Likewise, since

$$g^{-n-1}\alpha^{n-m+1}\delta(\alpha-Rg) = \alpha^{1-\epsilon}g^{-m-1+\epsilon}R^{n-m+\epsilon}\delta(\alpha-Rg) \leqq \alpha^{1-\epsilon}x^{m+1-\epsilon}\alpha'^{-m-1+\epsilon}\delta(\alpha-Rg), \quad \text{(A13)}$$

we obtain

$$I_2(z,\alpha) \leqq \frac{\lambda}{2}\alpha^{-\epsilon}\int_{-1}^{1} dz' \int_{0-}^{\infty} d\alpha' \, |\varphi_l^{[n]}(z',\alpha')|\alpha'^{-m-1+\epsilon} \times J(z,\alpha;z',\alpha') \quad \text{(A14)}$$

with

$$J(z,\alpha;z',\alpha') \equiv \int_0^1 dx\, x^{l+m-n-\epsilon}\alpha\delta(\alpha-Rg) \leqq \frac{2\gamma\theta(\gamma-\alpha'-\mu^2-2\mu(\alpha'+\rho)^{1/2})}{[(\gamma-\alpha'-\mu^2)^2-4\mu^2(\alpha'+\rho)]^{1/2}} \quad \text{(A15)}$$

on account of (A12). Therefore, the α' integral is finite as $\alpha\to\infty$, and hence $I_2(z,\alpha)\to 0$. Thus we have established (A7).

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Backscatter from Inhomogeneous Media*

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A WKB approximation is used to calculate cross sections for the 180° scattering of scalar and vector waves by a class of spherically symmetric, repulsive potentials. These potentials are such that the corresponding index of refraction has a unique zero. The scalar problem is discussed in the framework of quantum mechanics, and the result is just the classical cross section. Electromagnetic backscatter from a dielectric is found to be three-quarters of the scalar approximation in the extreme geometrical-optics limit.

I. INTRODUCTION

INTEREST in radar cross sections has encouraged investigations on the backscatter of waves from inhomogeneous media. In general, this is a difficult problem to analyze. Exact solutions are rare, and the Born approximation[1] is worthless when the index of refraction differs significantly from unity. The Schiff approximation[2] is expected to have a wider range of validity, but its usefulness hinges on the evaluation of a difficult volume integral. In this paper, we consider the simplest spherically symmetric systems to which a "semiclassical" approximation is applicable. Specifically, the index of refraction of such a system is a continuous function of r, and it has a unique zero at r_0.

The scalar-wave problem is studied by investigating the equivalent problem of electron backscatter from repulsive potentials. The correspondence principle is derived for 180° scattering; that is, a WKB scattering amplitude is obtained which gives the correct classical cross section. The classical result is shown to have an upper limit of $\frac{1}{4}r_0^2$. In addition, the inverse square-law potential is examined in some detail, for the phase shifts are known exactly, and corrections to the classical result can be derived.

It is known[3,4] that the problem of electromagnetic scattering from a spherically symmetric dielectric is reducible to the solution of two scalar problems; i.e., two radial differential equations must be solved for two sets of phase shifts. For our purpose, the amplitude for vector backscatter is proportional to the difference of the corresponding scalar amplitudes. While difficulties arise because of the zero in the index of refraction, these scalar amplitudes can be replaced by WKB approximations analogous to the one introduced earlier. This approximation is valid in the extreme geometrical optics limit. Here expressions simplify, with the differential cross section for electromagnetic backscatter reducing to three-quarters of the result predicted on the basis of the scalar wave equation.

II. THE SCALAR PROBLEM

The time-independent scalar wave equation is

$$[\nabla^2+k^2n^2(r)]\psi(\mathbf{r})=0, \quad (2.1)$$

where $n(r)$ is the (spherically symmetric) index of refraction of the medium, and $2\pi/k$ is the wavelength of the incident wave. The asymptotic scattering solution of Eq. (2.1) is

$$\psi(\mathbf{r}) \underset{r\to\infty}{\longrightarrow} e^{i\mathbf{k}\cdot\mathbf{r}} + \frac{e^{ikr}}{r}f(\theta), \quad |\mathbf{k}|=k, \quad (2.2)$$

* The research reported in this paper was sponsored by the Air Force Ballistic Systems Division, Air Force Systems Command, under contract No. AF 04(694)-1 with Space Technology Laboratories, Inc.

[1] D. S. Saxon, *Lectures on the Scattering of Light*, Scientific Report No. 9, Dept. of Meteorology, UCLA, 1955.

[2] L. I. Schiff, Phys. Rev. **104**, 1481 (1956).

[3] P. J. Wyatt, Phys. Rev. **127**, 1837 (1962).

[4] D. Arnush, Space Technology Laboratory Report No. 6110-7466-RU-001 (unpublished).

where θ is the scattering angle. The differential cross section is related to the angular amplitude f by

$$\sigma(\theta)=|f(\theta)|^2. \qquad (2.3)$$

We want to evaluate $\sigma(\pi)$ by geometrical optics. To be more specific, we rewrite Eq. (2.1) in the form

$$[\nabla^2+k^2-U(r)]\psi(\mathbf{r})=0. \qquad (2.4)$$

This is the Schrödinger equation provided that

$$k^2=(2m/\hbar^2)E, \qquad (2.5a)$$

$$U(r)=(2m/\hbar^2)V(r). \qquad (2.5b)$$

Equations (2.4) and (2.5) represent the scattering of nonrelativistic particles of mass m and energy E by a potential $V(r)$. Now the problem is to demonstrate the correspondence principle for backscattering; that is, to let $\hbar\rightarrow 0$ in such a way as to obtain the $\sigma(\pi)$ of classical mechanics.

The scattering amplitude is given by the relation

$$f(\theta)=(2ik)^{-1}\sum_{l=0}^{\infty}(2l+1)[\exp(2i\delta_l)-1]P_l(\cos\theta). \qquad (2.6)$$

The δ_l are the phase shifts which each of the elementary partial waves of angular momentum l has experienced because of the potential $V(r)$. The P_l are Legendre polynomials. The WKB or semiclassical approximation[5] for scattering is defined by the following steps:

(1) The approximation of δ_l by the WKB phase shift

$$\delta_l'=\lim_{R\rightarrow\infty}\left\{\int_{r_1}^{R}dr\left[k^2-U(r)-\frac{(l+\frac{1}{2})^2}{r^2}\right]^{1/2}-kR+(l+\tfrac{1}{2})\pi/2\right\}, \qquad (2.7)$$

$$k^2-U(r_1)-(l+\tfrac{1}{2})^2/r_1^2=0. \qquad (2.8)$$

(2) The replacement of the sum in Eq. (2.6) by an integral over l.

