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Regge Poles in Relativistic Wave Equations

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Solutions of the relativistic wave equation for three different coupling schemes to an external source are examined. The Regge trajectories are observed to exhibit a high degree of model dependence and to possess singularities which are often assumed absent in a fully relativistic theory. It is shown that these arise from multiple poles in the scattering amplitude and can occur in theories for which no collapse into the center is possible for physical values of the angular momentum.

I. INTRODUCTION

RECENT work of Regge and others[1] concerning the analytic properties of the radial Schrödinger equation has resulted in many attempts to determine general properties of the so-called Regge trajectories. In particular, it has been shown for the class of potentials

$$V(r)=\int_M^\infty d\mu \frac{e^{-\mu r}}{r}\sigma(\mu) \tag{1.1}$$

that the Regge poles and their residues are analytic functions in the E plane cut along the positive real axis (if there exist no multiple poles in the scattering amplitude). Although attempts have been made at the fully relativistic problem using the Mandelstam representation and unitarity,[2,3] as of yet no rigorous correspondence to the nonrelativistic case has been established. In view of the limited success of the latter program, it is natural to investigate a domain of intermediate complexity as represented by the relativistic wave equation. In this connection Oehme[4] and Singh[5] have found in the Coulomb problem that in addition to the expected right-hand cut, there exist complex branch points which are the result of the well-known modification of the centrifugal term of the radial wave equation. While this particular case is of limited applicability in the context of strong interactions, Oehme has observed that these complex singularities arise solely as a consequence of the behavior of the potential near the origin and are, therefore, expected to persist for the class of potentials (1.1) in a relativistic wave equation. He has thus suggested that these results might have an analog in the static limit of a vector meson theory.

In this paper we consider three possible schemes for the coupling of a relativistic particle to a $1/r$ type source. It will be shown that for each of these, the functions $\alpha_n(\nu)$ which prescribe the location of the Regge poles have singularities on the physical $\nu[\equiv(E^2-m^2)/m^2]$ sheet in addition to the expected right-hand cut. Oehme has suggested that these branch cuts could arise from the possibility of a "collapse into the center" and might, therefore, be absent in a fully relativistic theory.[6] We demonstrate that in a scalar coupling model such branch lines occur despite the fact that there is no collapse.[7] Their existence is seen to derive from the frequently overlooked possibility of multiple poles in the scattering amplitude.

II. A SCALAR COUPLING MODEL

The vector coupling model has enjoyed considerable popularity as a consequence of its application to electrodynamics. Another coupling scheme which is even simpler in structure, however, is the scalar model which also admits a solution for the $1/r$ type source. It is described by the Lagrangian

$$\mathcal{L}=\int dx\,\{\phi^{\mu*}\partial_\mu\phi-\tfrac{1}{2}[m+e\chi(x)]^2\phi\phi^* +\tfrac{1}{2}\phi^{\mu*}\phi_\mu+\text{c.c.}\}, \tag{2.1}$$

where we have used the metric (1, 1, 1, −1). In the fixed source limit $\chi=-e/r$. The Lagrangian (2.1) leads to the wave equation

$$[-\partial_\mu{}^2+(m-e^2/r)^2]\phi(x)=0, \tag{2.2}$$

which is equivalent to the radial equation

$$\frac{E^2-m^2}{2m}\phi(x)=\left[\frac{p_r{}^2}{2m}+\frac{l(l+1)+e^4}{2mr^2}-\frac{e^2}{r}\right]\phi(x). \tag{2.3}$$

Note that the Lagrangian for a classical point particle in interaction with a scalar field is

$$\mathcal{L}=-\int[m+e\chi(x)](-dx^\mu dx_\mu)^{1/2},$$

* National Science Foundation Cooperative Fellow.

[1] T. Regge, Nuovo Cimento 14, 951 (1959); 18, 947 (1960); A. Bottino, A. M. Longoni, and T. Regge, *ibid.* 23, 954 (1962).

