Comments on the Existing Theories of Magnetic Effects on Superconducting Films

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The Gupta-Mathur theory and the Douglass theory, following the Ginzburg-Landau approach, are shown to merge into each other if the conditions of validity are taken into more careful consideration.

THERE are two existing theories on the magnetic field effects on superconductivity of thin films. Gupta and Mathur[1] (GM) derive a perturbation theory; while Douglass[2] (D), following Gor'kov, Ginzburg, and Landau,[3] works out a theory in the temperature region $T_c-T\ll T_c$ and in the local limit. The calculated variation of energy gap ϵ with thickness of film d disagrees even qualitatively in the two cases.[2] The energy gap dependence on critical field given by Douglass,[2]

$$[\epsilon(H)/\epsilon(0)]^2=1-[H_0/H_c]^2, \tag{1}$$

is again different from the result of Mathur, Panchapakesan, and Saxena[4] (MPS). It is the purpose of this paper to show that the two theories do not differ so drastically as they appear to, and that they merge into each other if the local-nonlocal limits and regions of validity are taken into consideration more carefully.

The GM theory yields

$$\epsilon(H)/\epsilon(0)=1-\sum_q f(\mathbf{q}), \tag{2}$$

where

$$f(\mathbf{q})=\frac{1}{3}\frac{e^2}{m^2}\frac{k_F^2}{\epsilon(0)^2}|A(\mathbf{q})|^2 \quad \text{for} \quad \xi_0<\frac{1}{q}, \tag{3}$$

$$f(\mathbf{q})\simeq 2.7\frac{e^2}{m}\frac{k_F}{\epsilon(0)}\frac{|A(\mathbf{q})|^2}{q} \quad \text{for} \quad \xi_0>\frac{1}{q}. \tag{4}$$

For very thick films, Eq. (3) gives the main contribution and only the local limit is relevant, because $1/\xi\simeq 1/\xi_0+1/d$ with $d\to\infty$. For very thin films, the coherence length ξ is restricted by the size of the specimen and, in addition, the penetration depth λ increases with the decrease of thickness. Again the local limit applies.[5] In these cases, Eq. (2) can be written as[6]

$$\frac{\epsilon(H)}{\epsilon(0)}=1-\frac{1}{4}\frac{(\lambda/d)\sinh(d/\lambda)-1}{\cosh^2(d/2\lambda)}\left(\frac{H_0}{H_{cb}}\right)^2, \tag{5}$$

which differs only by a factor $\frac{1}{2}$ from the weak-field approximation, Eq. (8) of D.

Taking the relation

$$\left(\frac{H_c}{H_{cb}}\right)^2=\frac{1}{1-(2\lambda/d)\tanh(d/2\lambda)} \tag{6}$$

(the justification of this is to be discussed below), the GM result (dashed lines) is compared in Fig. 1 with the solution of the non-linear Ginzburg-Landau equations:

$$\left(\frac{H_0}{H_{cb}}\right)^2=\frac{4\phi_0^2(\phi_0^2-1)\cosh^2(\phi_0 d/2\lambda)}{1-(\lambda/\phi_0 d)\sinh(\phi_0 d/\lambda)}, \tag{7}$$

$$\left(\frac{H_c}{H_{cb}}\right)^2=\frac{\phi_c^2(2-\phi_c^2)}{1-(2\lambda/\phi_0 d)\tanh(\phi_c d/2\lambda)}. \tag{8}$$

The agreement in the weak-field region $H/H_c<0.6$ is quite good. The GM lines tend to curve towards the solid lines for very thin films, $d/\lambda<0.5$.

Equation (5) is derived by neglecting contributions from Eq. (4). In the intermediate thickness region, nonlocal effects have to be considered, and Eq. (4) does contribute. This, however, merely lifts the dashed curves a little in the middle but has very little effect on the large and small d/λ limiting parts.

It should be emphasized that Eq. (5) is expected to be valid only for very weak fields, because in GM's calculation the magnetic field is treated as a perturbation. All quantities are calculated up to the second order in $\mathbf{A}(\mathbf{q})$, the external vector potential. Therefore, Eq. (5) cannot predict the transition of the critical field H_c at $d/\lambda\simeq\sqrt{5}$ as in D. Notice that in the above comparison, H_c is not obtained from Eq. (5), but is given by the relation with H_{cb} in Eq. (7).

The GM theory is based on the Bardeen-Cooper-Schrieffer (BCS) theory, taking into consideration only the lowest excitations. Hence, it applies only for 0°K. The Ginzburg-Landau equations, on the other hand, are strictly valid only for temperatures very close to the critical T_c. Several suggestions for the latter case have been made to extrapolate into the lower $t=T/T_c$ range.