(3) The replacement of $P_l(\cos\theta)$ by a smooth function of l.

The approximation in step (1) is applicable to potentials which vary slowly in a wavelength. The result of differentiating Eq. (2.7) with respect to l is

$$\frac{2d\delta_l'}{dl}=\theta(L)=\pi-2\int_{r_1}^{\infty}dr\frac{L}{r^2[2m(E-V)-L^2/r^2]^{1/2}}, \qquad (2.9)$$

$$L=\hbar(l+\tfrac{1}{2}). \qquad (2.10)$$

$\theta(L)$ is the classical scattering angle[6] for energy E and angular momentum L.

[5] K. Ford and J. A. Wheeler, Ann. Phys. (N. Y.) **7**, 259 (1959).
[6] H. Goldstein, *Classical Mechanics* (Addison-Wesley Publishing Company, Cambridge, Massachusetts, 1950), pp. 82, 91–92.

The correspondence principle appears to have been demonstrated[7] only when $l\sin\theta\gtrsim 1$. We are interested in the case $l\sin\theta\rightarrow 0$. Using the above WKB prescription, we replace P_l by

$$P_l(\cos\theta)\simeq\exp(-i\pi l)J_0[(l^2+l)^{1/2}(\pi-\theta)], \qquad (2.11)$$

a result which is exact when $\theta=\pi$ and l is an integer. The error in replacing the sum by an integral can be determined in principle from the Euler summation formula[8]

$$K\equiv\sum_{l=0}^{\infty}g_l-\int_0^{\infty}dl\,g(l), \qquad (2.12)$$

of which the following form is most useful:

$$K\sim\tfrac{1}{2}(g_0+g_\infty)+\sum_{l=1}^{N}\frac{B_{2l}}{(2l)!}[g_\infty^{(2l-1)}-g_0^{(2l-1)}]+R_{N+1}. \qquad (2.13)$$

The superscripts on the g's refer to the number of derivatives. As $N\rightarrow\infty$, the remainder R_{N+1} may approach 0. In general, however, the sum over Bernoulli numbers B_{2l} will yield an asymptotic series.

When $\theta=\pi$, Eq. (2.6) simplifies to

$$f(\pi)=(2ik)^{-1}\sum_{l=0}^{\infty}(2l+1)[\exp(2i\delta_l)-1]\times\exp(-i\pi l). \qquad (2.14)$$

Since $f(\pi)$ is assumed to be well defined, Eq. (2.14) can be written as

$$f(\pi)=(2ik)^{-1}\lim_{\omega\rightarrow 0}\sum_{l=0}^{\infty}(2l+1)\times\exp[2i\delta_l-(\omega+i\pi)l], \qquad (2.15)$$

where we have used

$$\lim_{\omega\rightarrow 0}\sum_{l=0}^{\infty}(2l+1)\exp[-(\omega+i\pi)l]=0. \qquad (2.16)$$

The latter equation follows from the completeness relation for Legendre polynomials. It may also be derived from Eq. (2.12), for $K=2/\pi^2+i/\pi$ by Eq. (2.13), whereas

$$\lim_{\omega\rightarrow 0}\int_0^{\infty}dl\,(2l+1)\exp[-(\omega+i\pi)l]=-\frac{2}{\pi^2}-\frac{i}{\pi}. \qquad (2.17)$$

The limit on ω, a positive quantity, is carried out after summation. Equation (2.15) now becomes

$$f(\pi)=K+f_i(\pi), \qquad (2.18)$$

$$f_i(\pi)\equiv(2ik)^{-1}\lim_{\omega\rightarrow 0}\int_0^{\infty}dl\,(2l+1)\times\exp[2i\delta(l)-(\omega+i\pi)l]. \qquad (2.19)$$

[7] N. F. Mott and H. S. Massey, *Theory of Atomic Collisions* (Oxford University Press, London, 1949), 2nd ed., p. 120.
[8] K. Knopp, *Theory and Application of Infinite Series* (Hafner Publishing Co., New York, 1928), English ed., p. 520.

Clearly, the above result is most useful when K is a small correction; that is, when the replacement of sum by integral is a good approximation. In general, the phase of the summand in Eq. (2.15) changes rapidly with l, so that $f(\pi)$ will be the result of delicate cancellations between positive and negative terms. For this situation, K will be large. However, if the phase does not change appreciably over a large range of l's, the algebraic sign of the terms stays the same, and we get a large contribution to the sum. Hence, K is expected to be small when the phase is stationary for some l_0:

$$2d\delta(l_0)/dl_0-\pi=0. \quad (2.20)$$

Referring to Eqs. (2.9) and (2.10) for the WKB approximation to the phase shifts, we see that Eq. (2.20) is satisfied when $L_0=0$. The implication is that Eq. (2.19) can describe the backscatter of a classical particle with zero-angular momentum.

The above suggests that we replace $\delta(l)$ in Eq. (2.19) by an expansion of $\delta'(l)$ in powers of $\hbar$. The class of systems for which this makes sense can be determined by examining the potential-dependent term in Eq. (2.7),

$$\frac{2m}{\hbar^2}\int_{r_1(\hbar)}^{R} dr\left[(E-V)-\frac{\hbar^2(l+\frac{1}{2})^2}{2mr^2}\right]^{1/2}. \quad (2.21)$$

An expansion about $\hbar=0$ implies that the centrifugal potential term is a correction to V. Then the turning point r_1 is approximately r_0, where

$$E=V(r_0). \quad (2.22)$$

This immediately rules out attractive potentials and repulsive potentials for which E is always greater than V. The simplest remaining situation is a repulsive potential and an incident energy for which there is a single turning point; e.g., a monotonically decreasing potential. Equation (2.22) is valid for this case if only small l's are important, and if $kr_0\gg1$. Systems with multiple turning points are ignored as these involve nonclassical barrier penetration.

