Scattering, Absorption, and Emission of Light by Thin Metal Wires

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Scattering, absorption, and emission of light by thin metal wires are studied in a theory where the "plasma" properties of the metal are taken into account. The wires are assumed to have a diameter of the same order of magnitude as the free-space wavelength of the light and special attention is paid to polarization phenomena. Experiments on silver wires having a diameter down to 2000 Å have been carried out and compared with the theory. A degree of polarization of about 50% is observed for the light emitted by thin wires.

INTRODUCTION

A SYSTEM composed of a large number of positive ions and free electrons having zero total charge was first called a plasma by Langmuir in 1928. In 1929 Tonks and Langmuir gave a simple theory for the longitudinal plasma oscillations that can be set up in the electron gas in a gas discharge plasma. As early as 1932 Steenbeck suggested that such plasma oscillations should also exist in metals, because a metal is a system of positive ions arranged on a crystal lattice in which the valence electrons are more or less free to move. The full significance of plasma oscillations for the theory of metals was, however, not realized until the beginning of 1950, when Bohm and Pines began a series of papers on this subject.[1]

Plasma oscillations in metals have been studied experimentally by examining the energy-loss spectra of energetic electrons that are sent through thin metal films. The first experiments of this type were made by Ruthemann[2] and Lang.[3] A more recent experiment in this field has been made by Blackstock *et al.*[4]

The connection between plasma oscillations in metals and the optical properties of metals has been studied by a number of authors.[5] As early as 1933, Wood[6] and Zener[7] reported experiments on the optical transmission properties of alkali metals which showed that the dielectric constant vanished at a critical frequency. They did not, however, give any explanation of the observations. Recent experiments of this type have been reported by Taft and Philipp.[8]

Van de Hulst[9] has made a very extensive theoretical study of light scattering from small particles. In the treatment of specific problems in his book it is assumed that the optical parameters characterizing the particles are independent of the frequency. In the work presented here, the frequency dependence of these parameters is of basic importance.

The aim of our investigation is to study the properties of thin metal wires as regards scattering, absorption, and emission of light when the plasma properties of the model are taken into account. If the wires are thin enough one will get a strong resonance interaction between the light and the metal plasma at the light frequency for which the dielectric constant of the plasma, ϵ, is such that $\epsilon \cong -\epsilon_1$, where ϵ_1 is the dielectric constant for the medium surrounding the wire. This type of resonance, which was first studied in connection with gas discharge plasmas,[10] is called plasma resonance and occurs for waves having the electric vector perpendicular to the plasma column (wire).

The reasons for choosing silver wires in the experimental investigation are that the plasma resonance for silver falls in the visible range and that silver wires having diameters of a few thousand angstroms are relatively easy to make. The investigation, which was started also as an attempt to explain the polarization effects for thin tungsten wires reported by Öhman,[11] is closely connected with investigations on scattering and absorption properties of gas discharge plasmas[12] and semiconductor plasmas[13] made at our institute.

THEORY

Assumptions

In our studies of light scattering and absorption by thin metal wires we make the following assumptions:

1. A part of the wire is considered, which has the form of a circular cylinder of length l and radius $a \ll l$. The distance of observation, r, is such that $a^2 \ll \lambda r \ll l^2$, where λ is the wavelength of the light. The scattered field is

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[1] D. Bohm and D. Pines, Phys. Rev. **82**, 625 (1951); **85**, 338 (1952); **92**, 609 (1953).
[2] G. Ruthemann, Naturwissenschaften **29**, 648 (1941); **30**, 145 (1942); Ann. Physik **2**, 113 (1948).
[3] W. Lang, Optik **3**, 233 (1948).
[4] A. W. Blackstock, R. H. Ritchie, and R. P. Birkhoff, Phys. Rev. **100**, 1078 (1955).
[5] N. F. Mott, in *Proceedings of the Tenth Solvay Congress, Brussels, 1954* (R. Stoops, Brussels, 1955); H. Fröhlich and H. Pelzer, Proc. Phys. Soc. (London) **A68**, 525 (1955); D. Pines, Rev. Mod. Phys. **28**, 184 (1956).
[6] R. W. Wood, Phys. Rev. **92**, 18 (1933).
[7] C. Zener, Nature **132**, 968 (1933).
[8] E. A. Taft and H. R. Philipp, Phys. Rev. **121**, 1100 (1961).
[9] H. C. Van de Hulst, *Light Scattering by Small Particles* (John Wiley & Sons, Inc., New York, 1957).
[10] N. Herlofsson, Arkiv Fysik **3**, No. 15 (1951).
[11] Y. Öhman, Nature **192**, 254 (1961).
[12] B. Agdur, B. Kerzar, and F. Sellberg, Phys. Rev. **128**, 1 (1962).
[13] F. Sellberg, Royal Institute of Technology Internal Report, Stockholm, 1962 (unpublished).

