Rate of Capture of Electrons Injected into Superconducting Lead*

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The lifetime against capture into a Cooper pair of an electron injected into a superconductor is calculated. The electrons are assumed to have momenta close to the Fermi momentum, and the electron-phonon interaction, which causes a change in the momentum of the electron accompanied by emission of a phonon, is assumed to be responsible for the capture. Umklapp processes, in which the lattice takes up some momentum, seem to dominate the rate. The matrix element for umklapp processes is determined from the experimentally measured high-temperature resistivity. The lifetime of an electron injected into lead at 1.44°K, is calculated as 1.67×10^{-8} sec.

I. INTRODUCTION

THE lifetime of an injected electron in a superconductor is of interest in determining the tunneling current to be expected in superconducting sandwich experiments, and also must be taken into account in an accurate solution of the energy gap equation.[1] By "lifetime" we mean the time it takes for the injected electron to be captured into a Cooper pair and become incapable of tunneling. Burstein, Langenberg, and Taylor[2] discussed the possible use of sandwiches as detectors of microwave and submillimeter wavelength radiation. They also calculated by the principle of detailed balance the lifetime for radiative recombination, obtaining for lead at 2°K, 0.4 sec, which seems too long to be the dominant mode of decay. More recently, Schrieffer and Ginsberg[3] have calculated the lifetime for recombination by phonon emission and obtained for lead 0.43×10^{-7} sec at 1.44°K, a result consistent with the upper limit experimentally obtained by Ginsberg.[4] The Schrieffer-Ginsberg calculation involves an empirical constant, the Bloch constant, which measures the electron-phonon interaction. Schrieffer and Ginsberg evaluate the Bloch constant by fitting a theoretical formula to the measured high-temperature resistivity of lead. In the derivation of the resistivity formula, umklapp processes (in which the lattice takes up some momentum) are ignored. However, umklapp processes are generally believed[5] to be responsible for the major portion of the high-temperature resistivity. Nevertheless, the Schrieffer-Ginsberg calculation is consistent and probably basically correct; having neglected the role of umklapp processes in evaluating the electron-phonon interaction, they also neglect the role of umklapp processes in the recombination of electrons. The two effects essentially cancel, i.e., the electron-phonon coupling is overestimated by ignoring the role of umklapp processes in the resistivity, but the subsequent neglect of umklapp processes in the recombination calculation offsets the overestimate of the coupling.

In this paper we calculate the lifetime, attempting to calculate separately the rates of recombination via "normal" and umklapp processes. We find that the umklapp process is the dominant one, and the calculated lifetime is a bit shorter than that obtained by Schrieffer and Ginsberg.

The matrix element for umklapp processes is estimated in Sec. II from the measured high-temperature resistivity, regarding the matrix element for normal processes as known from theory. The rates of recombination via normal and umklapp processes are calculated in Secs. III and IV.

II. DETERMINATION OF MATRIX ELEMENT FOR UMKLAPP PROCESSES FROM HIGH TEMPERATURE RESISTIVITY

The electron-phonon interaction can be written as

$$H_{\text{el-ph}}=\sum_{\substack{\mathbf{k}\mathbf{k}'\mathbf{K}\lambda\sigma\\(\mathbf{q}=\mathbf{k}-\mathbf{k}'+\mathbf{K})}}\left(\frac{\hbar N}{2M\omega(\mathbf{q}\lambda)}\right)^{1/2}\mathbf{e}_{\mathbf{q}\lambda}\cdot\mathbf{I}(\mathbf{k},\mathbf{k}')\times c_{\mathbf{k}'\sigma}{}^{*}c_{\mathbf{k}\sigma}A_{\mathbf{q}\lambda}{}^{*}+\text{Hermitian conj.},\quad(1)$$