The Taylor expansion of Eq. (2.7) in powers of $\hbar$ yields

$$\delta'(l)=(l+\tfrac{1}{2})\pi/2+a/\hbar-\tfrac{1}{2}b\hbar(l+\tfrac{1}{2})^2+O[\hbar^3(l+\tfrac{1}{2})^4], \quad (2.23)$$

where

$$a=\lim_{R\to\infty}\left\{\int_{r_0}^{R} dr\,[2m(E-V)]^{1/2}-(2mE)^{1/2}R\right\}, \quad (2.24)$$

$$b=\int_{r_0}^{\infty} dr\,\frac{1/r^2}{[2m(E-V)]^{1/2}}. \quad (2.25)$$

If Eq. (2.23) is substituted for $\delta(l)$ in Eq. (2.19), and if a change of variables, $x^2=\hbar(l+\frac{1}{2})^2$, is made, the integral becomes

$$f_i(\pi)\simeq p^{-1}\exp(2ia/\hbar)\lim_{\omega\to0}\int_{(\hbar/4)^{1/2}}^{\infty} dx\,x\times\exp[-(\omega+ib)x^2+O(\hbar)], \quad (2.26)$$

$$p=\hbar k. \quad (2.27)$$

The convergence factor has been modified for convenience. In the limit $\hbar\to0$, Eq. (2.26) simplifies to the semiclassical result:

$$f^0(\pi)\equiv(2pib)^{-1}\exp(2ia/\hbar). \quad (2.28)$$

The corresponding cross section is obtained from Eqs. (2.3) and (2.28):

$$\sigma^0(\pi)=\frac{1}{4p^2}\left\{1\Big/\left[\int_{r_0}^{\infty} dr\,\frac{1/r^2}{[2m(E-V)]^{1/2}}\right]^2\right\}. \quad (2.29)$$

We must now show that Eq. (2.29) follows from the classical cross section, namely,[6]

$$\sigma_{\text{cl}}(\theta)=-\frac{1}{p^2}\frac{L}{\sin\theta}\frac{dL}{d\theta}. \quad (2.30)$$

For a repulsive potential and a single turning point r_1, the relation between θ and L is

$$\pi-\theta=2L\int_{r_1(L)}^{\infty} dr\,\frac{1/r^2}{[2m(E-V)-L^2/r^2]^{1/2}}. \quad (2.31)$$

The integral in Eq. (2.31) is greater than zero. Thus, when $\theta=\pi$, L must be zero. Expansion about $L=0$ gives the relations

$$L\underset{\theta\to\pi}{\longrightarrow}(2b)^{-1}(\pi-\theta), \quad (2.32)$$

$$\frac{dL}{d\theta}\underset{\theta\to\pi}{\longrightarrow}-(2b)^{-1}, \quad (2.33)$$

where b is defined by Eq. (2.25). By substituting the last two equations in Eq. (2.30), we again obtain Eq. (2.29).

Equation (2.28) applies to the combination of a repulsive potential and a single turning point. A validity condition relating wavelength and potential strength is most easily obtained from Eq. (2.13), the difference between sum and integral. Referring to Eq. (2.15), we find

$$g_l\simeq(2ik)^{-1}(2l+1)\exp[2i\delta_l'-(\omega+i\pi)l], \quad (2.34)$$

so that K contributes the following first-order correction in $\hbar$ (magnitude) to $f^0(\pi)$:

$$(\hbar/4i)p^{-1}\exp(2ia/\hbar). \quad (2.35)$$

The ratio of the preceding factor to Eq. (2.28) gives the following restriction:

$$\int_{r_0}^{\infty} dr\,\frac{1/r^2}{[k^2-U(r)]^{1/2}}=\int_0^{1/r_0}\frac{du}{[k^2-U(1/u)]^{1/2}}\ll1. \quad (2.36)$$

It is clear from a graph of the u integrand that the above integral has a minimum value of $(kr_0)^{-1}$ for repulsive potentials which vanish at infinity. Thus, we are again led to the conclusion that $kr_0 \gg 1$ for the validity of the classical result. The above integral also occurs in the classical cross section, Eq. (2.29). We obtain, therefore, the following upper limit to $\sigma_{\text{cl}}(\pi)$:

$$\sigma_{\text{cl}}(\pi)=\sigma^0(\pi)\leq\frac{1}{4k^2}\frac{1}{(kr_0)^{-2}}=\frac{r_0^2}{4}. \tag{2.37}$$

The form of Eq. (2.37) suggests division by 4π of a geometrical total cross section, πr_0^2.

The integral in Eq. (2.29) can be done exactly for inverse power-law potentials, $U(r)=\alpha_n r^{-n}$, $n>0$. Thus, by the substitution $r=r_0 t^{-1}$, we have

$$I=\int_{r_0}^{\infty}dr\,\frac{1/r^2}{[k^2-U(r)]^{1/2}}=\frac{1}{n(kr_0)}B\left(\frac{1}{n},\frac{1}{2}\right), \tag{2.38}$$

where B is the beta function.[9] In terms of gamma functions, Eq. (2.29) becomes

$$\sigma^0(\pi)=\frac{r_0^2}{4\pi}\left[\frac{\Gamma(1/n+\frac{1}{2})}{\Gamma(1/n+1)}\right]^2. \tag{2.39}$$

For the Coulomb potential, we have $\sigma^0(\pi)=\frac{1}{16}r_0^2$, $r_0=\alpha_1/k^2$, while the result for the inverse square law is $\sigma_{\text{cl}}(\pi)=(1/\pi^2)r_0^2$, $r_0=(\alpha_2)^{1/2}/k$. The backscatter cross section for the inverse square law can be approximated by the classical result when $(\alpha_2)^{1/2}\gg 1$. Finally, as $n\to\infty$ in Eq. (2.39), σ^0 approaches the upper limit of Eq. (2.37).

III. THE INVERSE SQUARE-LAW POTENTIAL

In Sec. II, we could not write down explicit corrections to $\sigma^0(\pi)$ because, in general, the error in replacing δ_l by δ_l' is not known. However, the phase shifts are known exactly[7] for the potential $U=\gamma r^{-2}$, and they are equal to the WKB phase shifts:

$$\delta_l=\delta_l'=(\pi/2)\{(l+\tfrac{1}{2})-[(l+\tfrac{1}{2})^2+\gamma]^{1/2}\}. \tag{3.1}$$