then obtained as the far-field solution for an infinitely long circular cylinder.

2. The wire is assumed to be homogeneous with a permittivity (dielectric constant), ϵ, which is, in general, frequency dependent and complex. Temperature effects, other than those inherent in the imaginary part of ϵ, are disregarded.

3. The wire surface is treated as a sharp boundary between materials having permittivities 1 and ϵ, respectively.

The permittivity of silver is discussed in Appendix I. The complex refractive index of silver measured by Schulz[14] and by Taft and Philipp[8] can be closely approximated, for wavelengths down to 3000 Å, by a formula based on a simplified plasma model of the metal. The permittivity of silver according to this model is shown in Fig. 1.

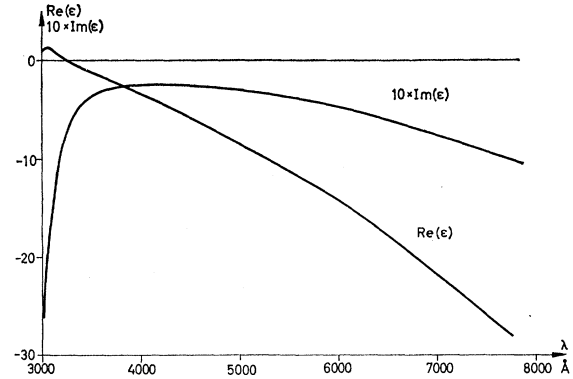

FIG. 1. The complex permittivity of silver at optical wavelengths according to a modified plasma model.

General Expressions

For the theory of light scattering by circular cylinders we refer to the textbook by van de Hulst.[9] For light incident perpendicularly upon a wire, one obtains[9]

$$I=(2/\pi kr)|T(\Theta)|^2 I_0,$$

where I_0 = intensity of incident light, I = intensity of light scattered in the direction Θ, Θ = scattering angle ($\Theta=180°$ for backscattered light), $k=2\pi/\lambda$ = wave number in vacuum, and r = radial distance from the wire. $T(\Theta)$ is a function defined below, which is different for different directions of the electric field vector.

Incident Electric Field Vector Perpendicular to Cylinder Axis

$$T^{\perp}(\Theta)=a_0+2\sum_{n=1}^{\infty} a_n \cos n\Theta,$$

where

$$a_n=\frac{J_n'(y)J_n(x)-mJ_n(y)J_n'(x)}{J_n'(y)H_n^{(2)}(x)-mJ_n(y)H_n^{(2)\prime}(x)},$$

with $x=ka$; $y=mka$; $m=\sqrt{\epsilon}$ = the complex refractive index of the wire.

We shall use efficiency factors, Q, of extinction, scattering, and absorption, where, e.g., Q_{ext} is the ratio between cross section of extinction and the geometrical cross section of the wire. The cross section of extinction is a measure of the total energy withdrawn from the incident beam.

$$Q_{\text{ext}}^{\perp}=\frac{2}{x}\operatorname{Re}(a_0+2\sum_{n=1}^{\infty} a_n),$$

$$Q_{\text{sca}}^{\perp}=\frac{2}{x}(|a_0|^2+2\sum_{n=1}^{\infty}|a_n|^2),$$

$$Q_{\text{abs}}^{\perp}=(Q_{\text{ext}}^{\perp}-Q_{\text{sca}}^{\perp}),$$

[14] L. G. Schulz, J. Opt. Soc. Am. 44, 357 (1954).

