(4.5)$$

where r depends on the type of experiment and the

[15] Dr. Krass agrees with these remarks in a private communication.

resolution

$$r=\Delta p_i/p_i \qquad \text{for type (a) experiment,} \tag{4.6a}$$

$$r=\frac{p_i \sin\theta_i}{m}\Delta\theta_i$$

$$\cong 2\left(1-\frac{E_i}{E_1}\right)\frac{\Delta\theta_i}{\sin\theta_i} \qquad \text{for type (c) experiment.} \tag{4.6b}$$

It is also a simple matter to revise (4.5) for a clashing beam experiment ($\mathbf{p}_1+\mathbf{p}_2=0$) where only one of the particles is detected with energy resolution. Simply replace E_l by the invariant $p_l\cdot p_2/m$; r is then given by (4.6a) with the understanding that Δp_i and p_i are center-of-mass quantities. As usual, the energy distribution may be obtained by differentiation; it will not be reproduced here. Equation (4.5) differs from the result of Tsai.[3] His calculation uses approximations adequate for the experimental conditions he envisaged, but the present results are valid for more general conditions.

D. Electron-Positron Scattering

The results are contained in a single formula:

$$\delta\cong\frac{\alpha}{\pi}\left\{\left[\ln\left(\frac{2E_1E_4}{mE_3}\right)-1\right]\ln\left(\frac{E_1{}^2}{E_3E_4}r^3\right)-\tfrac{1}{2}\ln^2\left(\frac{E_3}{E_4}r\right)+\frac{11}{3}\ln\left(\frac{2E_4}{m}\right)-\tfrac{3}{4}\ln\left(\frac{E_1}{m}r\right)\right\}, \tag{4.7}$$

where r is given by (4.6). The result may also be used in the center-of-mass frame using the rules given in the preceding subsection.

E. Compton Scattering with Photon Detected

In this case, particle (1) has mass and charge zero; as in electron-electron scattering, $\delta\tilde{B}$ is quite important.

The result is

$$\delta\cong\frac{\alpha}{\pi}\left\{\left[\ln\left(\frac{2E_4}{m}\right)-1\right]\ln\left(\frac{E_1}{E_4}r\right)-\tfrac{1}{2}\ln^2\left(\frac{E_1}{E_4}r\right)+\tfrac{3}{2}\ln\left(\frac{2E_4}{m}\right)-\tfrac{3}{4}\ln\left(\frac{E_1}{m}r\right)\right\}. \tag{4.8}$$

From the discussion given in IIIA, this formula is not expected to be very reliable for backward scattering in the center-of-mass system, corresponding here to $E_1\gg E_3$.

F. Compton Scattering with Electron Detected

There is now a slight complication in that the undetected particle has zero mass; the separation of $\tilde{B}'$ into $\tilde{B}$ and $\tilde{B}\delta$ is. therefore, apparently meaningless. However, we may use our previous result by taking the limit of the expressions for $\tilde{B}$ and $\delta\tilde{B}$ as $m_1\to 0$. Only $\tilde{B}$ contributes, and the correction is

$$\delta\cong\frac{\alpha}{\pi}\left\{\left[\ln\left(\frac{2E_4}{m}\right)-1\right]\ln\left(\frac{E_1}{E_3}r^3\right)+\tfrac{3}{2}\ln\left(\frac{2E_4}{m}\right)-\tfrac{1}{2}\ln^2\left(\frac{E_1}{E_3}\right)\right\}. \tag{4.9}$$

The validity of the limiting procedure ($m_1\to 0$) has been confirmed by a direct calculation in which m_1 is taken to be zero and $\tilde{B}'$ is evaluated directly. As in the preceding example, this equation is not expected to be reliable for $E_1\gg E_3$.

V. DISCUSSION

Two basically different approximations have been made in obtaining the results of the preceding section. The first of these, which was discussed with the aid of numerous examples in Sec. III, is the neglect of all terms which are not obviously large because of the confluence of several small denominators. The terms retained can be studied without reference to the details of the basic interaction; we have termed them *external radiative corrections*. The *internal radiative corrections*, whose complete evaluation would be many more times difficult, have in most cases been estimated to be of lesser importance. A general estimate of the error made in neglecting the internal radiative corrections cannot be made. In the case of Compton scattering with nearly complete interchange of momentum between the electron and the photon, the corrections appear to be large. On the other hand, in those cases where the basic interaction is due to the exchange of a photon between charged particles, there is no reason to suppose that these corrections are important relative to the second type of approximation, which is the neglect of terms of order unity (times α/π) in the external radiative corrections. This second approximation is expected to introduce an error of order 1% and probably not more than 2% in the calculated cross section.