[2] A. O. Barut and D. E. Zwanziger, Phys. Rev. 127, 974 (1962).

[3] K. Bardakci, Phys. Rev. 127, 1832 (1962).

[4] R. Oehme, Nuovo Cimento 25, 183 (1962).

[5] V. Singh, Phys. Rev. 127, 632 (1962).

[6] R. Oehme, Phys. Rev. Letters 9, 358 (1962).

[7] It should be emphasized at this point that throughout this paper the term "collapse into the center" will refer to the failure of a given Hamiltonian to possess a lower bound for physical values of the energy and angular momentum.

and that the corresponding Hamiltonian

$$H^2=p^2+(m-e^2/r)^2 \tag{2.4}$$

gives rise to (2.2) by the usual canonical substitutions. This shows that the replacement of m in the free Lagrangian by $m+e\chi$ is indeed the simplest and most natural definition of a scalar coupling. It is to be noted that the Hamiltonian (2.4) is positive definite in contrast to the corresponding expression (3.3) for the case of a vector coupling and is, therefore, a more reasonable model of a fully relativistic theory.

Comparison of the wave equation (2.3) with the usual Schrödinger equation for the Coulomb field,

$$E\psi(x)=\left[\frac{p_r^2}{2m}+\frac{l(l+1)}{2mr^2}-\frac{e^2}{r}\right]\psi(x),$$

and the corresponding S matrix,[8]

$$S(l,E)=\frac{\Gamma[l+1-ie^2(m/2E)^{1/2}]}{\Gamma[l+1+ie^2(m/2E)^{1/2}]},$$

yields for the scalar theory

$$S(l,E)=\frac{\Gamma\{[(l+\frac{1}{2})^2+e^4]^{1/2}+\frac{1}{2}-ie^2[m/(E^2-m^2)^{1/2}]\}}{\Gamma\{[(l+\frac{1}{2})^2+e^4]^{1/2}+\frac{1}{2}+ie^2[m/(E^2-m^2)^{1/2}]\}}\times\exp\{i\pi(l+\tfrac{1}{2}-[(l+\tfrac{1}{2})^2+e^4]^{1/2})\},$$

where the physical sheet is defined by the condition $\mathrm{Im}[(E^2-m^2)^{1/2}]>0$. Since $\Gamma(z+1)$ is meromorphic in the finite z plane with poles at $z=-n$ $(n=1, 2, \cdots)$, the singularities of the S matrix are trivially found to occur at

$$\frac{E^2-m^2}{m^2}=-\frac{e^4}{\{n+[(l+\frac{1}{2})^2+e^4]^{1/2}-\frac{1}{2}\}^2}, \tag{2.5}$$

which for positive integral values of l yields the eigenvalue spectrum. The inversion of (2.5) gives the Regge trajectories:

$$\alpha_n(E)=\left[\left(\frac{ie^2m}{(E^2-m^2)^{1/2}}-n+\frac{1}{2}\right)^2-e^4\right]^{1/2}-\frac{1}{2}. \tag{2.6}$$

It follows from (2.6) that $\alpha_n(E)$ has in the ν plane both the expected cut on the positive real axis and, corresponding to the vanishing of $\alpha_n(E)+\frac{1}{2}$, real branch points at

$$\nu=-e^4/(n-\tfrac{1}{2}\pm e^2)^2. \tag{2.7}$$

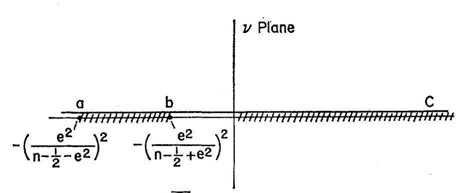

FIG. 1. Cuts of $\alpha_n(\nu)$ for scalar model.