where $c_{\mathbf{k}\sigma}$ annihilates an electron of quasimomentum $\mathbf{k}$ (in the extended zone scheme) and spin σ and $A_{\mathbf{q}\lambda}{}^{*}$ creates a phonon of momentum $\mathbf{q}$ (restricted to first Brillouin zone), polarization λ, and frequency $\omega(\mathbf{q}\lambda)$. We shall assume that the unit polarization vector $\mathbf{e}_{\mathbf{q}\lambda}$ is parallel to $\mathbf{q}$ for $\lambda=1$, and perpendicular to $\mathbf{q}$ for $\lambda=2$, 3. The vector $\mathbf{K}$ is any reciprocal lattice vector; $\mathbf{K}=0$ for normal processes, $\mathbf{K}\neq0$ for umklapp processes. The number density of ions is N and M is the mass per ion. Reliable values for the vector matrix element $\mathbf{I}(\mathbf{k},\mathbf{k}')$ are hard to obtain theoretically, especially when $\mathbf{k}-\mathbf{k}'$ is as large as a reciprocal lattice vector (which is just the case which arises when studying umklapp processes). For small values of $\mathbf{k}-\mathbf{k}'$, the matrix elements obtained from a free-electron picture ought to be reasonably accurate. Bardeen[6] and Bardeen and

* This research was sponsored in part by the National Science Foundation.
[1] J. R. Schrieffer and Y. Wada (to be published).
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[3] J. R. Schrieffer and D. M. Ginsberg, Phys. Rev. Letters 8, 207 (1962).
[4] D. M. Ginsberg, Phys. Rev. Letters 8, 204 (1962).
[5] J. M. Ziman, *Electrons and Phonons* (Oxford University Press, New York, 1960), Chap. 9.
[6] J. Bardeen, Phys. Rev. 52, 688 (1937).

Pines[7] obtain, for small $\mathbf{k}-\mathbf{k}'$,

$$\mathbf{I}(\mathbf{k},\mathbf{k}')=(4\pi Ze^2/k_s^2)(\mathbf{k}-\mathbf{k}'), \tag{2}$$

where Z is the number of conduction electrons per ion (which we take as 4 for lead) and k_s is the inverse Fermi-Thomas screening length defined by $k_s/k_0=0.814\ r_s^{1/2}$; k_0=Fermi momentum; $r_sa_0=(3/4\pi NZ)^{1/3}$; a_0=Bohr radius. The form of $\mathbf{I}$, plus the assumption that $\mathbf{e}_{q1}=\mathbf{q}/|\mathbf{q}|$, $\mathbf{q}\cdot\mathbf{e}_{q2}=\mathbf{q}\cdot\mathbf{e}_{q3}=0$, implies that in normal processes, for which $\mathbf{q}=\mathbf{k}-\mathbf{k}'$, only longitudinal phonons ($\lambda=1$) can be emitted and absorbed. It will shortly be seen that if Eq. (2) is even approximately correct, then normal processes account for only a negligible fraction of the high-temperature electrical resistivity.

The "relaxation time" approximation[8] gives an accurate solution of the Boltzmann equation if the following conditions are fulfilled: (a) the single-particle energy depends only on $|\mathbf{k}|$, (b) the relaxation time $\tau(\mathbf{k})$ depends only on $|\mathbf{k}|$, and (c) $\tau(\mathbf{k})$ is nearly the same as $\tau(\mathbf{k}')$ if transitions are possible between states $\mathbf{k}$ and $\mathbf{k}'$.

It is hard to estimate the error introduced by the failure of assumption (a) near zone boundaries. For simplicity, we shall assume spherical energy surfaces with $m^*=m$. Despite the predominance of umklapp processes, (b) holds within 10% or less; the abundance of reciprocal lattice vectors makes all directions almost completely equivalent.

Note added in proof. Dr. Y. Wada has pointed out that $\tau(\mathbf{k})$ may become very small on certain small portions of the Fermi surface, in the neighborhood of points $\mathbf{k}$ which can be connected to another point $\mathbf{k}'$ on the Fermi surface by means of a reciprocal lattice vector. Whether or not this phenomenon occurs depends on the behavior of $\mathbf{I}(\mathbf{k},\mathbf{k}')$ when $\mathbf{k}-\mathbf{k}'$ is near a reciprocal lattice vector, but the "pathological" region on the Fermi surface is so small as to have little effect on the validity of the relaxation-time approximation, and produces no divergence in Eq. (3).