There is one other important question to be discussed; namely, to what extent can the higher order corrections be estimated by assuming that the factor $(1+\delta)$ is actually the beginning of the series expansion of e^δ, where e^δ provides a good estimate of the corrections to all orders. It is known that the infrared part of the radiative correction should be exponentiated in this manner.[16] The doubly logarithmic terms in the virtual photon contribution B clearly are related to the infrared divergence since they depend on the type of cutoff (λ or $k_{\min}$) used in the calculation. With very good resolution (Γ_i sufficiently small), the real photon contribution $\tilde{B}'$ ($\cong\tilde{B}$) is purely infrared, and so its doubly logarithmic

[16] See reference 1 and other references given there; a more recent work on the same subject is K. T. Mahanthappa, Phys. Rev. **126**, 329 (1962).

contribution may also be exponentiated. With poorer energy resolution, $\delta\tilde{B}$ becomes important and terms involving the square of the logarithm of the energy resolution arise. Although these terms are not of the typical infrared form, their main contribution does arise from the smaller values of the energy loss [see (2.26)]. We, therefore, make the following *conjecture*. If δ' is the doubly logarithmic part of δ, the expression

$$e^{\delta'}(1+\delta-\delta')$$

yields a better estimate of the radiative corrections than does the original estimate $(1+\delta)$.

ACKNOWLEDGMENTS

One of us (D.R.Y.) wishes to thank Professor Maurice Lévy for the hospitality of the Laboratoire de Physique Theorique et Hautes Energies during the academic year 1960–1961, when this work was initiated. Both authors enjoyed the hospitality of CERN during the summer months of 1961, and one of us (D.R.Y.) spent a brief but productive period at the Aspen Physics Institute during August 1962. We are indebted to Professor H. Suura for his careful criticisms of the manuscript.

APPENDIX A. KINEMATICAL DETAILS

As discussed in Sec. II, it is convenient to carry out the final-state integration over the phase space of the final unobserved photon and particle by first integrating over angles in a special Lorentz frame in which the total momentum of the unobserved constituents is zero. There is a different Lorentz frame for each value of the four-momentum loss of the observed particle. This section of the Appendix will be concerned primarily with the details of how various kinematical quantities, as seen in the special frame, depend on the momentum loss of the observed particle. Unfortunately, the analysis is complicated and uninteresting, but it is straightforward and it seems unavoidable if all the terms of logarithmic order are to be properly identified.

The special Lorentz frame is defined by the relation

$$\mathbf{p}_j'+\mathbf{k}=0. \tag{A1}$$

In this equation, and in the remainder of the Appendix, we shall always let the subscript j refer to the final unobserved particle and i to the observed particle; the spatial parts of four-vectors in this frame are indicated by boldface type. The energies of the unobserved constituents in the special frame are

$$\tilde{k}=\gamma_i/C \quad \text{and} \quad \tilde{E}_j'=(m_j^2+\gamma_i)/C, \tag{A2}$$

with

$$C=(m_j^2+2\gamma_i)^{1/2},$$

and

$$\gamma_i=(p_1+p_2)\cdot\delta p_i.$$

When it is necessary to distinguish, we use $C=C_1$ or C_2 in Experiment I or Experiment II, respectively. The energy $\tilde{E}_l'$ of any other particle as seen in this special frame can then be determined from

$$C\tilde{E}_l'=(k+p_j')\cdot p_l'=(p_j+\delta p_i)\cdot p_l'. \tag{A3}$$

The resulting values of E_l' for the two experiments are the following:

Experiment I:

$$\tilde{E}_1'=\frac{m_2E_3+\gamma_1-m_2\delta E_3}{C_1}\cong\frac{m_2E_3+\gamma_1}{C_1}, \tag{A4a}$$

$$\tilde{E}_2'=\frac{m_2E_4+m_2\delta E_3}{C_1}, \tag{A4b}$$

$$\tilde{E}_3'=\frac{m_2E_1}{C_1}. \tag{A4c}$$

Experiment II:

$$\tilde{E}_1'=\frac{m_2(E_4-m_2)+\gamma_2-m_2\delta E_4}{C_2}\cong\frac{m_2(E_4-m_2)+\gamma_2}{C_2}, \tag{A5a}$$

$$\tilde{E}_2'=\frac{m_2E_3+m_2\delta E_4}{C_2}, \tag{A5b}$$

$$\tilde{E}_4'=\frac{m_2E_1}{C_2}. \tag{A5c}$$

Obviously, in these expressions δE_i is zero for Experiments I_c and II_c. In order to make suitable approximations in evaluating the final integral over the momentum loss, it is necessary to know whether these various energies are relativistic or nonrelativistic. The situation is complicated by the fact that as δp_i varies, some of the energies may vary between relativistic and nonrelativistic values. This variation is to be studied under the general experimental restrictions we have imposed, namely, that the energy of the incident and scattered particle be extremely relativistic as seen in the laboratory, that $|q^2|\gg m_1^2$, and the resolution of the detected particle be reasonably good (say $|\delta\mathbf{p}_i|/|\mathbf{p}_i|\lesssim 0.05$). Also, we assume $m_1\leqslant m_2$ and if $m_1=m_2$ that both final particles are extremely relativistic in the laboratory system. As δp_i tends toward zero, the various $\tilde{E}_l'$ approach limits $\tilde{E}_l$ which are simply the energies of the elastically scattered particles in the Lorentz frame in which the unobserved particle is at rest. For the two experiments these are:

I. $\tilde{E}_1=E_3$, $\tilde{E}_2=E_4$, $\tilde{E}_3=E_1$.

II. $\tilde{E}_1=m_2(E_4-m_2)/m_1$, $\tilde{E}_2=m_2E_3/m_1$, $\tilde{E}_4=m_2E_1/m_1$.

Now we can discuss the situation in the two experiments. In Experiment I it is easy to see that $\tilde{E}_1'$ and $\tilde{E}_3'$ are always much greater than m_1. It is also easy to see that if $\tilde{E}_2$ is large compared to m_2 (i.e., $E_4\gg m_2$),

$\bar{E}_2'$ will be large compared to m_2 for all δp_3. The only case that might cause trouble then is that $\bar{E}_2'$ might be comparable to m_2 for small δp_3, but might become relativistic for larger δp_3. From (A4b), this could happen only for Experiment I_a when $\delta E_3 \gg E_4$, which implies $E_3 \gg E_4$ and, hence, $E_1/E_3 \cong 1$. Now the condition that E_4 be a nonrelativistic energy is simply that $|q^2|/m_2^2$ be much less than one. But since $|q^2| \gg m_1^2$, this may happen only if $m_2 \gg m_1$. Thus, we need simultaneously

$$m_1^2 \ll |q^2| \ll m_2^2 \quad \text{and} \quad E_3 \gg \delta p_3 \gg m_2.$$

In the most "favorable" case of electron-proton scattering, one could reach these conditions by the following experimental parameters:

$$E_1 = 50 \text{ BeV}, \quad \theta_3 = 0.5 \times 10^{-3} \text{ rad}, \quad \text{and} \quad \delta p_3 = 5 \text{ BeV}.$$

It seems unlikely that precision work will be carried out in this region in the near future; even if it was, it would not be justified to investigate this point in further detail in view of other more serious approximations that have been made in the calculations. Thus, for Experiment I we conclude that the $\bar{E}_l'$ always have the same character (relativistic or nonrelativistic) as the corresponding $\bar{E}_l$.

In Experiment II, $\bar{E}_1'$ and $\bar{E}_4'$ are always extremely relativistic; $\bar{E}_2$ is also extremely relativistic, but under certain conditions, $\bar{E}_2'$ may become nonrelativistic. For electron-electron scattering these conditions cannot be attained because of the restriction that both final particles to be extremely relativistic. For electron-proton scattering they can be attained for Experiment II_a when $E_1 \gg m_2$ and θ_4 is small. We shall not present a detailed analysis here, but in the calculations special attention will be paid to those integrals which depend on $\bar{E}_2'$ in Experiment II.