In addition, the integration in Eq. (2.19) can be carried out exactly, even if Eq. (2.11) is used to extend (approximately) the result to angles close to 180°:

$$f_i(\theta)=(1/2ik)\lim_{\omega\to 0}\int_0^{\infty}dl\,(2l+1)$$
$$\times\exp[2i\delta(l)-(\omega+i\pi)l]J_0[(l^2+l)^{1/2}(\pi-\theta)] \tag{3.2}$$
$$=\left(\frac{1}{k}\right)\lim_{\omega\to 0}\int_0^{\infty}dx\,x$$
$$\times\exp\{-(\omega+i\pi)[x^2+(\gamma+\tfrac{1}{4})]^{1/2}\}J_0[x(\pi-\theta)], \tag{3.3}$$

where the substitution $x^2=l(l+1)$ is made in the last step. Equation (3.3) integrates to[10]

$$f_i(\theta)=\frac{-\pi}{k}\frac{\exp\{-i(\gamma+\frac{1}{4})^{1/2}[\pi^2-(\pi-\theta)^2]^{1/2}\}}{[\pi^2-(\pi-\theta)^2]^{3/2}}$$
$$\times\{1+i(\gamma+\tfrac{1}{4})^{1/2}[\pi^2-(\pi-\theta)^2]^{1/2}\}. \tag{3.4}$$

By multiplying Eq. (3.4) by its complex conjugate and by keeping only the term proportional to γ, we obtain the classical cross section[6] for $\theta\simeq\pi$:

$$|f(\theta)|^2=[\gamma/(k\pi)^2][(\theta/\pi)(2-\theta/\pi)]^{-2}. \tag{3.5}$$

The ratio γ/k^2 is independent of $\hbar$.

When $\theta=\pi$, we use Eqs. (2.18) and (3.4) to write the following exact result:

$$f(\pi)=K-(k\pi^2)^{-1}$$
$$\times\exp[-i\pi(\gamma+\tfrac{1}{4})^{1/2}][1+i\pi(\gamma+\tfrac{1}{4})^{1/2}]. \tag{3.6}$$

However, as we now show, the asymptotic expansion for K, Eq. (2.13), is valid only when $\gamma>1$. To obtain K, we set

$$g_l=(2k)^{-1}(2l+1)\exp\{-(\omega+i\pi)[\gamma+(l+\tfrac{1}{2})^2]^{1/2}\}, \tag{3.7}$$

and note that $g_\infty^{(2l-1)}=0$, because of the convergence factor ω. Then, Eq. (2.13) yields

$$\exp[i\pi(\gamma+\tfrac{1}{4})^{1/2}]kK\sim\frac{1}{6}-\frac{\pi^2}{630}(\gamma+\tfrac{1}{4})^{-1}+O[(\gamma+\tfrac{1}{4})^{-2}]$$
$$+i\left\{\frac{\pi}{60(\gamma+\frac{1}{4})^{1/2}}+\frac{\pi}{630}\frac{(1-\pi^2/8)}{(\gamma+\frac{1}{4})^{3/2}}+O[(\gamma+\tfrac{1}{4})^{-5/2}]\right\}. \tag{3.8}$$

The first three terms of an asymptotic expansion for $\sigma(\pi)$ now follow from Eqs. (2.3), (3.6), and (3.8):

$$k^2\pi^2\sigma(\pi)\sim\gamma+\alpha+\beta\gamma^{-1}+O(\gamma^{-2}), \tag{3.9}$$

where

$$\alpha=1/\pi^2-1/12-\pi^2/180\simeq-0.037, \tag{3.10}$$

$$\beta=(11/75600)\pi^4\simeq 0.014. \tag{3.11}$$

Equation (3.9) can be interpreted as an expansion in powers of $\hbar^2$. A graph of $\sigma(\pi)$ versus γ approaches a straight line for large γ, with α as its projected intercept.

IV. THE VECTOR PROBLEM

The preceding formulation can be applied, in a suitable limit, to the problem of electromagnetic backscatter from an inhomogeneous dielectric $\epsilon(r)$. With the assumption of sinusoidal time dependence of the

[9] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. I, p. 425.

[10] A. Erdélyi, editor, *Tables of Integral Transforms* (McGraw-Hill Book Company, Inc., New York, 1954), Vol. II, p. 9.

fields, the pertinent Maxwell equations are

$$\nabla\cdot(\epsilon\mathbf{E})=0, \tag{4.1a}$$

$$\nabla\cdot\mathbf{H}=0, \tag{4.1b}$$

$$\nabla\times\mathbf{E}=ik\mathbf{H}, \tag{4.1c}$$

$$\nabla\times\mathbf{H}=-ik\epsilon\mathbf{E}. \tag{4.1d}$$

It has been shown by Arnush[4] that $\mathbf{H}$ is derivable from two scalar functions,

$$\mathbf{H}=\nabla\times(\epsilon^{\frac{1}{2}}\phi\mathbf{r})+(1/k)\nabla\times[\nabla\times(\psi\mathbf{r})], \tag{4.2}$$

where ϕ and ψ satisfy the following partial differential equations:

$$\nabla^2\psi+k^2\epsilon(r)\psi=0, \tag{4.3}$$

$$\nabla^2\phi+[k^2\epsilon(r)-W(r)]\phi=0, \tag{4.4}$$

$$W(r)=\frac{3}{4\epsilon^2}\left(\frac{d\epsilon}{dr}\right)^2-\frac{1}{2\epsilon}\frac{d^2\epsilon}{dr^2}. \tag{4.5}$$

The boundary conditions on ψ and ϕ must be such that

$$\mathbf{E}\underset{r\to\infty}{\longrightarrow}\hat{x}\exp(ikz)+\mathbf{A}(\theta,\Phi)\frac{1}{r}\exp(ikr). \tag{4.6}$$

Here, $\hat{x}$ is the initial polarization and $\mathbf{A}$ is the vector scattering amplitude. The absolute square of $\mathbf{A}$ determines the differential cross section. The radial equations associated with Eqs. (4.3) and (4.4) are

$$\frac{d^2}{dr^2}(rR_l)+\left[k^2\epsilon(r)-\frac{l(l+1)}{r^2}\right](rR_l)=0, \tag{4.7}$$

$$\frac{d^2}{dr^2}(rS_l)+\left[k^2\epsilon(r)-W(r)-\frac{l(l+1)}{r^2}\right](rS_l)=0, \tag{4.8}$$

with boundary conditions

$$rR_l,rS_l\underset{r\to\infty}{\longrightarrow}0, \tag{4.9}$$

$$rR_l\underset{r\to\infty}{\longrightarrow}\sin(kr-l\pi/2+\delta_l), \tag{4.10}$$

$$rS_l\underset{r\to\infty}{\longrightarrow}\sin(kr-l\pi/2+\eta_l). \tag{4.11}$$

The phase shifts δ_l and η_l determine the scattering.