Condition (c) will be fulfilled if the equilibrium occupation factors are approximately equal for all states between which transitions are possible. Since the maximum phonon energy is $k_B\Theta_D$ (k_B=Boltzmann constant, Θ_D=Debye temperature) and the Fermi distribution varies on an energy scale determined by k_BT, condition (c) will hold if T is large compared to Θ_D ($\Theta_D=95°$ for lead). The formula for the high-temperature electrical resistivity, in the relaxation time approximation, can be written as

$$\rho=\frac{m^2k_0k_BT}{\pi\hbar^3Ze^2M}\left\langle\sum_{\lambda\mathbf{K}}\frac{1}{\omega^2(\mathbf{q}\lambda)}[\mathbf{e}_{q\lambda}\cdot\mathbf{I}(\mathbf{k},\mathbf{k}')]^2(1-\cos\theta)\right\rangle. \tag{3}$$

The symbol $\langle\ \rangle$ denotes that $\mathbf{k}$ and $\mathbf{k}'$ are to be averaged over the Fermi surface; θ is the angle between $\mathbf{k}$ and $\mathbf{k}'$; for given $\mathbf{k}$ and $\mathbf{k}'$ the sum over $\mathbf{K}$ is restricted to values of $\mathbf{K}$ such that $\mathbf{q}=\mathbf{k}-\mathbf{k}'+\mathbf{K}$ is in the first Brillouin zone, which will be taken as a sphere of radius q_D. Condition (b) is equivalent to the statement that for fixed direction of $\mathbf{k}$ the average over directions of $\mathbf{k}'$ yields a value essentially independent of the direction of $\mathbf{k}$.

It is easy to show that if $\mathbf{k}$ and $\mathbf{k}'$ are randomly chosen vectors on a spherical surface of radius k_0, then the vector $\mathbf{p}=\mathbf{k}-\mathbf{k}'$ may lie anywhere inside a sphere of radius $2k_0$, with probability distribution $f(\mathbf{p})d^3p=(8\pi k_0^2p)^{-1}d^3p$. Furthermore, we have $1-\cos\theta=p^2/2k_0^2$. We assume the dispersion relation $\omega(\mathbf{q}1)=s_Lq$, $\omega(\mathbf{q}2)=\omega(\mathbf{q}3)=s_Tq$, where s_L and s_T are the longitudinal and transverse sound velocities (in lead[9] $s_L=2.35\times10^5$ cm/sec, $s_T=1.27\times10^5$ cm/sec). Thus, the contribution to the resistivity from normal processes ($\mathbf{K}=0$) is

$$\rho_N=\frac{m^2k_0k_BT}{\pi\hbar^3Ze^2M}\int_{p<q_D}f(\mathbf{p})d^3p\frac{1}{s_L^2p^2}\left(\frac{4\pi Ze^2}{k_s^2}\right)^2p^2\frac{p^2}{2k_0^2}$$

$$=\frac{\pi m^2k_0Ze^2}{\hbar^3k_s^4}\frac{k_BT}{Ms_L^2}\left(\frac{q_D}{k_0}\right)^4. \tag{4}$$

Using the "free electron" values $k_0=(3\pi^2NZ)^{1/3}=1.57\times10^{-8}$ cm^{-1}, $q_D/k_0=(2/Z)^{1/3}$, we obtain when $T=273°$K, $\rho_N=1.8\times10^{-19}$ esu which is to be compared with the experimental value[10] $\rho=2.1\times10^{-17}$ esu. Thus, if one takes the matrix element (2) seriously, only 1% of the high-temperature resistivity arises from normal processes.

The crystal structure of lead is face-centered cubic; the edge of the unit cubic cell is 4.94 Å and the nearest-neighbor distance is $4.94/\sqrt{2}=3.49$ Å. The reciprocal lattice is body-centered cubic. Sitting at any lattice point, one sees eight nearest neighbors at a distance $K=2.20\times10^8$ cm^{-1}, six second-nearest neighbors at distance $K=2.54\times10^8$ cm^{-1}, twelve at $K=3.6\times10^8$ cm^{-1}, twenty-four at 4.22×10^8 cm^{-1}, eight at 4.4×10^8 cm^{-1}, etc. Since the maximum value of $\mathbf{k}-\mathbf{k}'$ in (3) is $2k_0=3.14\times10^8$ cm^{-1}, and the Debye cutoff is $q_D=1.25\times10^8$ cm^{-1}, most of the contribution to (3) comes from the $\mathbf{K}$ vectors of length 2.20×10^8 cm^{-1} and 2.54×10^8 cm^{-1}. If one draws the Debye sphere of radius 1.25×10^8 cm^{-1} centered at a $\mathbf{K}$ vector of length 3.6×10^8 cm^{-1}, less than 20% of the volume of the sphere lies inside a sphere of radius 3.14×10^8 cm^{-1} centered at the origin in $\mathbf{K}$ space (see Fig. 1). Furthermore, simple theories suggest that the matrix element $\mathbf{I}$ falls off as $\mathbf{k}-\mathbf{k}'$ increases. Accordingly, we neglect all reciprocal lattice vectors except the two shortest ones. About 83% of the volume of a Debye sphere centered at $K=2.54\times10^8$