Finally, we present a simple approximation which is useful in evaluation of certain integrals which will occur in the next section. Angular integrations often result in expressions of the form

$$\frac{E}{p} \ln\left(\frac{E+p}{E-p}\right).$$

It is easy to show that this function can be bounded from above and below in the following manner:

$$2 \ln\left(\frac{2E}{m}\right) < \frac{E}{p} \ln\left(\frac{E+p}{E-p}\right) < 2\left(1+\frac{m^2}{2E^2}\right) \ln\left(\frac{2E}{m}\right). \quad \text{(A6)}$$

If $E \gg m$, we can approximate the function by the lower bound; the upper bound is numerically a better approximation than the lower bound (error is less than 5% for all E), but it leads to slightly more complicated integrals. These bounds will be particularly helpful in the case of $\bar{E}_2'$ in Experiment II since it will enable us to determine the consequences of its variation from relativistic values to nonrelativistic ones.

APPENDIX B. CALCULATION OF $\delta\tilde{B}$

We have to evaluate the integrals of the form given in (2.22):

$$I_{kl} = \int d\tilde{\Omega} \left\{ \frac{\tilde{k}^2 p_l' \cdot p_k'}{k \cdot p_l' k \cdot p_k'} - \frac{\tilde{k}^2 p_l \cdot p_k}{k \cdot p_l k \cdot p_k} \right\}. \quad \text{(B1)}$$

The angular integration in the first term is carried out in the Lorentz frame defined by (A1); that of the second term is defined by $\tilde{\mathbf{p}}_j = 0$. We have given arguments in Sec. II that all terms with l and k combinations of 1 and 2, and the term with $l = k = i$ are identically zero. Now we shall show that the terms with $l = i$ and $k = 1$ or 2 are also unimportant as a consequence of the restriction $|\Delta \mathbf{p}_i| \ll |\mathbf{p}_i|$. The integral (2.34) yields a logarithm only if $p_l \cdot p_i \gg m_l m_i$; in that case it is approximately

$$I_{ki} = 4\pi \ln\left(\frac{p_k \cdot p_i'}{p_k \cdot p_i}\right) \cong 4\pi\left(-\frac{p_l \cdot \delta p_i}{p_l \cdot p_i}\right).$$

The resulting contribution to $\delta\tilde{B}$ is always of order $(\Delta p_i/p_i)$.

We must therefore consider only two cases for (B1):

1. The case $k = j$, $l \neq j$.

Using (2.35), we find for the integrand

$$\frac{\bar{E}_l'}{\bar{E}_l' - \bar{p}_l' \cos\theta} - \frac{\bar{E}_l}{\bar{E}_l - \bar{p}_l \cos\theta} - \frac{\tilde{k}^2}{\gamma_i},$$

which yields

$$I_{lj} = 2\pi \left\{ \frac{\bar{E}_l'}{\bar{p}_l'} \ln\left(\frac{\bar{E}_l' + \bar{p}_l'}{\bar{E}_l' - \bar{p}_l'}\right) - \frac{\bar{E}_l}{\bar{p}_l} \ln\left(\frac{\bar{E}_l + \bar{p}_l}{\bar{E}_l - \bar{p}_l}\right) - \frac{2\gamma_i}{m_j^2 + 2\gamma_i} \right\}. \quad \text{(B2)}$$

2. The case $k = l = j$.

The integrand is

$$\frac{m_j^2 \tilde{k}^2}{\gamma_i} - \frac{m_j^2}{(\bar{E}_j - \bar{p}_j \cos\theta)^2},$$

which yields

$$I_{jj} = 4\pi\left[\frac{m_j^2}{m_j^2 + 2\gamma_i} - 1\right] = -8\pi \frac{\gamma_i}{m_j^2 + 2\gamma_i}. \quad \text{(B3)}$$

The results may now be combined and the final integrals evaluated for the two experiments.

Experiment I.

It is convenient to split $\delta\tilde{B}$ into three parts associated with the decomposition of $\tilde{S}'$. For Experiment I, $\delta\tilde{B}(1)$ vanishes and we need consider only $\delta\tilde{B}(12)$ and $\delta\tilde{B}(2)$.