In contrast to the vector coupling model, the eigenvalues defined by (2.5) remain real for arbitrarily large values of e^2 as a consequence of the positivity of H^2. It is therefore of interest to examine the analytic properties of Eq. (2.6) for all real values of e. It is clear from Eq. (2.7) that $e^2=n-\frac{1}{2}$ defines that critical value of the coupling for which the left-hand branch point of $\alpha_n(E)$ moves to $\nu=-\infty$. We thus distinguish between the two domains in which e^2 is less than and greater than $n-\frac{1}{2}$. For the former of these, Fig. 1 shows

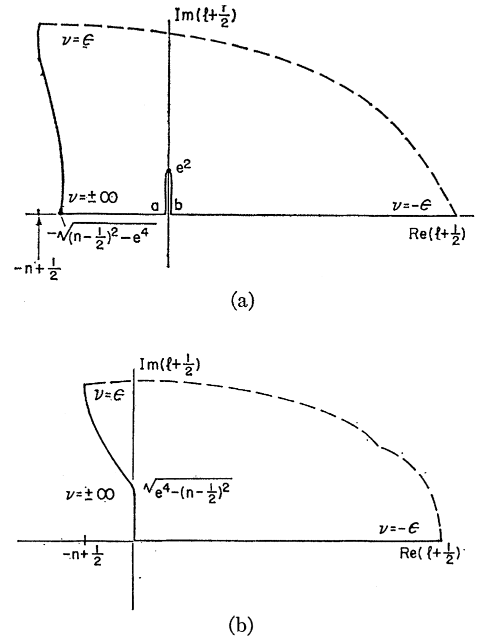

FIG. 2(a). Trajectory of a Regge pole in scalar model for $e^2<n-\frac{1}{2}$. (b) Trajectory of a Regge pole in scalar model for $e^2>n-\frac{1}{2}$.

the cuts of $\alpha_n(\nu)$ in the complex ν plane. For e^2 approaching the critical value of $n-\frac{1}{2}$ the left-hand branch point moves to $\nu=-\infty$ and emerges on the unphysical sheet. Further increase in the value of e^2 has no effect on the branch cuts on the physical sheet. In Fig. 2(a) is shown the Regge trajectory as defined by the contour C of Fig. 1. Finally, Fig. 2(b) indicates the corresponding trajectory for the case $e^2>n-\frac{1}{2}$.

We remark that while the spin-zero wave equation has been the basis for the above discussion, the extension to spin one-half is straightforward. In this latter case the Dirac equation

$$\left[\gamma^\mu\frac{1}{i}\partial_\mu+m-\frac{e^2}{r}\right]\psi(x)=0,$$

[8] H. Bethe and E. Salpeter, *Quantum Mechanics of One and Two Electron Atoms* (Academic Press Inc., New York, 1957), p. 34.

which follows from the Lagrangian

$$\mathcal{L}=\int dx\left\{\frac{i}{2}\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi-\tfrac{1}{2}[m+e\chi(x)]\bar{\psi}\psi\right\},$$

can be solved by the usual techniques to yield the energy spectrum

$$\frac{E^2-m^2}{m^2}=-\frac{e^4}{\{n+[(J+\frac{1}{2})^2+e^4]^{1/2}\}^2}.$$

The preceding discussion is extended to this additional case with no difficulty. Except for the replacement of $n-\frac{1}{2}$ in Eq. (2.6) by n, the results are identical.

In concluding this section, we draw attention to the fact that Oehme has used the nonrelativistic Schrödinger equation with a repulsive $1/r^2$ potential to obtain results similar to those described here. These, however, emerge in a more natural way using a relativistic wave equation with a scalar coupling.

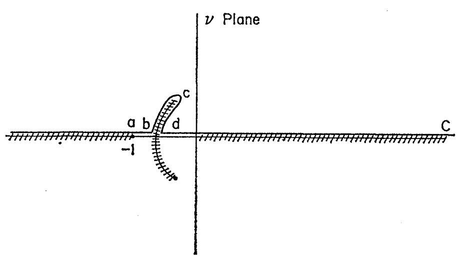

FIG. 3. Cuts of $\alpha_n(\nu)$ for vector model.