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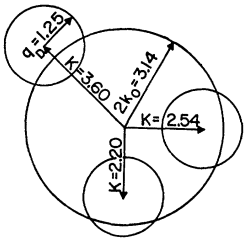

FIG. 1. Kinematics of Umklapp processes. The change of electron momentum is represented by a vector inside a sphere of radius $2k_0=3.14\times10^8$ cm^{-1}, and must be equal to the sum of a reciprocal lattice vector $\mathbf{K}$ plus a phonon momentum less than $q_D=1.25\times10^8$ cm^{-1}. Only the two shortest reciprocal lattice vectors $K=2.20\times10^8$ cm^{-1} and $K=2.54\times10^8$ cm^{-1} contribute appreciably.

cm^{-1} lies inside a sphere of radius 3.14×10^8 cm^{-1}. We pretend that the entire Debye sphere lies inside, but compensate by taking the number of reciprocal lattice vectors of length 2.54×10^8 cm^{-1} as five rather than six. The corresponding correction for the reciprocal lattice vectors $K=2.20\times10^8$ cm^{-1} would be negligible.

If, as in the Bardeen[6] theory, one approximates the unit cell of the crystal by a Wigner-Seitz sphere centered at an ion, then $\mathbf{I}$ must be of the form

$$\mathbf{I}(\mathbf{k},\mathbf{k}')=\frac{\mathbf{k}-\mathbf{k}'}{|\mathbf{k}-\mathbf{k}'|}I(|\mathbf{k}-\mathbf{k}'|). \tag{5}$$

With this form for $\mathbf{I}$, the contribution to (3) from a particular reciprocal lattice vector $\mathbf{K}$ is

$$\rho_U(\mathbf{K})=\frac{m^2k_0k_BT}{\pi\hbar^3Ze^2M}\sum_\lambda\int_{q<q_D}d^3q\,f(\mathbf{q}-\mathbf{K})\frac{1}{\omega^2(\mathbf{q}\lambda)}\times\left(\frac{\mathbf{e}_{\mathbf{q}\lambda}\cdot(\mathbf{q}-\mathbf{K})}{|\mathbf{q}-\mathbf{K}|}\right)^2[I(|\mathbf{q}-\mathbf{K}|)]^2(\mathbf{q}-\mathbf{K})^2/2k_0^2. \tag{6}$$

We follow Morel[11] in replacing all "slowly varying" factors in the integral by their values at the center of the Debye sphere, i.e., $\mathbf{q}=0$. The rapidly varying factors, which require more accurate treatment, are $[\omega(\mathbf{q}\lambda)]^{-2}$ and the polarization factor (which varies rapidly because of the variation of the direction of $\mathbf{e}_{\mathbf{q}\lambda}$; the vector $\mathbf{q}-\mathbf{K}$ may be replaced by $\mathbf{K}$). We obtain

$$\rho_U(\mathbf{K})=\frac{m^2k_BT}{4\pi Ze^2\hbar^3M}\frac{q_D|\mathbf{K}|}{k_0^3}\left[\frac{1}{3}\frac{1}{s_L^2}+\frac{2}{3}\frac{1}{s_T^2}\right][I(|\mathbf{K}|)]^2. \tag{7}$$

One might question the accuracy of the approximations leading from (6) to (7) on the ground that $|\mathbf{q}-\mathbf{K}|$ varies considerably as $\mathbf{q}$ ranges over the Debye sphere; for example, if $K=2.20\times10^8$ cm^{-1} then $|\mathbf{q}-\mathbf{K}|$ varies from 0.95×10^8 cm^{-1} to 3.45×10^8 cm^{-1}. The other factors, especially I^2, may vary considerably over this range. This question was investigated carefully under the assumption that $x[I(x)]^2$ is proportional to $\exp(-\alpha x)$, and it was found that (7) differs from (6) by 15% or less even in the "extreme" case $\alpha=1/q_D$ (note that in this case the factor I^2 varies by a factor of 25 over the region of integration).