The arguments of Appendix A showed that $\bar{E}_1'$ and $\bar{E}_3'$ are always relativistic, so it is permissible to approximate the functions of (B2) by the lower bound of (A6). It follows that

$$\delta\tilde{B}(12)\cong\frac{Z_1Z_2}{\pi}\int_0^{\Gamma_1}\frac{d\gamma_1}{\gamma_1}\ln\left(\frac{\bar{E}_1'\bar{E}_3}{\bar{E}_1\bar{E}_3'}\right)$$

$$=\frac{Z_1Z_2}{\pi}\int_0^{\Gamma_1}\frac{d\gamma_1}{\gamma_1}\ln\left(1+\frac{\gamma_1}{m_2E_3}\right)$$

$$\cong\frac{Z_1Z_2}{2\pi}\ln^2\left(1+\frac{\Gamma_1}{m_2E_3}\right). \qquad \text{(B4)}$$

There are no experimental conditions for which this term is likely to be important. The expression for $\delta\tilde{B}(2)$ reduces to

$$\delta\tilde{B}(2)=-\frac{Z_2^2}{2\pi}\int_0^{\Gamma_1}\frac{d\gamma_1}{\gamma_1}\left\{\frac{\bar{E}_2}{\bar{p}_2}\ln\left(\frac{\bar{E}_2+\bar{p}_2}{\bar{E}_2-\bar{p}_2}\right)-\frac{\bar{E}_2'}{\bar{p}_2'}\ln\left(\frac{\bar{E}_2'+\bar{p}_2'}{\bar{E}_2'-\bar{p}_2'}\right)\right\}.$$

If we use the simple lower bound approximation from (A6), the result is

$$\delta\tilde{B}(2)\cong\frac{Z_2^2}{4\pi}\left\{-\ln^2\left(1+\frac{2\Gamma_1}{m_2^2}\right)+2\ln^2\left(1+\frac{\Delta E_3}{E_4}\right)\right\}. \qquad \text{(B5)}$$

The additional contribution that would be obtained by using the upper bound of (A6) rather than the lower bound can, after considerable labor, be reduced to

$$\frac{Z_2^2}{8\pi}\frac{m_2^2}{E_4^2}\left\{-\ln^2\left(1+\frac{2\Gamma_1}{m_2^2}\right)+\ln^2\left(1+\frac{\Delta E_3}{E_4}\right)+2\ln\left(1+\frac{\Delta E_3}{E_4}\right)\ln\left(\frac{\Gamma_1}{2E_4\Delta E_3}\right)\right\}. \qquad \text{(B6)}$$

We want to show that this can safely be neglected. Note first that it can be comparable to (B5) only if E_4 is comparable to m_2. The logarithms can then have a large argument only if $E_3\cong E_1\gg\Delta p_3\gg m_2$. As discussed in Appendix A, there are no practical experimental situations where these and the other restrictions are likely to be met. The approximation (B5) is, therefore, adequate. In practice, only the first term of (B5) is likely to be important.

Experiment II

This time $\delta\tilde{B}(2)$ vanishes. The results for $\delta\tilde{B}(1)$ and $\delta\tilde{B}(12)$, using the lower bound approximation of (A6) are

$$\delta\tilde{B}(1)=\frac{Z_1^2}{4\pi}\left\{-\ln^2\left(1+\frac{2\Gamma_2}{m_1^2}\right)+2\ln^2\left[1+\left(\frac{E_1}{E_4}-1\right)\frac{\Delta p_4}{p_4}\right]\right\}, \qquad \text{(B7)}$$

$$\delta\tilde{B}(12)=\frac{Z_1Z_2}{2\pi}\ln^2\left(1+\frac{\Delta E_4}{E_3}\right). \qquad \text{(B8)}$$

The contribution resulting from the difference between the upper and lower bounds of (A6) is of order (m_1^2/E_3^2) compared with (B8) and is, hence, completely negligible. As in Experiment I, only the first term of (B7) is important in practice; the unimportant terms are dropped in the result quoted in Sec. IIB.