III. THE VECTOR COUPLING MODEL

The familiar Coulomb problem is represented by the Lagrangian

$$\mathcal{L}=\int dx\,[\phi^{\mu*}\partial_{\mu}\phi-\tfrac{1}{2}m^2\phi^*\phi+\tfrac{1}{2}\phi^{\mu}\phi_{\mu}^* -ie\phi^{\mu*}\phi A_{\mu}+\text{c.c.}]. \quad (3.1)$$

While the form (3.1) emphasizes the vector character of the coupling, we shall consider only the usual fixed source limit $\mathbf{A}=0$, $A^0=-e/r$. Corresponding to (3.1) one has the wave equation

$$\frac{E^2-m^2}{2m}\phi(x)=\left[\frac{p_r^2}{2m}+\frac{l(l+1)-e^4}{2mr^2}-\frac{E}{m}\frac{e^2}{r}\right]\phi(x), \quad (3.2)$$

and the nonpositive definite Hamiltonian

$$H=-\frac{e^2}{r}+(\mathbf{p}^2+m^2)^{1/2}. \quad (3.3)$$

Because the e^4/r^2 term in (3.2) represents an attractive term in the effective potential, there exists a possibility of collapse in this theory.

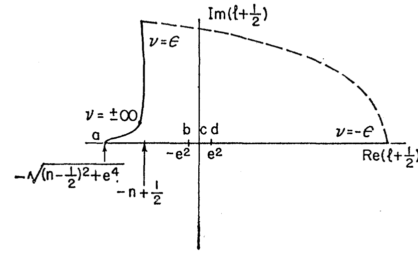

FIG. 4. Trajectory of a Regge pole in vector model.

A significant departure of this paper from previous considerations[4,5] of the Regge poles in the Coulomb problem arises from our choice of E^2-m^2 as the analog of the usual invariant square energy variable rather than E itself. The models considered here have all been solved by comparison with the nonrelativistic Schrödinger equation for a $1/r$ potential, and one observes that $(E^2-m^2)/2m$ does in fact correspond to the nonrelativistic energy variable. In Eq. (3.2) the $1/r$ term is proportional to $(E^2)^{1/2}$ and one would, therefore, expect a left-hand cut for $\alpha_n(\nu)$ originating at $\nu=-1$. While the choice of E as the basic variable is sufficient to eliminate such a kinematical cut, the utility of this device is unique to the vector theory.

The equation for the Regge trajectory,

$$\alpha_n(\nu)=-\tfrac{1}{2}+\left\{e^4+\left[i\frac{e^2E}{(E^2-m^2)^{1/2}}-n+\tfrac{1}{2}\right]^2\right\}^{1/2},$$

shows immediately the existence of branch points at $\nu=0$ and $\nu=-1$. In addition, the vanishing of $\alpha_n(\nu)+\frac{1}{2}$ yields complex branch points at

$$\nu=-\frac{[e^2/(n-\frac{1}{2})]}{1\pm 2i[e^2/(n-\frac{1}{2})]}.$$

The branch cuts are chosen as in Fig. 3 with the curve connecting the complex branch points defined by

$$\nu=\frac{1}{1+[e^2/(n-\frac{1}{2}+i\lambda e^2)]^2}-1$$

as λ varies in the interval

$$-1<\lambda<1.$$

This choice of the cut guarantees that

$$\mathrm{Im}\alpha_n(\nu)\geqslant 0$$

for $\mathrm{Im}\nu\geqslant 0$. A trajectory corresponding to the path C is illustrated in Fig. 4. It is interesting to note that in the scalar theory the trajectories closed at

$$l=-[(n-\tfrac{1}{2})^2-e^4]^{1/2}-\tfrac{1}{2},$$

while for the vector coupling they close at the necessarily complex values

$$l=-[(n-\tfrac{1}{2})^2-2ie^2(n-\tfrac{1}{2})]^{1/2}-\tfrac{1}{2}.$$

This contrasts with the usual Schrödinger theory in which the trajectories approach the negative integers for E approaching plus and minus infinity.