The total resistance arising from umklapp processes is obtained by summing (7) over the eight $\mathbf{K}$ vectors of length 2.20×10^8 cm^{-1} and the "five" $\mathbf{K}$ vectors of length 2.54×10^8 cm^{-1}. We assume that the matrix elements are equal, i.e., $I(2.20\times10^8)=I(2.54\times10^8=I_U$. The total resistance from umklapp processes is

$$\rho_U=13\frac{m^2k_BT}{4\pi Ze^2\hbar^3M}\frac{q_D\bar{K}}{k_0^3}\left[\frac{1}{3}\frac{1}{s_L^2}+\frac{2}{3}\frac{1}{s_T^2}\right]I_U^2, \tag{8}$$

where 13 is the number of reciprocal lattice vectors which contribute and $\bar{K}=2.33\times10^8$ cm^{-1} is a weighted average of the lengths of the reciprocal lattice vectors. If we believe that essentially all the observed high-temperature resistance arises from umklapp processes, then Eq. (8) can be used to provide an experimental determination of I_U^2, and we find

$$I_U^2=6.1\times10^{-52}\text{ erg}^2\text{ cm}^4.$$

Pines[12] has proposed the formula

$$[I(\mathbf{k},\mathbf{k}')]^2=\left(\frac{4\pi Ze^2|\mathbf{k}-\mathbf{k}'|}{|\mathbf{k}-\mathbf{k}'|^2+k_s^2}\right)^2 \tag{9}$$

which gives, when $|\mathbf{k}-\mathbf{k}'|=\bar{K}$, $I^2=8.7\times10^{-52}$ erg^2 cm^4 in good agreement with the experimentally determined value.

III. THE "NORMAL" PROCESS LIFETIME

We are interested in a process in which an electron (momentum and spin $\mathbf{k}\uparrow$) injected into a superconductor combines with an unpaired electron ($-\mathbf{k}'\downarrow$) to form a Cooper pair with the emission of a phonon. Using the BCS[13] theory of superconductivity, one can represent the initial state by a typical excited-state wave function for the temperature in question in which the states $\mathbf{k}\uparrow$ and $-\mathbf{k}'\downarrow$ are definitely occupied, and $-\mathbf{k}\downarrow$ and $\mathbf{k}'\uparrow$ are unoccupied.

$$\Psi_i=\prod_{\substack{\mathbf{k}_1(G)\\ \mathbf{k}_1\neq\mathbf{k},\mathbf{k}'}}[(1-h_{\mathbf{k}_1})^{1/2}+h_{\mathbf{k}_1}{}^{1/2}b_{\mathbf{k}_1}{}^*]\times\prod_{\substack{\mathbf{k}_2(P)\\ \mathbf{k}_2\neq\mathbf{k},\mathbf{k}'}}[(1-h_{\mathbf{k}_2})^{1/2}b_{\mathbf{k}_2}{}^*-h_{\mathbf{k}_2}{}^{1/2}]\times\prod_{\substack{\mathbf{k}_3(S)\\ \mathbf{k}_3\neq\mathbf{k},\mathbf{k}'}}c_{\mathbf{k}_3\sigma_3}{}^*c_{\mathbf{k}\uparrow}{}^*c_{-\mathbf{k}'\downarrow}{}^*\Phi_0. \tag{10}$$

The first product contains ground pairs, the second

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the excited pairs in which both members of a pair state happen to be occupied by excited particles, the third product contains singly excited particles. The pair operators are defined by

$$b_{\mathbf{k}}^*=c_{\mathbf{k}\uparrow}^* c_{-\mathbf{k}\downarrow}^*, \quad b_{\mathbf{k}}=c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow},$$

and

$$h_{\mathbf{k}}=\tfrac{1}{2}[1-\epsilon/(\epsilon^2+\epsilon_0^2)^{1/2}], \quad \epsilon=(\hbar^2/2m)(k^2-k_0^2),$$

where ϵ_0 is the temperature-dependent energy gap parameter and ϵ the Bloch energy measured relative to the Fermi energy. We assume that we will be working at temperatures well below the transition temperature so that $\epsilon_0=\epsilon_0(0)$. Φ_0 is the vacuum state. The final-state wave function has a component in which the pair state $\mathbf{k}$ $(\mathbf{k}\uparrow, -\mathbf{k}\downarrow)$ (a pair state is labeled by the spin-up member) is occupied by a ground pair and a component in which the pair state $\mathbf{k}$ is completely unoccupied. This is true also for the pair state $\mathbf{k}'$. The final state is