The scattering amplitude is derived by substituting expansions of the form

$$\sum_{l,m}a_{l,m}R_lY_l^m(\theta,\Phi), \tag{4.12}$$

for ψ and ϕ in Eq. (4.2), and then by using Eqs. (4.1c) and (4.6). In general, $\mathbf{A}$ is a complicated function of angles, but it simplifies for the forward and backward directions:

$$4ik\mathbf{A}(0)=\hat{x}\sum_{l=1}^{\infty}(2l+1)[\exp(2i\delta_l)-1+\exp(2i\eta_l)-1], \tag{4.13}$$

$$4ik\mathbf{A}(\pi)=\hat{x}\sum_{l=1}^{\infty}(-1)^l(2l+1)[\exp(2i\delta_l)-\exp(2i\eta_l)]. \tag{4.14}$$

The latter equation may clearly be written in the form

$$2A(\pi)=f_\psi(\pi)-f_\phi(\pi)-(2ik)^{-1}[\exp(2i\delta_0)-\exp(2i\eta_0)]. \tag{4.15}$$

Here, $f_{\psi,\phi}$ are backscatter amplitudes for the scalar problems represented by Eqs. (4.3) and (4.4), with boundary conditions of the type Eq. (2.2). We derive an approximate expression for $A(\pi)$ by replacing $f_{\psi,\phi}$ by their relevant WKB scattering amplitudes.

Equation (4.3) can be rewritten in Schrödinger form:

$$\nabla^2\psi+[k^2-U(r)]\psi=0, \tag{2.4}$$

where

$$\epsilon(r)=1-U(r)/k^2. \tag{4.16}$$

The restrictions on $U(r)$ are the same as those applied to the potentials of Sec. II; namely, that U is positive-definite and slowly varying in a wavelength, and that $U=k^2$ is satisfied uniquely at $r=r_0$. With this choice, $\epsilon(r_0)=0$, and $\epsilon(r)\leq 1$; e.g., a plasma. The WKB phase shifts for δ_l are then given by Eq. (2.7), and the scattering amplitude by Eq. (2.28). In terms of k and ϵ, we have

$$f_\psi^0(\pi)=\exp\left\{2i\lim_{R\to\infty}\left[\int_{r_0}^{R}dr\,(k^2\epsilon)^{1/2}-kR\right]\right\}\Bigg/2ik\left[\int_{r_0}^{\infty}dr\frac{1/r^2}{(k^2\epsilon)^{1/2}}\right]. \tag{4.17}$$

Equation (4.17) requires for its validity that $kr_0\gg 1$.

The derivation of the analogous expression for f_ϕ is more subtle, for it is not true that one merely replaces $k^2\epsilon$ in Eq. (4.17) by $k^2\epsilon-W$. Instead, we show that

$$f_\phi^0(\pi)=\exp\left\{2i\lim_{R\to\infty}\left[\int_{\rho_0}^{R}dr\,(k^2\epsilon-\tfrac{4}{3}W)^{1/2}-kR\right]\right\}\Bigg/2ik\left[\int_{\rho_0}^{\infty}dr\frac{1/r^2}{(k^2\epsilon-\frac{4}{3}W)^{1/2}}\right], \tag{4.18}$$

$$k^2\epsilon(\rho_0)-\tfrac{4}{3}W(\rho_0)=0. \tag{4.19}$$

An understanding of this modification is arrived at by examining the nature of $W(r)$. In particular, this "effective potential" is singular at $r=r_0$, for ϵ is zero there. An expansion of Eq. (4.5) about r_0 gives

$$W\simeq\tfrac{3}{4}(r-r_0)^{-2}, \tag{4.20}$$

which implies that W drops, in magnitude, from infinity to order k^2 in the distance of a wavelength. It is convenient to partially rewrite W in terms of U by use of Eq. (4.16):

$$W(r)=\tfrac{3}{4}(\epsilon k^2)^{-2}[U'(r)]^2+\tfrac{1}{2}(\epsilon k^2)^{-1}U''(r). \quad (4.21)$$

The primes denote derivatives of U. The function $U(r)$ changes appreciably from its value at r_0 in some characteristic distance $R(\gg 1/k)$. At this relative distance, ϵ is of order unity, and as an order of magnitude estimate of Eq. (4.21) reveals, $W\ll U$. Thus, Eq. (4.8) represents scattering by a long-range potential U and a short-range potential W, the latter rising very steeply near r_0, where it presents an impenetratable sphere. The form of W for $r<r_0$ is immaterial since the incident wave cannot penetrate this region. The above remarks are most easily illustrated in the case $U=\gamma r^{-2}$, where

$$W(r)=3r_0^2(r^2-r_0^2)^{-2}, \quad kr_0=\gamma^{1/2}. \quad (4.22)$$

The WKB formalism cannot be expected to hold in the region where W is varying rapidly. The turning point ρ_0 must be enough wavelengths from r_0 such that W appears as a slowly varying perturbation to the incoming scalar wave. To show that this is possible, we evaluate the fractional change of W in a wavelength for the example of Eq. (4.22); that is,

$$\frac{|W'(\rho_0)|}{kW(\rho_0)}=\frac{4\rho_0/k}{(\rho_0+r_0)(\rho_0-r_0)}. \quad (4.23)$$

To determine ρ_0 for this example, we write Eq. (4.19) in the form

$$(\rho_0^2-r_0^2)^3=4k^{-2}r_0^2\rho_0^2. \quad (4.24)$$

A first approximation is obtained by replacing ρ_0 on the right-hand side of Eq. (4.24) by r_0:

$$\rho_0\simeq r_0[1+\tfrac{1}{2}(4/\gamma)^{1/3}]. \quad (4.25)$$

This step is clearly justified for γ very large. The difference between ρ_0 and r_0 is

$$k(\rho_0-r_0)=(\gamma/4)^{1/6}, \quad (4.26)$$

so Eq. (4.23) becomes

$$|W'(\rho_0)|/kW(\rho_0)\propto(kr_0)^{-1/3}. \quad (4.27)$$

Thus, the left-hand side of Eq. (4.27) can be made small. The generality of this result can be inferred from Eqs. (4.19) and (4.20). The factor to the right of the proportionality sign appears again in Sec. V as an expansion parameter in the perturbation treatment of Eq. (4.18).