Incident Electric Field Vector Parallel to Cylinder Axis

$$T^{\parallel}(\Theta)=b_0+2\sum_{n=1}^{\infty} b_n \cos n\Theta,$$

where

$$b_n=\frac{mJ_n'(y)J_n(x)-J_n(y)J_n'(x)}{mJ_n'(y)H_n^{(2)}(x)-J_n(y)H_n^{(2)\prime}(x)}.$$

The efficiency factors are

$$Q_{\text{ext}}^{\parallel}=\frac{2}{x}\operatorname{Re}(b_0+2\sum_{n=1}^{\infty} b_n),$$

$$Q_{\text{sca}}^{\parallel}=\frac{2}{x}(|b_0|^2+2\sum_{n=1}^{\infty}|b_n|^2),$$

$$Q_{\text{abs}}^{\parallel}=(Q_{\text{ext}}^{\parallel}-Q_{\text{sca}}^{\parallel}).$$

Polarization of Scattering and Absorption

We define the polarization of scattering, P_Θ, in a direction Θ as

$$P_\Theta=\frac{|T^{\perp}(\Theta)|^2-|T^{\parallel}(\Theta)|^2}{|T^{\perp}(\Theta)|^2+|T^{\parallel}(\Theta)|^2}.$$

This quantity is a measure of the degree of polarization of the scattered light, when the incident light is unpolarized.

The polarization of absorption is defined by the formula

$$P_{\text{abs}}=\frac{Q_{\text{abs}}^{\perp}-Q_{\text{abs}}^{\parallel}}{Q_{\text{abs}}^{\perp}+Q_{\text{abs}}^{\parallel}}.$$

According to Kirchhoff's law of radiation, the ratio between emissivity and absorptivity is a constant for a body in thermal equilibrium. Applying this law, P_{abs} should also be the polarization of the light emitted from a hot wire.

Special Cases Studied

The computation of a_n and b_n for complex values of y, as occurs for metals in the optical range, is generally difficult, but simple solutions can be obtained for some cases.

Case I: $|y|\gg 1$ *and* x^2

This case, which is equivalent to $|m|\gg 1/x$ and x, is easily solved by using asymptotic expansions of the Bessel functions. It is treated in Appendix II, and we will here discuss only the degree of polarization of the absorbed (or emitted) light. Figure 2 shows P_{abs} as a function of $x=2\pi a/\lambda$. This polarization is independent of the actual value of the absorption and depends only on the parameter $x=2\pi a/\lambda$. Figure 2 also shows the validity range of the approximation $|y|\gg 1$ and x^2 for silver. We see that the validity range in this case is restricted to infrared wavelengths.

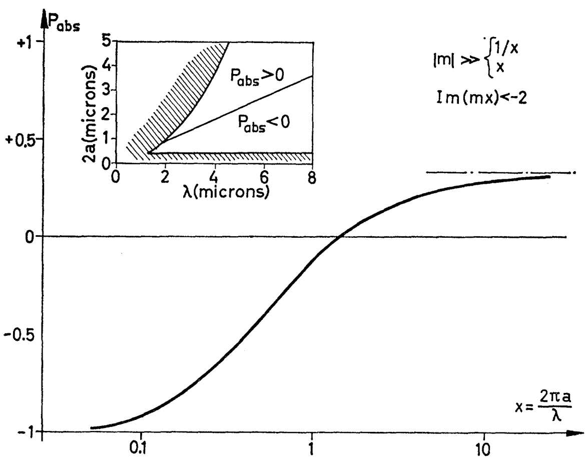

FIG. 2. The polarization of the absorption (or emission) for a cylinder of radius a. Inserted figure shows the range of validity of the approximation for silver.

Case II: $|Re(y)|\ll 1$ *or* $|Im(y)|\ll 1$

When either the real or the imaginary part of y is small, a Taylor expansion of the Bessel functions leads to useful approximate solutions. These conditions can, of course, always be fulfilled, if the wire diameter is small enough. For a silver wire with a diameter of 1.2 μ the real part of y is smaller than 0.5 for wavelengths in the near infrared and visible range. The formulas for $|\text{Re}(y)|\ll 1$ are developed in Appendix II.