IV. THE SCALAR PARTICLE IN GENERAL RELATIVITY

Another case for which a solution can be found for a fixed source is that of a scalar particle in the general theory of relativity. Here, one considers the Lagrangian

$$\mathcal{L}=\int dx\,(-g)^{1/2}[\phi^{\mu *}\partial_\mu\phi+\tfrac{1}{2}\phi^{\mu *}g_{\mu\nu}\phi^\nu-\tfrac{1}{2}m^2\phi^*\phi+\text{c.c.}].$$

The field equations are

$$\partial_\mu\phi+g_{\mu\nu}\phi^\nu=0,$$
$$\partial_\mu(-g)^{1/2}\phi^\mu+m^2(-g)^{1/2}\phi=0,$$

which is equivalent to

$$-(-g)^{1/2}\partial_\mu(-g)^{1/2}g^{\mu\nu}\partial_\nu\phi+m^2\phi=0. \tag{4.1}$$

A metric which admits a solution to this equation is

$$ds^2=\left(1+\frac{e^2}{mr}\right)(dx^2+dy^2+dz^2)-\frac{dt^2}{1+e^2/mr},$$

which, for the classical Lagrangian,

$$L=-\int m(-g_{\mu\nu}dx^\mu dx^\nu)^{1/2}$$

leads to the canonical equations of motion

$$\frac{d\mathbf{r}}{dt}=\frac{\mathbf{p}A}{(m^2A^{-1}+\mathbf{p}^2)},$$

$$d\mathbf{p}/dt=-[m^2A^{-1}+\mathbf{p}^2]^{-1/2}[\mathbf{p}^2+\tfrac{1}{2}m^2A^{-1}](e^2\mathbf{r}A^2/mr^3),$$

where $A^{-1}=(1+e^2/mr)$. For completeness, we remark that the energy momentum tensor corresponding to this metric in spherical coordinates is[8a]

$$\begin{aligned}&8\pi T_r{}^r=-8\pi T_\theta{}^\theta=-8\pi T_\phi{}^\phi=-A^3(e^2/2mr^2)^2,\\&8\pi T_0{}^0=-3A^3(e^2/2mr^2)^2.\end{aligned} \tag{4.2}$$

With the above choice of metric, Eq. (4.1) becomes

$$[-(1/A)(\partial^2/\partial t^2)+A\nabla^2-m^2]\phi(x)=0.$$

The corresponding eigenvalue problem,

$$\frac{E^2-m^2}{2m}\phi(x)=\left[\frac{p_r{}^2}{2m}+\frac{l(l+1)-e^4E^2/m^2}{2mr^2}-\frac{e^2}{r}\left(\frac{E^2}{m^2}-\tfrac{1}{2}\right)\right]\phi(x),$$

yields the S matrix

$$S(l,E)=\frac{\Gamma\{[(l+\frac{1}{2})^2-(E^2/m^2)e^4]^{1/2}+\frac{1}{2}-i(e^2/m)(E^2-\frac{1}{2}m^2)/(E^2-m^2)^{1/2}\}}{\Gamma\{[(l+\frac{1}{2})^2-(E^2/m^2)e^4]^{1/2}+\frac{1}{2}+i(e^2/m)(E^2-\frac{1}{2}m^2)/(E^2-m^2)^{1/2}\}}\exp\{i\pi(l+\tfrac{1}{2}-[(l+\tfrac{1}{2})^2-e^4E^2/m^2]^{1/2})\},$$

and the spectrum of bound states

$$\frac{E^2-m^2}{m^2}=-\frac{e^4(E^2/m^2-\frac{1}{2})^2}{\{n+[(l+\frac{1}{2})^2-e^4E^2/m^2]^{1/2}-\frac{1}{2}\}^2}.$$

As in the vector theory, there exists the possibility of a collapse into the center because of the attractive $1/r^2$ term in the wave equation.