In order to derive the WKB phase shifts, we observe that the singularity at $r=0$ is responsible for the factor $(l+\frac{1}{2})^2$ in Eq. (2.7), as opposed to the familiar $l(l+1)$. This was first proved by Langer,[11] who introduced the following approximation to the wave function:

$$\omega_l(r)=\left[\left(\frac{k\pi}{6}\right)\left(\frac{\zeta_l}{Q_l}\right)\right]^{1/2}[J_{1/3}(\zeta_l)+J_{-1/3}(\zeta_l)], \quad (4.28)$$

$$\zeta_l(r)=\int_{\rho_l}^{r}dr'\,Q_l(r'), \quad (4.29)$$

$$Q_l(\rho_l)=0. \quad (4.30)$$

The J's are Bessel functions. Equation (4.28) satisfies the differential equation

$$d^2\omega_l/dr^2+[Q_l^2(r)-q_l(r)]\omega_l=0, \quad (4.31)$$

$$q_l(r)=-\left(\frac{5}{36}\right)\frac{Q_l^2}{\zeta_l^2}-\frac{1}{4Q_l^2}(Q_l^2)''+\left(\frac{5}{16}\right)\left[\frac{(Q_l^2)'}{Q_l^2}\right]^2. \quad (4.32)$$

This approximation is valid when the WKB approximation is valid ($q_l\ll Q^2$), and, because $q_l(\rho_l)$ is finite, it may be used throughout an interval containing the turning point.

We tentatively take the following form for Q_l:

$$Q_l^2=k^2\epsilon(r)-cW(r)-(l+\tfrac{1}{2})^2/r^2, \quad (4.33)$$

where c is a constant. For discussion purposes, Eq. (4.8) can be thought of as representing electron scattering by potentials U and W. Then, the wave number k is proportional to $\hbar^{-1}$, W is independent of $\hbar$, and q_l is independent of $\hbar$ to first order. Thus, in the limit $\hbar\to 0$, the first term on the right-hand side of Eq. (4.33) will be dominant except near r_0, where it goes to zero while W becomes infinite. Equation (4.33) clearly takes the following form near $r=r_0$:

$$Q_l^2\underset{r\to r_0}{\longrightarrow}-c_1(r-r_0)^{-2}-c_2(r-r_0)^{-1}, \quad (4.34)$$

where c_1, c_2 are constants which depend on c. By substituting Eq. (4.34) into Eq. (4.32), and by noting that

$$\zeta_l(r)\underset{r\to r_0}{\longrightarrow}i(c_1)^{1/2}\ln(r-r_0), \quad (4.35)$$

we find

$$q_l(r)\underset{r\to r_0}{\longrightarrow}-\tfrac{1}{4}(r-r_0)^{-2}-\frac{1}{4}\frac{c_2}{c_1}(r-r_0)^{-1}. \quad (4.36)$$

Thus, the differential equation for ω_l has the following form near r_0:

$$\omega_l''+\left[\frac{-(c_1-\frac{1}{4})}{(r-r_0)^2}-\frac{c_2}{c_1}\frac{(c_1-\frac{1}{4})}{(r-r_0)}\right]\omega_l=0. \quad (4.37)$$

Also, Eq. (4.8) takes the form

$$\frac{d^2(rS_l)}{dr^2}+\left[-\frac{3}{4}\frac{1}{(r-r_0)^2}-\frac{\alpha}{4(r-r_0)}\right](rS_l)=0, \quad (4.38)$$

$$\alpha=U''(r_0)[U'(r_0)]^{-1}. \quad (4.39)$$

[11] R. E. Langer, Phys. Rev. **51**, 669 (1937).

A comparison of Eqs. (4.37) and (4.38) yields the results,

$$c_1=1,\quad c_2=\tfrac{1}{3}\alpha. \tag{4.40}$$

These values are consistent with the choice $c=4/3$ in Eq. (4.33). Thus, Eq. (4.29) becomes

$$\zeta_l(r)=\int_{\rho_l}^{r} dr'\,[k^2\epsilon-\tfrac{4}{3}W-(l+\tfrac{1}{2})^2/r'^2]^{1/2}. \tag{4.41}$$

If the asymptotic forms of the Bessel functions for ζ_l large and real are substituted into Eq. (4.28), the asymptotic form of ω_l is obtained:

$$\omega_l(r)\xrightarrow[r\to\infty]{}\sin(kr-l\pi/2+\eta_l'), \tag{4.42}$$

$$\eta_l'=\lim_{R\to\infty}[\zeta_l(R)-kR+(l+\tfrac{1}{2})\pi/2]. \tag{4.43}$$

As a check, we note that the required solution to Eq. (4.38) approaches zero at r_0 as

$$rS_l\xrightarrow[r\to r_0]{}(r-r_0)^{3/2}. \tag{4.44}$$

For $r<\rho_l$ and $|\zeta_l|$ large, Eq. (4.28) takes the form[11]

$$\omega_l\xrightarrow[r\to r_0]{}(Q_l)^{-1/2}\exp(-|\zeta_l|). \tag{4.45}$$

From Eqs. (4.35) and (4.40), we have

$$\exp(-|\zeta_l|)\xrightarrow[r\to r_0]{}\exp[\ln(r-r_0)]=(r-r_0). \tag{4.46}$$

The factor $(Q_l)^{-1/2}$ contributes $(r-r_0)^{1/2}$, and ω_l approaches zero as $(r-r_0)^{3/2}$, in agreement with Eq. (4.44).

The l dependence of the η_l' is the same as in Eq. (2.7). Hence, the approximate scattering amplitude f_ϕ^0, obtained by substituting Eq. (4.43) into Eq. (2.19), must have the same form as (4.17), but with $k^2\epsilon$ replaced by $k^2\epsilon-\frac{4}{3}W$. This is just Eq. (4.18). From the standpoint of quantum mechanics, k is proportional to $\hbar^{-1}$, and W is independent of $\hbar$. Thus, it might be argued that the term $\frac{4}{3}W$ must be dropped for consistency. This argument is erroneous, again because W is singular at r_0. In Sec. V, it is shown that the amplitudes f_ψ^0 and f_ϕ^0 have equal magnitudes, but differing phases in the limit $\hbar\to 0$.