Using the values of the index of refraction for silver obtained according to Appendix I, first a_n and b_n, and subsequently the different efficiency factors and polarizations for wires of different diameters have been calculated on a computer. The results are shown in Figs. 3–8. The curves for the diameters 0.8 and 1.2 μ are slightly incorrect because the condition for the approximations to be valid is only crudely fulfilled.

Figures 3 and 6 show that the absorption and the extinction of perpendicularly polarized light have

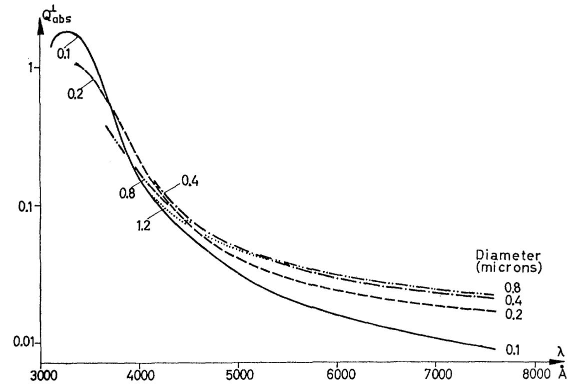

FIG. 3. Efficiency factor of absorption for perpendicularly polarized light by a silver wire. Parameter is wire diameter. Computed curves.

maxima at about 3500 Å, which become more pronounced as the diameter of the wire decreases. For cylindrical geometry, plasma resonance occurs when the real part of ϵ takes the value -1 and the damping and the diameter are small enough, which is the case for a thin silver wire at 3500 Å (see Fig. 1). If the wire is surrounded by a thick sheath of lossless dielectric (permittivity ϵ_1), the plasma resonance will be displaced to a wavelength, where $\epsilon=-\epsilon_1$. This means that plasma resonance can be obtained at light frequencies much smaller than the plasma frequency of the "free" electrons in a metal.

EXPERIMENTS

Preparation of Filaments

Earlier experiments on polarized thermal emission[11] were made on tungsten filaments with a thickness ranging from 3.5 to 6.5 μ. These wires were prepared by Lumalampan Aktiebolag in Stockholm. The present work has been made mainly with much thinner filaments. A special technique has been developed for producing silver wires as thin as 0.2 μ. The procedure is as follows:

A Vycor quartz tube with an inner diameter of about

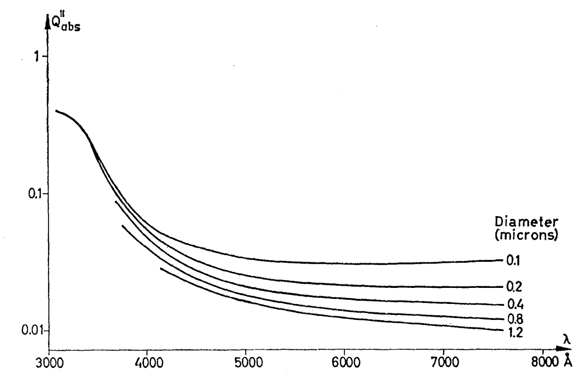

FIG. 4. Efficiency factor of absorption for parallel polarized light by a silver wire. Parameter is wire diameter. Computed curves.

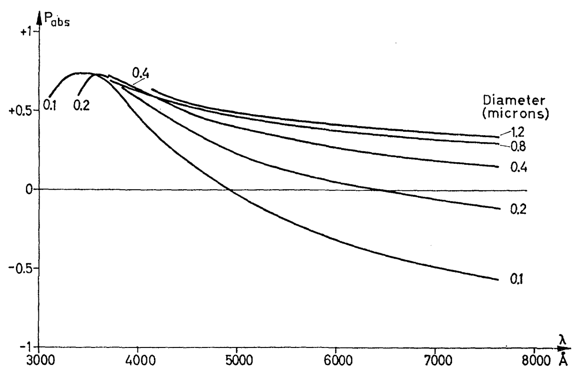

FIG. 5. Polarization of light absorbed (or emitted) by a silver wire. Parameter is wire diameter. Computed curves.