APPENDIX C. SPIN-CONVECTION TERMS INVOLVING A HEAVY PARTICLE

The purpose of this Appendix is to derive some of the results of Sec. IIC. In effect, it is an exercise in the manipulation of relatively complicated expressions to reduce them to a simple form by neglecting terms of order unity. Consider first the contribution arising from the convection current of particle 2 and the spin current of particle 1. Following the procedure used in deriving $C(1)$, the result may be written

$$C(12)=\frac{Z_1Z_2}{4\pi}\int_{-1}^{1}(1+x)dx\left\{\frac{p_1\cdot p_2}{p_x^2(12)}+\frac{p_3\cdot p_2}{p_x^2(32)}\right\}, \qquad \text{(C1)}$$

where

$$p_x^2(12)=\tfrac{1}{4}[(1+x)^2m_1^2+(1-x)^2m_2^2-2p_1\cdot p_2(1-x^2)],$$
$$p_x^2(32)=\tfrac{1}{4}[(1+x)^2m_1^2+(1-x)^2m_2^2+2p_3\cdot p_2(1-x^2)].$$

It is understood that (C1) is a principal-value integral, since the imaginary part will not contribute to the cross section to order α. The general restrictions assumed on the parameters are

$$p_1\cdot p_2,\ p_3\cdot p_2\gg m_1m_2,$$

however, it is possible that $m_2^2\gg p_1\cdot p_2,\ p_3\cdot p_2$.

Neglecting terms of order unity, we find

$$C(12)=-\frac{Z_1Z_2}{2\pi}\left\{\ln\left(\frac{p_1\cdot p_2}{p_3\cdot p_2}\right)+\frac{m_2^2}{m_2^2+2p_1\cdot p_2}\ln\left(\frac{2p_1\cdot p_2}{m_2^2}\right)-\frac{m_2^2}{m_2^2-2p_3\cdot p_2}\ln\left(\frac{2p_2\cdot p_3}{m_2^2}\right)\right\}, \qquad \text{(C2a)}$$

$$\cong-\frac{Z_1Z_2}{2\pi}\ln\left(\frac{p_1\cdot p_2}{p_3\cdot p_2}\right). \qquad \text{(C2b)}$$

The approximation is justified as follows: It is clear that the second and third terms of (C2a) are important only when the arguments of the logarithms are small; the third term is of order one when $2p_3\cdot p_2 \cong m_2^2$ in spite of the small denominator. The scattering kinematics yield

$$\frac{2p_3\cdot p_2}{m_2^2}=\frac{2E_3}{m_2}=\frac{2E_1/m_2}{1+(E_1/m_2)(1-\cos\theta_3)}\geqslant\frac{2p_1\cdot p_2/m_2^2}{1+(2p_1\cdot p_2/m_2^2)}.$$

Thus, if $2p_1\cdot p_2 \gg m_2^2$, the second term of (C2a) is small and the third of order unity. If $2p_1\cdot p_2 \ll m_2^2$, the second and third terms tend to cancel; and, in fact, $C(12)$ itself is small.

In calculating the contributions involving the spin of particle 1, the approximation

$$p_1 u(p_1)=m_1 u(p_1)\cong 0$$

was used repeatedly. In fact, detailed examination of the trace shows these terms are actually of relative order (m_1^2/E_1^2). In evaluating the contributions from the spin of particle 2, more care is needed since these terms may not be negligible if m_2 is large. The contribution from the convection current of particle 1 and the spin current of particle 2 yields a contribution $C(21)$ which is identical to $C(12)$ (to order unity) plus a residue

$$-\frac{m_2Z_1Z_2}{8\pi}\int_{-1}^{1}(1-x)dx\left\{\frac{p_1\Gamma+\Gamma p_3}{p_x^2(32)}+\frac{p_3\Gamma+\Gamma p_1}{p_x^2(12)}\right\}$$
$$\cong\frac{m_2Z_1Z_2}{2\pi}\left\{[p_1\Gamma+\Gamma p_3]\frac{\ln(2p_3\cdot p_2/m_2^2)}{m_2^2-2p_3\cdot p_2}+[p_3\Gamma+\Gamma p_1]\frac{\ln(2p_1\cdot p_2/m_2^2)}{m_2^2+2p_1\cdot p_2}\right\}. \quad \text{(C3)}$$

If $m_2^2 \ll 2p_1\cdot p_2$, $2p_3\cdot p_2$ these terms are of relative order $(m_2^2/2p_1\cdot p_2)\times\ln(2p_1\cdot p_2/m_2^2)$ and hence negligible. If $m_2^2 \gg 2p_1\cdot p_2$, $2p_3\cdot p_2$, their relative order is E_1/m_2 or E_3/m_2, which is again negligible. In the intermediate region, they are of order unity.