Finally, we investigate the error in replacing the sum over l by an integral. The difference between the exact amplitudes is given by Eq. (4.15), so the first-order correction as determined by Eq. (2.12) and (2.13) is

$$(4ik)^{-1}[\exp(2i\delta_0)-\exp(2i\eta_0)]. \tag{4.47}$$

But Eq. (4.47) has the same form as the second term in Eq. (4.15). In semiclassical terms, Eq. (4.47) is of order $\hbar$ and must be dropped. Therefore, the WKB approximation for $A(\pi)$ takes the form

$$2A^0(\pi)=f_\psi^0(\pi)-f_\phi^0(\pi). \tag{4.48}$$

V. APPROXIMATE EVALUATION OF INTEGRALS

The integrations in Eq. (4.18) can be carried out approximately in the extreme geometrical-optics limit. We first consider the phase factor

$$\Lambda=2\lim_{R\to\infty}\left[\int_{\rho_0}^{R} dr\,(k^2\epsilon-\tfrac{4}{3}W)^{1/2}-kR\right] \tag{5.1}$$

$$=2\lim_{R\to\infty}\left[\int_{r_0}^{R} dr\,(k^2\epsilon)^{1/2}-kR\right]+2I, \tag{5.2}$$

where

$$I=-\int_{r_0}^{\rho_0} dr\,(k^2\epsilon)^{1/2}+\int_{\rho_0}^{\infty} dr\,(k^2\epsilon)^{1/2}\left[\left(1-\tfrac{4}{3}\frac{W}{k^2\epsilon}\right)^{1/2}-1\right]. \tag{5.3}$$

The first term in Eq. (5.2) is common to both f_ψ^0 and f_ϕ^0, and, hence, can be ignored when deriving cross sections. If the second term of Eq. (5.3) is integrated by parts, the boundary term is zero at the upper limit and is the negative of the first term at the lower limit, by Eq. (4.19). Therefore, we are left with

$$I=-\int_{\rho_0}^{\infty} dr\left[\frac{d}{dr}\left(1-\tfrac{4}{3}\frac{W}{k^2\epsilon}\right)^{1/2}\right]\int_{r_0}^{r} dr'\,(k^2\epsilon)^{1/2}. \tag{5.4}$$

It is shown that the main contributions to I come from values of r near ρ_0. To emphasize this feature, we change variables to

$$y^2=\tfrac{4}{3}W(k^2\epsilon)^{-1}=(k^6\epsilon^3)^{-1}(U')^2A^2(r), \tag{5.5}$$

$$A(r)=[1+\tfrac{2}{3}k^2\epsilon(U'')(U')^{-2}]^{1/2}, \tag{5.6}$$

where the second equality of Eq. (5.5) follows from Eq. (4.21). The derivative of y is

$$dy/dr=-\tfrac{3}{2}(k^5\epsilon^{5/2})^{-1}(U')^2A(r)B(r), \tag{5.7}$$

$$B(r)=1+\tfrac{2}{3}(k^2\epsilon)[(U')^2A(r)]^{-1}\frac{d}{dr}[U'A(r)]. \tag{5.8}$$

The minus sign in Eq. (5.7) arises from the convention that y be positive for monotonically decreasing $U(r)$. From the definition of y^2, we see that $r=r_0,\ \rho_0,\ \infty$ correspond to $y=\infty$, 1, 0, respectively. In terms of y, Eq. (5.4) can be written as

$$I=-\tfrac{2}{3}\int_0^1 dy\,y(1-y^2)^{-1/2}\int_y^\infty dy'\,(y')^{-2}f(y'), \tag{5.9}$$

where

$$f(y)=A[r(y)]/B[r(y)]. \tag{5.10}$$

The inverse transformation, r as a function of y, is obtained, in part, by a Taylor expansion of Eq. (5.5) about $r=r_0$:

$$y^2=[-U'(r_0)]^{-1}(r-r_0)^{-3}[1-\tfrac{1}{6}\alpha(r-r_0)+\cdots], \tag{5.11}$$

where α is given by Eq. (4.39). As a first approximation, we set $r=r_0$ inside the brackets:

$$r-r_0\simeq[-U'(r_0)y^2]^{-1/3}. \quad (5.12)$$

An improved result can now be obtained by successive approximations, the first of which leads to

$$y^2\simeq[-U'(r_0)]^{-1}(r-r_0)^{-3}(1+\tfrac{1}{6}\lambda^2y^{-2/3}), \quad (5.13)$$

where

$$\lambda^2\equiv U''(r_0)[U'(r_0)]^{-4/3}. \quad (5.14)$$

Thus, if $y\gg\lambda^3$, Eq. (5.12) is a good approximation to the inverse transformation.

The quantity λ is our perturbation parameter. When $\lambda\ll1$, the integral of Eq. (5.9) can be evaluated approximately. The magnitude of λ is easily estimated in the case of inverse power laws, for $U'(r_0)$ is of order $U(r_0)/r_0=k^2/r_0$, while $U''(r_0)$ has magnitude k^2/r_0^2. Thus, λ has magnitude $(kr_0)^{-1/3}$. In general, the condition is $(kR)^{-1/3}\gg1$, where R is the range over which U changes appreciably. We have seen in Sec. IV that this restriction insures the slow variation of the effective potential W near the turning point. Hence, the size of λ determines the applicability of the WKB formulation.

Since λ is small, Eq. (5.12) holds very well for $y\geq\lambda$. The value $y=\lambda$ defines the corresponding value $r=r_1$:

$$(r_1-r_0)/r_0\simeq[-\lambda^2r_0^3U'(r_0)]^{-1/3}. \quad (5.15)$$

An estimate of the magnitude of the right-hand side of Eq. (5.15) yields $(kr_0)^{-4/9}\ll1$. Thus, when $r_0<r<r_1$, a function of r can be expanded about r_0 and then converted, in first order, to a function of y by Eq. (5.12). Application of this procedure to Eq. (5.10) yields

$$f(y)\simeq1+\tfrac{1}{9}\alpha(r-r_0)\simeq1-\tfrac{1}{9}\lambda^2y^{-2/3}, \quad y\geq\lambda. \quad (5.16)$$

We now rewrite Eq. (5.9) as the sum of two integrals, the first of which is

$$I_1=-\tfrac{2}{3}\int_\lambda^1 dy\,y(1-y^2)^{-1/2}\int_y^\infty dy'\,(y')^{-2}f(y'). \quad (5.17)$$

Since $y'\geq\lambda$, we can use the relation of Eq. (5.16):

$$I_1\simeq-\tfrac{2}{3}\int_\lambda^1 dy\,y(1-y^2)^{-1/2}\times\int_y^\infty dy'\,(y')^{-2}[1-\tfrac{1}{9}\lambda^2(y')^{-2/3}], \quad (5.18)$$

$$=-\pi/3+\tfrac{2}{3}\lambda+O(\lambda^2). \quad (5.19)$$

The second integral,

$$I_2=-\tfrac{2}{3}\int_0^\lambda dy\,y(1-y^2)^{-1/2}\int_y^\infty dy'\,(y')^{-2}f(y'), \quad (5.20)$$

can be carried out by ignoring the variation of $(1-y^2)^{-1/2}$ with y. This introduces an error in the final result of $O(\lambda^3)$. The simplified integral can be integrated by parts:

$$I_2=-\tfrac{1}{3}y^2\int_y^\infty dy'\,(y')^{-2}f(y')\bigg|_{y=0}^{y=\lambda}-\tfrac{1}{3}\int_0^\lambda dy\,f(y). \quad (5.21)$$