1 mm and an outer diameter of about 7 mm is filled with small pieces of pure silver. The tube is evacuated to about 10^{-3} mm Hg and sealed up in a quartz burner. A small part of the tube, where silver is present, is then heated to about 1700°C so that the silver melts. The tube is rapidly removed from the burner and a first reduction of the outer diameter to about 0.5 mm is made by drawing out the two ends of the tube. The filament is then put into a hydrogen-oxygen burner, giving a very narrow flame with a temperature of about 1900°C. When the melted silver shows a strong thermal emission, one end of the tube is rapidly pulled out until the tube breaks into two parts. The part of the tube that has been removed from the burner may then contain a silver filament which at one end is as thin as 0.2 μ or less.

The useful part of the wire is now cemented to a copper support shaped like a small fork with an opening of 1.5 mm. The Vycor quartz is then removed from the filament by carefully dipping the wire into a drop of hydrofluoric acid.

Polarization Measurements of Scattered Light

The first measurements of the polarization of scattered light were made in broad spectral regions with a lamp giving a continuous spectrum combined with selected filters. The polarization was measured mainly

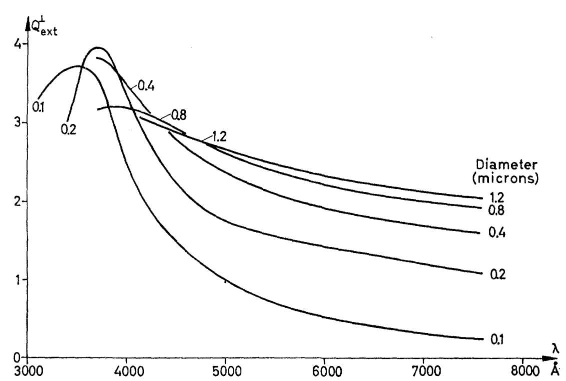

FIG. 6. Efficiency factor of extinction for perpendicularly polarized light scattered by a silver wire. Parameter is wire diameter. Computed curves.

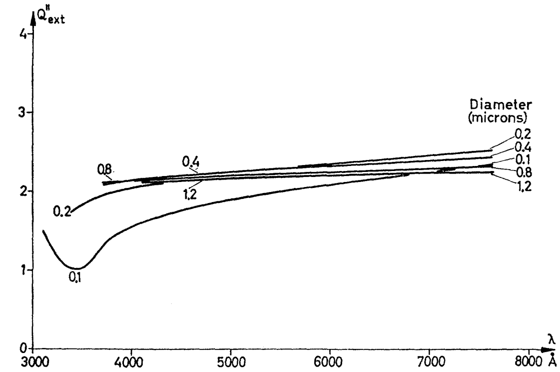

FIG. 7. Efficiency factor of extinction for parallel polarized light scattered by a silver wire. Parameter is wire diameter. Computed curves.

by using a Cornu polarimeter, but also by means of a photographic method using a four-image polarigraph.[15]

As the change of polarization with wavelength was found to be very rapid in the case of light scattering from narrow silver filaments, the final measurements have been made with monochromatic light. A high-pressure Philips mercury lamp combined with selected Schott filters was found very satisfactory for this purpose.

The final arrangement for measuring the polarization of scattered light is shown in Fig. 9. A parallel beam of light from the mercury lamp, L, illuminates the filament, F, an image of which is formed by the objective, O, at a distance giving an enlargement of about 20 times. This image is viewed through an eyepiece, E, combined with a polaroid film, P. In front of the eyepiece are a colorscreen, C, and a fixed plate of Iceland Spar, I, producing two images with a separation of about 0.5 mm. As the eyepiece can be rotated and the setting read on a scale, the arrangement works as a Cornu polarimeter. In fact, the degree of polarization, p, is obtained directly from the readings v_1 and v_2, giving equal intensity of the two images of the filament according to the formula:

$$p=\cos(v_1-v_2).$$

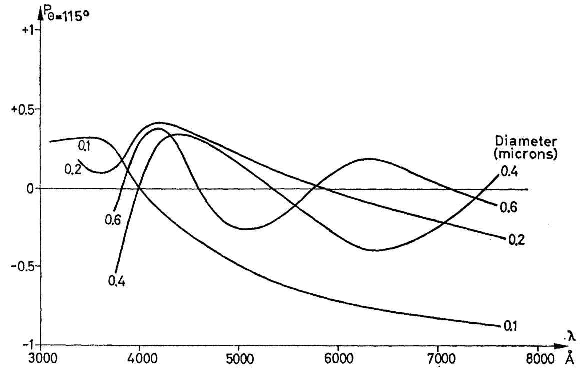

FIG. 8. Polarization of the light scattered in the angle 115° by a silver wire. Parameter is wire diameter. Computed curves.