The Regge trajectories,

$$\alpha_n(E)=-\tfrac{1}{2}+\left\{e^4E^2+\left[ie^2m\frac{E^2/m-m/2}{(E^2-m^2)^{1/2}}-n+\tfrac{1}{2}\right]^2\right\}^{1/2},$$

have the usual branch cut from $E^2=m^2$ to $E^2=\infty$ and additional branch points corresponding to the solution of

$$\frac{E^2-m^2}{m^2}=-e^4\frac{(E^2/m^2-\frac{1}{2})^2}{[n+(-e^4E^2/m^2)^{1/2}-\frac{1}{2}]^2}.$$

The substitution $x=-im/E$ yields

$$x[x^3(n'^2-e^4/4)+2e^2n'x^2+n'^2x+2e^2n']=0, \tag{4.3}$$

where we have introduced $n'=n-\frac{1}{2}$. Equation (4.3) immediately exhibits the branch point at $|E|=\infty$.

The remaining cubic equation is conveniently discussed in terms of the discriminant

$$d\equiv(a^3/27)+(b^2/4).$$

Here

$$a=\tfrac{1}{3}(1-\tfrac{1}{4}\beta^2)^{-2}[3(1-\tfrac{1}{4}\beta^2)-4\beta^2],$$
$$b=(2/27)(1-\tfrac{1}{4}\beta^2)^{-3}[8\beta^2-9(1-\tfrac{1}{4}\beta^2)+27(1-\tfrac{1}{4}\beta^2)^2],$$

and we have defined $\beta=e^2/n'$. One finds for d the expression

$$d=(1/432)(1-\tfrac{1}{4}\beta^2)^{-4}[16+31\beta^2+28\beta^4+27\beta^6].$$

From the general theory of the cubic equation[9] it is deduced that because of the positive definite property

[8a] *Note added in proof.* It is perhaps useful to remark here that this choice does, in fact, represent a physically realizable possibility. Such an energy momentum tensor can arise, for example, from a spherically symmetrical distribution of a uniformly charged "fluid" which acts as the source of the radial electric field

$$F^{or}=A^{3/2}\frac{e^2}{2mr}.$$

By associating with this fluid a sufficiently large charge-to-mass ratio, the combination $T_{\text{matter}}{}^{\mu\nu}+T_{\text{emf}}{}^{\mu\nu}$ can be made to yield the energy momentum tensor of Eq. (4.2).

[9] R. S. Burington, in *Handbook of Mathematical Tables and Formulas* (Handbook Publishers, Inc., Sandusky, Ohio), p. 7.

of d for all β, Eq. (4.3) has one real and two complex conjugate solutions in addition to the null root already observed. These correspond to a branch point on the negative E^2 axis and a complex conjugate pair.

The positions of the branch points can be extracted to lowest order in e^2 by using the following form of (4.3):

$$(1+x^2)(xn'^2+2e^2n')=e^4x^3/4.$$

From the factor xn'^2+2e^2n', it is seen that one of these occurs at

$$E^2=-m^2(n-\tfrac{1}{2})^2/4e^4.$$

Expansion around $x=\pm i$ yields the remaining branch points at

$$E^2=m^2\left[1-\frac{e^4}{4(n-\frac{1}{2})^2}\pm i\frac{e^6}{2(n-\frac{1}{2})^3}+O(e^8)\right].$$

It can readily be shown that the real solution of (4.3) is on the unphysical sheet of the E^2 plane. However, for the repulsive case which is formally obtained by the replacement of e^2 by $-e^2$, this branch cut appears on the physical sheet. In this latter case for small e^2 there is a cut from $E^2=-\infty$ to $E^2=-m^2(n-\frac{1}{2})^2/4e^4$. As e^2 increases this branch point approaches the origin which it reaches at the critical value of $e^2=2n'$. Further increase in e^2 causes it to move back on the negative axis and to approach minus infinity for very large coupling.