The contribution involving the spin and convection current of particle 2 may be evaluated in a similar manner. Again, the terms involving a factor m_2 are of order unity or less and the surviving contribution is given by (2.29).

APPENDIX D. FORMULAS FOR CALCULATION OF THE EFFECT OF TRACE VARIATION

In this Appendix the formulas needed in the analysis of Sec. IIIF will be derived. The integrals required in the evaluation of $\tilde{B}'$ take the form

$$\int_0^{\Gamma_i}\frac{d\gamma_i}{\gamma_i}\int d\tilde{\Omega}\,\frac{\tilde{k}^2p_a'\cdot p_b'}{k\cdot p_a'k\cdot p_b'}.$$

The modification of these integrals produced by factors of form $k\cdot p_c'$ in the numerator is to be determined for various choices of a and b. It may be helpful to preface the following analysis with a brief outline of the arguments which will be used to determine which terms are significant. The main point is to observe the γ_2 dependence resulting after the angular integration. If m_1^2 can be neglected without causing a divergence for small γ_2, and if there are powers of γ_2 in the numerator, the final integral will be some positive power of Γ_2. Such terms will be of relative order $(|\Delta\mathbf{p}_4|/E_1)^n$, $n\geqslant 1$, and hence negligible. We, therefore, retain only the terms with the smallest degree in γ_2 (as $m_1^2\to 0$). How this is used may be illustrated by the following example:

$$\mathbf{p}_a\cdot\mathbf{p}_b=\tilde{E}_a\tilde{E}_b-p_a\cdot p_b,$$
$$\cong\frac{p_a\cdot(p_3+\delta p_4)p_b\cdot(p_3+\delta p_4)}{m_1^2+2\gamma_2}.$$

Case 1. $a=b=3$

$$\int d\tilde{\Omega}\,\frac{\tilde{k}^2m_1^2}{(k\cdot p_3')^2}=4\pi\frac{\tilde{k}^2m_1^2}{\gamma_2^2}=4\pi\frac{m_1^2}{m_1^2+2\gamma_2}$$
$$=4\pi-4\pi\frac{2\gamma_2}{m_1^2+2\gamma_2}.$$

Here the first term yields a contribution to $\tilde{B}$ and the second to $\delta\tilde{B}$. Suppose there is an extra factor $k\cdot p_c'$ in the numerator. If $c=3$, this factor will yield an extra factor γ_2 after the angular integration, and it may therefore be neglected. Suppose $c\neq 3$; then

$$k\cdot p_c'=\tilde{k}(\tilde{E}_c-\tilde{p}_c\cos\tilde{\theta}),$$
$$\doteq\tilde{k}\tilde{E}_c=\frac{\gamma_2p_c\cdot(p_3+\delta p_4)}{m_1^2+2\gamma_2},$$

where a term drops out in the angular integration. The final integration yields a result of order unity. Thus, an extra power of $k\cdot p_c'$, gives a result which may be ignored. The same is true of higher powers as well, so in this case the trace variation may be neglected.

Case 2. $a=b\neq 3$

If the extra factor is $k\cdot p_3'=\gamma_2$ or $k\cdot p_a$ the resulting contribution is clearly negligible. If it is $k\cdot p_c$ $(c\neq 3, b)$, we write

$$k\cdot p_c=\tilde{k}\tilde{E}_c-\mathbf{k}\cdot\mathbf{p}_c,$$
$$\cong\tilde{k}\tilde{E}_c-(\mathbf{k}\cdot\mathbf{p}_a\mathbf{p}_c\cdot\mathbf{p}_a/\tilde{E}_a^2)-(\mathbf{k}\times\mathbf{p}_a)\cdot(\mathbf{p}_c\times\mathbf{p}_a)/\tilde{E}_a^2,$$
$$\cong\frac{\tilde{E}_c}{\tilde{E}_a}k\cdot p_a=\frac{p_c\cdot(p_3+\delta p_4)}{p_a\cdot(p_3+\delta p_4)}k\cdot p_a.$$

The second term in the second line was transformed by the argument given in the introductory paragraph and the third drops out upon angular integration. The final contribution is negligible.