[15] Y. Öhman, Stockholms Observatoriums Annaler **19**, 4 (1956).

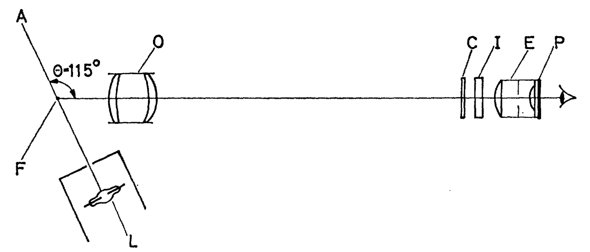

FIG. 9. Arrangement for measuring polarization of scattered light.

The simple arrangement of Fig. 9 has proved very useful in the present experiments. In Figs. 10 and 11 some representative measurements are given for the polarization produced by silver filaments with a thickness of 0.2 and 0.4 μ, respectively. In these experiments the angle, AFO, was 115°. The same figures show curves for the theoretically predicted values of the degree of polarization as taken from Fig. 8. It is seen that there is a good general agreement between theory and observation. The apparent discrepancies are probably due to deviations from a circular section. Measurements of wire diameters have been made with the electron microscope of the Physical Institution of the Caroline Institute.

Polarization Measurements of Thermal Emission

A number of observations have been made also for the purpose of extending the preliminary measurements reported[11] on polarized thermal emission. The material obtained is still not sufficient to allow rigorous comparison between theory and observation. Measurements with platinum filaments with diameters ranging from 1.0–10.0 μ show increasing polarization with decreasing thickness and a maximum polarization of about 50%. These filaments were electrically heated in vacuum.

We have also made experiments by observing the thermal emission from silver filaments enclosed in narrow quartz tubes. In this case the heating was made in a hydrogen burner. Silver filaments with a diameter of about 0.8 μ have been found to give a polarization of about 50%. This rather difficult work is still in progress.

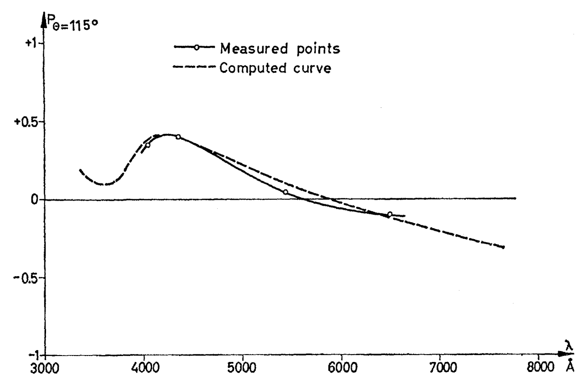

FIG. 10. Polarization of the light scattered in the angle 115° by a silver wire 0.2 μ thick.

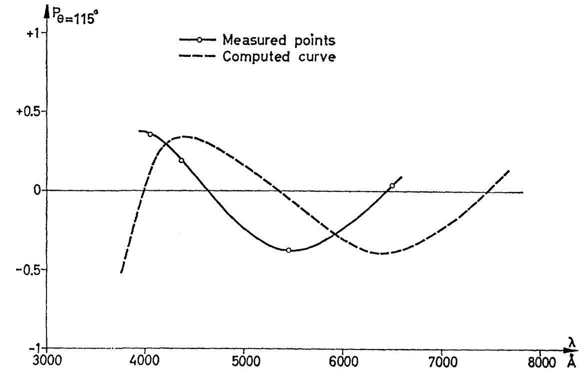

FIG. 11. Polarization of the light scattered in the angle 115° by a silver wire 0.4 μ thick.