The boundary term is $-\frac{1}{3}\lambda$ at the upper limit, by Eq. (5.16), and it is zero at the lower limit provided $W(r)$ goes to zero at infinity faster than r^{-1}. The remaining integral is trivial when transformed back to the r variable, so

$$I_2=-\tfrac{1}{3}\lambda-\tfrac{1}{3}\int_{r_1}^\infty dr\,\tfrac{3}{2}(k^5\epsilon^{5/2})^{-1}(U')^2A^2(r) \quad (5.22)$$

$$=-\tfrac{1}{3}\lambda-\tfrac{1}{3}[-U'(r_1)][k^3\epsilon^{3/2}(r_1)]^{-1}. \quad (5.23)$$

If the second term of Eq. (5.23) is expanded about r_0, we obtain, with the help of Eq. (5.15),

$$I_2=-\tfrac{2}{3}\lambda+O(\lambda^{7/3}). \quad (5.24)$$

The sum of Eqs. (5.19) and (5.24) yields

$$I=-\pi/3+O(\lambda^2). \quad (5.25)$$

The remaining integral in Eq. (4.18) can be evaluated in a similar way, the derivation being left to an Appendix . The result is

$$J=\int_{\rho_0}^\infty dr\,\frac{1/r^2}{(k^2\epsilon-\frac{4}{3}W)^{1/2}}=\left[\int_{r_0}^\infty dr\,\frac{1/r^2}{(k^2\epsilon)^{1/2}}\right]\times[1+O(\lambda)]. \quad (A12)$$

Thus, Eq. (4.18) becomes

$$(2ik)Jf_\phi{}^0(\pi)=\exp(i\Lambda), \quad (5.26)$$

with Λ and J given by Eqs. (5.2), (5.25), and (A12). If Eqs. (4.17) and (5.26) are substituted into Eq. (4.48), the following approximation is obtained in the limit of very small λ:

$$|A^0(\pi)|=|[1-\exp(-2i\pi/3)]|\bigg/\left[4k\int_{r_0}^\infty dr\,\frac{1/r^2}{(k^2\epsilon)^{1/2}}\right]. \quad (5.27)$$

The cross section is the square of Eq. (5.27):

$$\sigma^0(\pi)=\tfrac{3}{4}\sigma_\psi(\pi). \quad (5.28)$$

The quantity σ_ψ is the absolute square of Eq. (4.17); that is, it is the cross section associated with a scalar wave equation of the form Eq. (4.3), with boundary conditions given by Eq. (2.2).

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APPENDIX

The integral,

$$J=\int_{\rho_0}^{\infty} dr \frac{1/r^2}{(k^2\epsilon-\frac{4}{3}W)^{1/2}}, \tag{A1}$$

can be written as

$$J=\int_{r_0}^{\infty} dr \frac{1/r^2}{(k^2\epsilon)^{1/2}}-\int_{r_0}^{\rho_0} dr \frac{1/r^2}{(k^2\epsilon)^{1/2}} +\int_{\rho_0}^{\infty} dr \frac{1/r^2}{(k^2\epsilon)^{1/2}}\left[\left(1-\frac{4}{3}\frac{W}{k^2\epsilon}\right)^{-1/2}-1\right]. \tag{A2}$$

We designate the three integrals as J_1, J_2, and J_3, respectively. The second integral in Eq. (A2) can be evaluated, to first order, by expanding r about r_0:

$$J_2\simeq(r_0)^{-2}[-U'(r_0)]^{-1/2}\int_{r_0}^{\rho_0} dr\ (r-r_0)^{-1/2} \tag{A3}$$

$$\simeq 2(r_0)^{-2}[-U'(r_0)]^{-1/2}(\rho_0-r_0)^{1/2}. \tag{A4}$$

By remembering that $r=\rho_0$ corresponds to $y=1$, we can use Eq. (5.12) to simplify Eq. (A4),

$$J_2\simeq 2(r_0)^{-2}[U'(r_0)]^{-2/3}\equiv 2g. \tag{A5}$$

To evaluate J_3, we split up the range of integration into two parts, $\rho_0\leq r\leq r_1$ and $r_1\leq r\leq\infty$, with r_1 defined by Eq. (5.15). An order-of-magnitude estimate for the latter range gives

$$J_3^{(2)}\leq[O(\lambda^2)]J_1. \tag{A6}$$

The remaining segment of J_3 may be transformed to the y variable defined by Eq. (5.5):

$$J_3^{(1)}=\frac{2}{3}\int_{\lambda}^{1} dy\, y^{-4/3}g(y)[(1-y^2)^{-1/2}-1], \tag{A7}$$

where

$$g(y)=(U')^{-2/3}r^{-2}A^{1/3}(r)B^{-1}(r). \tag{A8}$$

Equations (5.6) and (5.8) define A and B, respectively. Since $y\geq\lambda$, we can expand the right-hand side of Eq. (A8) about $r=r_0$, and then use Eq. (5.12). To first order, $g(y)$ is just the constant g of Eq. (A5). Further, in first approximation, the lower limit in Eq. (A7) can now be set equal to zero, with the result

$$J_3^{(1)}\simeq\frac{2}{3}g\int_0^1 dy\, y^{-4/3}[(1-y^2)^{-1/2}-1]. \tag{A9}$$

The order of magnitude of g can be estimated from Eq. (A5), in the manner of Sec. IV:

$$g=O[(kr_0)^{-4/3}]=O'(\lambda^4). \tag{A10}$$

In Sec. II, a minimum was established for J_1; that is,

$$J_1\geq(kr_0)^{-1}=O(\lambda^3). \tag{A11}$$

We assume that the actual value of J_1 has the same order of magnitude as its minimum. Then, Eq. (A6) can be neglected relative to Eqs. (A5) and (A9), and Eq. (A2) becomes

$$J=\left[\int_{r_0}^{\infty} dr \frac{1/r^2}{(k^2\epsilon)^{1/2}}\right][1+O(\lambda)]. \tag{A12}$$