Figure 5 shows a trajectory of this model for the analog of contour C of Fig. 3. In contrast to the scalar coupling theory for which $\alpha_n(\nu)$ approaches a path independent value for $|E|\to\infty$ and the vector theory for which it approaches a path dependent value, $\alpha_n(\nu)$ has no finite limit for $|E|\to\infty$.

V. MULTIPOLE POLES IN SCATTERING AMPLITUDES

Because of the fact that the "pathological" features we have found are a consequence of the modification of the centrifugal term in the wave equation at small distances, it is to be expected that they will persist even when the potential has no long-range tail. In the case of the usual Schrödinger equation for the potential (1.1) with the representation of the scattering amplitude as

$$a(l,k)=N(l,k)/D(l,k),$$

it has been shown that $D(l,k)$ is analytic in the product of the right-half l plane with the E plane cut along its positive real axis. Since the relativistic wave equation for the scalar theory considered here is formally mapped into the nonrelativistic case by the replacement of E by $(E^2-m^2)/2m$ and l by $[(l+\frac{1}{2})^2+e^4]^{1/2}-\frac{1}{2}$, one can infer in this case that the corresponding denominator function is analytic in the product of the ν plane cut along the positive real axis with the l plane cut from $l=-\frac{1}{2}+ie^2$ to $l=-\frac{1}{2}-ie^2$.

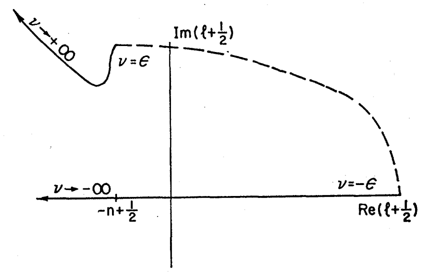

FIG. 5. Trajectory of a Regge pole in tensor model.

The Regge poles occur for

$$D(l,\nu)=0,$$

the inversion of which yields the trajectories $\alpha_n(\nu)$. By the implicit function theorem it follows that in a domain of analyticity in l and ν, singularities of $\alpha_n(\nu)$ can arise only if there exist multiple roots of $D(l,\nu)$, i.e., from solutions of

$$D(l,\nu)=0,$$
$$\partial D(l,\nu)/\partial l=0.$$

As has been previously noted, the left-hand branch points of $\alpha_n(\nu)$ occur for $l=-\frac{1}{2}$ which, by a suitable choice of the branch cut of $D(l,\nu)$ connecting the points $l=-\frac{1}{2}\pm ie^2$, does indeed lie within the domain of analyticity of $D(l,\nu)$. Thus, the left-hand cut of the function $\alpha_n(\nu)$ must originate in the existence of multiple roots. This can be seen directly by showing that $D(l,\nu)$ has in fact a double root at $l=-\frac{1}{2}$. Because this function is analytic in $[(l+\frac{1}{2})^2+e^4]^{1/2}-\frac{1}{2}$, it follows that

$$\begin{aligned}\frac{\partial}{\partial l}D([(l+\tfrac{1}{2})^2+e^4]^{1/2}-\tfrac{1}{2},\,\nu)\\ &=D'\frac{\partial}{\partial l}\{[(l+\tfrac{1}{2})^2+e^4]^{1/2}-\tfrac{1}{2}\}\\ &=D'\frac{(l+\frac{1}{2})}{[(l+\frac{1}{2})^2+e^4]^{1/2}},\end{aligned}\qquad(5.1)$$

demonstrating the double root at $l=-\frac{1}{2}$. In writing (5.1) we have used a dash to indicate the derivative of D with respect to $[(l+\frac{1}{2})^2+e^4]^{1/2}-\frac{1}{2}$. Similar considerations hold for the vector coupling model and in fact we have shown above in quite general terms that a modification of the wave equation such as that considered here will give rise to multiple roots in $D(l,\nu)$.