APPENDIX I

The complex refractive index, $m=\nu-i\kappa$, and the permittivity are connected by the formula

$$\epsilon=m^2=(\nu^2-\kappa^2)-i(2\nu\kappa).$$

Theoretical models that explain the behavior of the refractive index of metals have been proposed; in the infrared region a free-electron model is adequate, but for shorter wavelengths there are significant contributions to the permittivity from interband electron transitions in the metal.[16] The permittivity of silver can be adequately described down to a wavelength of about 3000 Å with the help of a simplified model including contributions from free conduction electrons, from one type of resonant transition just below 3000 Å and from vibrating, nonresonant electrons. This model gives

$$\epsilon=\left(1+\frac{\omega_{p0}^2}{(-\omega^2+i\omega/\tau_0)}+\frac{\omega_{p1}^2}{(\omega_1^2-\omega^2+i\omega/\tau_1)}+\epsilon_b\right),$$

where $\omega_{p0}^2=e^2n_0/\epsilon_0m^*$ is proportional to the density of the free electrons, n_0; ω_{p1}^2 is proportional to the density of electrons in the resonant transition; ω_1^2 is a measure of the bond strength of the resonant electrons; ϵ_b is the constant contribution from bound, nonresonant electrons; τ_0, τ_1 are the relaxation times for free and resonant electrons, respectively.

Schulz[14] and Taft and Philipp[8] have measured the complex index of refraction of silver over a wide frequency range. The best fit to their data for wavelengths down to 3000 Å is given by the following choice of constants in the formula for ϵ:

$$\omega_{p0}=13.7\times10^{15}\text{ rad/sec},\quad \tau_0=13.5\times10^{-15}\text{ sec},$$
$$\omega_{p1}=4.0\times10^{15}\text{ rad/sec},\quad \tau_1=1.6\times10^{-15}\text{ sec},$$
$$\omega_1=6.5\times10^{15}\text{ rad/sec},\quad \epsilon_b=3.2.$$

The value of ω_{p0} is computed by assuming that the density of free electrons equals the atom density in

[16] M. Suffczynski, Phys. Rev. **117**, 663 (1960).

silver, and that the effective mass of these electrons equals the free-electron mass. From the value for the low-frequency conductivity of silver one can calculate $\tau_0=37\times10^{-15}$ sec, which is and should be higher than the value for optical frequencies.

APPENDIX II

Case I: $|y|\gg 1$ and x^2

In this case we can use asymptotic forms of the Bessel functions:

$$J_n(y)=\left(\frac{2}{\pi y}\right)^{1/2}\cos\left(y-\frac{\pi}{4}-\frac{n\pi}{2}\right),$$

$$J_n'(y)=-\left(\frac{2}{\pi y}\right)^{1/2}\sin\left(y-\frac{\pi}{4}-\frac{n\pi}{2}\right).$$

These are valid only when $|y|\gg n^2$, which gives at least the validity range $n\lesssim x$. We now make the further simplification of assuming $\mathrm{Im}(y)<-2$ and get

$$J_n'(y)/J_n(y)\approx i$$

from which follows:

$$a_n=\frac{1}{(1-i\cot\alpha_n)};\qquad b_n=\frac{1}{(1-i\cot\beta_n)};$$

$$\cot\alpha_n\approx\frac{Y_n(x)+imY_n'(x)}{J_n(x)+imJ_n'(x)};\quad \cot\beta_n\approx\frac{mY_n(x)+iY_n'(x)}{mJ_n(x)+iJ_n'(x)}.$$

We finally get

$$Q_{\mathrm{ext}}^{\mathrm{I}}\approx Q_{\mathrm{sca}}^{\mathrm{I}}\approx\frac{2}{x}\sum_{n=-\infty}^{+\infty}\left|\frac{J_n'(x)}{H_n'(x)}\right|^2,$$

$$Q_{\mathrm{abs}}^{\mathrm{I}}\approx\frac{4\,\mathrm{Re}(m)}{\pi x^2|m|^2}\sum_{n=-\infty}^{+\infty}\left|\frac{1}{H_n'(x)}\right|^2,$$

$$Q_{\mathrm{ext}}^{\mathrm{II}}\approx Q_{\mathrm{sca}}^{\mathrm{II}}\approx\frac{2}{x}\sum_{n=-\infty}^{+\infty}\left|\frac{J_n(x)}{H_n(x)}\right|^2,$$

$$Q_{\mathrm{abs}}^{\mathrm{II}}\approx\frac{4\,\mathrm{Re}(m)}{\pi x^2|m|^2}\sum_{n=-\infty}^{+\infty}\left|\frac{1}{H_n(x)}\right|^2.$$