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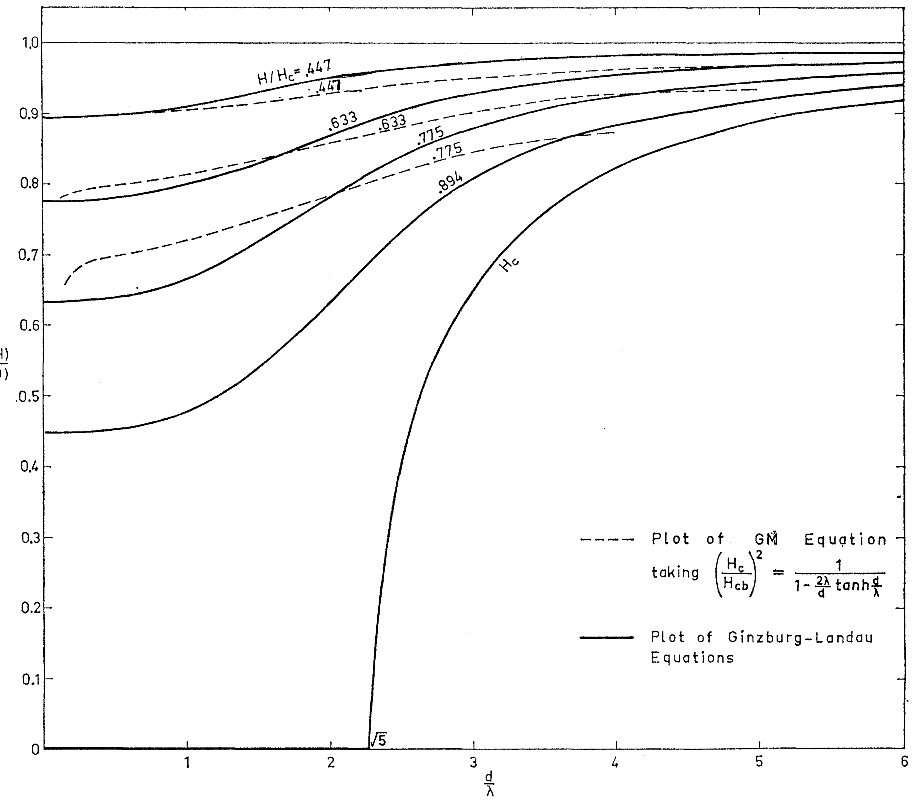

FIG. 1. Comparison of GM result with solution of Ginzburg-Landau equations.

For example, instead of Eq. (8), Ginzburg[7] gives

$$(H_c/H_{cb})^2 = 24\eta[\lambda(T,d)/d]^2, \qquad (9)$$

$$\eta = -\frac{4\pi}{\phi}\left[\frac{\partial}{\partial\phi}\left(\frac{F_{s0}-F_{n0}}{H_{cb}^2(T)}\right)\right]_{\phi=0} \qquad (10)$$

for very thin films, where F_{s0} is the free energy in the absence of a magnetic field, F_{n0} is the free energy of the normal state, and $\phi = \psi(T,H)/\psi(T,0)$. The explicit form of η as a function of T depends on the choice of F_{s0}. Taking[8]

$$F_{s0} = F_{n0} + [H_{cb}^2(0)/8\pi] \times\{2t^2[1-(1-|\chi|^2)^{1/2}]-|\chi|^2\}, \qquad (11)$$

where $\chi = \psi(T,H)/\psi(0,0)$, we see that $\eta = \frac{1}{2}(1+t^2)$. This gives

$$(H_c/H_{cb})^2 = 24(\lambda/d)^2 \quad \text{for} \quad t\simeq 1, \qquad (12)$$

$$= 12(\lambda/d)^2 \quad \text{for} \quad t\simeq 0. \qquad (13)$$

The $t\simeq 0$ result agrees with the thin-film approximation of Eq. (6). A recent calculation[9] on minimizing a general F_{s0} gives $(H_c/H_{cb})^2 = 0.68\times 24(\lambda/d)^2$ at $t\simeq 0$ by taking $N(0)V = 0.3$, $\ln y_c \simeq -0.80$, and Eq. (12) for $t\simeq 1$. Up to the present, although we cannot be sure of the exact temperature dependence of $(H_c/Hc_b)^2$, both theoretical and experimental evidence seems to indicate that we can take Eq. (6) for $t\simeq 0$, and Eq. (8) for $t\simeq 1$.

If this is the case, both theories give

$$\epsilon(H)/\epsilon(0) = 1 - \tfrac{1}{2}(H_0/H_c)^2 \qquad (14)$$

in the weak-field, thin-film approximation.

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