Case 3. $a\neq 3$, $b\neq 3$, $a\neq b$

It is necessary to consider only the special cases $c=3$, a, or b, since all other cases may be reduced to these by momentum conservation. If $c=3$, the extra factor of γ_2 yields an unimportant contribution. If $c=a$, the angular integral yields

$$4\pi\frac{\tilde{k}p_a\cdot p_b}{\bar{E}_b}\ln\left(\frac{2\bar{E}_b}{m_b}\right).$$

But $(\tilde{k}/\bar{E}_b)=\gamma_2/p_b\cdot(p_3+\delta p_4)$, and the resulting contribution may be neglected.

Case 4. $a=3$, $b\neq 3$, one factor $k\cdot p_c$ in numerator

This is now the only case which can yield an important contribution. A factor $k\cdot p_3'$ in the numerator may be neglected. Thus consider

$$\int d\tilde{\Omega}\frac{\tilde{k}^2 k\cdot p_c p_3'\cdot p_b}{k\cdot p_3' k\cdot p_b}.$$

As in Case 2,

$$k\cdot p_c \cong \frac{\bar{E}_c}{\bar{E}_b}k\cdot p_b - \frac{(\mathbf{k}\times\mathbf{p}_b)\cdot(\mathbf{p}_c\times\mathbf{p}_b)}{\bar{E}_b^2},$$

$$\cong \frac{\bar{E}_c}{\bar{E}_b}k\cdot p_b.$$

This is just what is obtained by the substitution

$$k\cdot p_c/k\cdot p_b \rightarrow p_3\cdot p_c/p_3\cdot p_b.$$

Case 5. $a=3$, $b\neq 3$, factor $k\cdot p_c k\cdot p_d$ in numerator

If c or $d=3$, the result is immediately negligible. For c, $d\neq 3$, the factors $k\cdot p_c$ and $k\cdot p_d$ are rewritten as in Case 4. This time the last term involving $(\mathbf{k}\times\mathbf{p}_b)\cdot(\mathbf{p}_c\times\mathbf{p}_b)/\bar{E}_b^2$ in $k\cdot p_c$ and a similar term in $k\cdot p_d$ cannot be eliminated by angular symmetry since there are two such factors in the numerator. However, they may be eliminated by another argument:

$$\left|\frac{(\mathbf{k}\times\mathbf{p}_b)\cdot(\mathbf{p}_c\times\mathbf{p}_b)}{\bar{E}_b^2}\right| \leqslant \frac{\tilde{k}}{\bar{E}_b^2}\{[\mathbf{p}_b\times(\mathbf{p}_c\times\mathbf{p}_b)]^2\}^{1/2},$$

$$=\frac{\tilde{k}}{\bar{E}_b}[\mathbf{p}_c^2\mathbf{p}_b^2-(\mathbf{p}_c\cdot\mathbf{p}_b)^2]^{1/2},$$

$$\cong \tilde{k}\left[p_c\cdot p_b\frac{2\bar{E}_c}{\bar{E}_b}\right]^{1/2}.$$

Because of the factor $\tilde{k}\sim(\gamma_2/2)^{1/2}$, this term may be neglected.

$$k\cdot p_c \cong (\bar{E}_c/\bar{E}_b)k\cdot p_b,$$

and

$$\int d\tilde{\Omega}\frac{\tilde{k}^2 p_3'\cdot p_b k\cdot p_c k\cdot p_d}{k\cdot p_3' k\cdot p_b}$$

$$\cong 4\pi\frac{\tilde{k}^3 p_3'\cdot p_b}{\gamma_2}\frac{\bar{E}_c\bar{E}_d}{\bar{E}_b},$$

$$=4\pi\frac{\gamma_2^2}{(m_1^2+2\gamma_2)^2}\frac{p_3'\cdot p_b p_c\cdot p_3 p_d\cdot p_3}{p_b\cdot p_3}.$$

The resulting term logarithmic in Γ_2 is precisely what would be obtained by the substitution

$$\frac{k\cdot p_c k\cdot p_d}{k\cdot p_b}\rightarrow\frac{1}{2}\frac{p_3\cdot p_c p_3\cdot p_d}{p_3\cdot p_d}$$

in the original expression.