The a_n and b_n are correctly given only for $|n|\lesssim x$, but it can be shown that all the terms are rapidly converging towards zero for $|n|>x$. The Q_{ext} are identical with those derived for an ideal metal ($|m|\rightarrow\infty$) in reference 9, p. 313. For large values of x ($x\gg1$) the efficiency factors become simple:

$$Q_{\mathrm{ext}}^{\mathrm{I}}\approx 2,\qquad Q_{\mathrm{ext}}^{\mathrm{II}}\approx 2,$$

$$Q_{\mathrm{abs}}^{\mathrm{I}}\approx 2\pi\frac{\mathrm{Re}(m)}{|m|^2},\qquad Q_{\mathrm{abs}}^{\mathrm{II}}\approx \pi\frac{\mathrm{Re}(m)}{|m|^2}.$$

These values may also be derived from geometrical optics by calculating the intensity of the reflected and absorbed rays through every point of the wire surface and integrating.

We get for the polarization of the absorption (Fig. 2):

$$P_{\mathrm{abs}}=\sum_{n=-\infty}^{+\infty}\left[\left|\frac{1}{H_n'(x)}\right|^2-\left|\frac{1}{H_n(x)}\right|^2\right]\Big/\sum_{n=-\infty}^{+\infty}\left[\left|\frac{1}{H_n'(x)}\right|^2+\left|\frac{1}{H_n(x)}\right|^2\right].$$

Thus, $P_{\mathrm{abs}}\rightarrow\frac{1}{3}$ when $x\rightarrow\infty$ and $|m|\rightarrow\infty$. It may be shown that in the range of geometrical optics ($x\gg1$), P_{abs} diminishes from $\frac{1}{3}$ as $|m|$ decreases from ∞.

Case II: $\mathrm{Re}(y)\ll 1$

In this case we expand the Bessel functions in Taylor series and keep only the zeroth and first-order terms in $\mathrm{Re}(y)$. Put $m=\nu-i\kappa$, where ν is the index of refraction and κ is the extinction coefficient:

$$J_n(y)\approx(-i)^nI_n(\kappa x)+(-i)^{n-1}(\nu x)I'_n(\kappa x),$$
$$J_n'(y)\approx(-i)^{n-1}I_n'(\kappa x)-(-i)^n(\nu x)I_n''(\kappa x),$$

from which follows:

$$a_n=\frac{1}{(1-i\cot\alpha_n)},\quad b_n=\frac{1}{(1-i\cot\beta_n)},$$

where

$$\cot\alpha_n\approx\frac{[I_n'(\kappa x)Y_n(x)+\kappa I_n(\kappa x)Y_n'(x)]+i(\nu x)[I_n''(\kappa x)Y_n(x)+(1/x)I_n(\kappa x)Y_n'(x)+\kappa I_n'(\kappa x)Y_n'(x)]}{[I_n'(\kappa x)J_n(x)+\kappa I_n(\kappa x)J_n'(x)]+i(\nu x)[I_n''(\kappa x)J_n(x)+(1/x)I_n(\kappa x)J_n'(x)+\kappa I_n'(\kappa x)J_n'(x)]},$$

$$\cot\beta_n\approx\frac{[\kappa I_n'(\kappa x)Y_n(x)-I_n(\kappa x)Y_n'(x)]+i(\nu x)[\kappa I_n''(\kappa x)Y_n(x)+(1/x)I_n'(\kappa x)Y_n(x)-I_n'(\kappa x)Y_n'(x)]}{[\kappa I_n'(\kappa x)J_n(x)-I_n(\kappa x)J_n'(x)]+i(\nu x)[\kappa I_n''(\kappa x)J_n(x)+(1/x)I_n'(\kappa x)J_n(x)-I_n'(\kappa x)J_n'(x)]}.$$