VI. CONCLUSION

In this paper we have considered three coupling schemes which are the simplest possible relativistic models. The scalar model is a particularly satisfactory one by virtue of the positive definite character of its Hamiltonian. Viewing these models as static limits of a fully relativistic theory, three points emerge which might have application to a complete theory.

(1) We have shown in various examples the existence of multiple poles in the scattering amplitude, the possibility of which has frequently been ignored.

(2) It has been suggested that the singularities of $\alpha_n(\nu)$ which occur off the positive real axis might be absent in a true field theory because of their connection with the fall into the center.[6] However, the scalar coupling theory considered here has displayed such singularities in spite of the fact that it has no possibility of collapse for physical l. The occurrence of these additional branch cuts in a complete theory cannot be excluded, and it would be almost remarkable if the consideration of recoil could completely eliminate them.

(3) We have noted that the trajectories associated with the models considered in this paper display marked differences in their qualitative behavior and analytic properties. All of these display analytic properties in conflict with those which have been expected to occur in a real field theory. It might well be anticipated, therefore, that the problem of analytic continuation in the complex angular momentum plane is not independent of the nature of the coupling.

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Artificial Singularity in the N/D Equations of the New Strip Approximation*

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The singularity introduced artificially into the equations of the new strip approximation, in order to bridge the gap between low and high energies, is investigated in detail. By explicit construction, it is shown that a necessary and sufficient condition for a (unique) solution of the N/D equations to exist is that the unitarity constraint on the cross section just above the strip boundary should be obeyed. The only singularities of the solution in the right-half angular momentum plane ($\mathrm{Re}J\geq 0$) are Regge poles.

I. INTRODUCTION

A SET of approximate dynamical equations based on the strip concept has recently been proposed for determining the self-consistent strong-interaction S matrix with Regge asymptotic behavior.[1] This paper is concerned with the singularity at the strip boundary introduced as a consequence of the approximation procedure. We propose to show that in spite of its artificial character this singularity plays a useful physical role and does not prevent a numerical solution of the equations. It also does not affect analyticity properties in angular momentum. The reader is assumed to be familiar with reference 1, whose notation is maintained here.

The integral equation in question is (III.11) of reference 1:

$$N_l(s)=B_l^P(s)+\frac{1}{\pi}\int_{s_0}^{s_1} ds'\,\frac{B_l^P(s')-B_l^P(s)}{s'-s}\rho_l(s')N_l(s'). \quad \text{(I.1)}$$

The singularity arises in the kernel because at $s=s_1$ the function $B_l^P(s)$ has a logarithmic branch point:

$$B_l^P(s)\underset{s\to s_1}{\longrightarrow}\frac{1}{\pi}\,\mathrm{Im}B_l^P(s_1)\ln(s_1-s). \quad \text{(I.2)}$$

Let us split off the singular part of the integral in (I.1):

$$N_l(s)=B_l^P(s)+\int_{s_0}^{s_1} ds'\,K_l(s,s')N_l(s')-\frac{\lambda_l}{\pi^2}\int_{s_0}^{s_1} ds'\,k(s,s')N_l(s'), \quad \text{(I.3)}$$

where

$$k(s,s')=\frac{\ln(s_1-s')-\ln(s_1-s)}{s'-s}, \quad \text{(I.4)}$$

$$\lambda_l=\rho_l(s_1)\,\mathrm{Im}B_l^P(s_1), \quad \text{(I.5)}$$

and where $K_l(s,s')$ is the residual part of the kernel obtained by comparison of Eqs. (I.1) and (I.3). In the dangerous region, $s\to s_1$, $s'\to s_1$,

$$K_l(s,s')\propto\frac{(s_1-s')\ln(s_1-s')-(s_1-s)\ln(s_1-s)}{s'-s}, \quad \text{(I.6)}$$

a behavior that causes no trouble. Equation (I.3) may

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[1] G. F. Chew, Phys. Rev. **129**, 2363 (1963).