$$\Psi_f=\prod_{\mathbf{k}_1(G)}[(1-h_{\mathbf{k}_1})^{1/2}+h_{\mathbf{k}_1}^{1/2}b_{\mathbf{k}_1}^*] \times\prod_{\substack{\mathbf{k}_2(P)\\ \mathbf{k}_2\neq\mathbf{k},\mathbf{k}'}}[(1-h_{\mathbf{k}_2})^{1/2}b_{\mathbf{k}_2}^*-h_{\mathbf{k}_2}^{1/2}]\prod_{\substack{\mathbf{k}_3(S)\\ \mathbf{k}_3\neq\mathbf{k},\mathbf{k}'}}c_{\mathbf{k}_3\sigma_3}^*\Phi_0. \quad (11)$$

The $\mathbf{k}'$ pair state in the third product of the initial state can be any pair state occupied singly by $-\mathbf{k}'\downarrow$. The transitions which will be possible from the initial to the final state with an interaction of the type c^*c are $\mathbf{k}\uparrow$ to $\mathbf{k}'\uparrow$ or $-\mathbf{k}'\downarrow$ to $-\mathbf{k}\downarrow$. In the matrix element these two cases are coherent. In Ψ_f, $\mathbf{k}$ and $\mathbf{k}'$ both appear in the ground pair product.

We are assuming that transitions are due to the electron-phonon interaction and, since the minimum energy to form a pair is $2\epsilon_0$, the phonons will have energies large compared to k_BT; thus only spontaneous emission is important for formation of ground pairs. For normal processes, in which $\mathbf{q}=\mathbf{k}-\mathbf{k}'$, we use the matrix element (2) which allows only longitudinal phonons to be emitted. The transition probability per unit time for the injected excited electron to decay into a ground pair can be written as

$$W_N=\frac{2\pi}{\hbar}A\sum_{\mathbf{k}'}|\langle\Psi_f|c_{\mathbf{k}'\uparrow}^*c_{\mathbf{k}\uparrow}+c_{-\mathbf{k}\downarrow}^*c_{-\mathbf{k}'\downarrow}|\Psi_i\rangle|^2 \times|\mathbf{k}-\mathbf{k}'|f'\delta(E+E'-\hbar s_L|\mathbf{k}-\mathbf{k}'|), \quad (12)$$

where

$$A=(\hbar N/2Ms_L)(4\pi Ze^2/k_s^2)^2, \quad E=(\epsilon^2+\epsilon_0^2)^{1/2},$$

and f' represents the result of averaging over a statistical distribution of initial states and is the probability that the state $\mathbf{k}'$ is occupied singly in a typical initial-state configuration.

In Eq. (12) the matrix element squared as evaluated from BCS[13] is $\frac{1}{2}[1+(\epsilon_0^2-\epsilon\epsilon')/EE']$. When the sum on $\mathbf{k}'$ is carried out for values of k' above and below k_0, ϵ' will take on plus and minus values and the term linear in ϵ' drops out. We can pass from the sum to an integral and calculating for a unit volume we obtain

$$W_N=\frac{2\pi}{\hbar}\frac{A}{(2\pi)^3}\int_{k'>k_0}\left[1+\frac{\epsilon_0^2}{EE'}\right]\exp(-E'/k_BT) \times|\mathbf{k}-\mathbf{k}'|\delta(E+E'-\hbar s_L|\mathbf{k}-\mathbf{k}'|)d^3k', \quad (13)$$

where we have replaced $f'=[\exp(E'/k_BT)+1]^{-1}$ by $\exp(-E'/k_BT)$ since the minimum value of E' is ϵ_0 and, for $T<T_c$, $\exp(\epsilon_0/k_BT)\gg1$. Approximating $|\mathbf{k}-\mathbf{k}'|$ by $2k_0\sin(\theta/2)$ and integrating over angles removes the delta function and introduces the factors $2\pi(E+E')^2/(\hbar s_L)^3k_0^2$. Changing variables from k' to E', and taking slowly varying factors out of the integral since $\exp(-E'/k_BT)$ acts as a sharp cutoff factor, we obtain

$$W_N=(8\pi)^{-1/2}mA(\epsilon_0k_BT)^{1/2}\hbar^{-6}s_L^{-3}k_0^{-1}(E+\epsilon_0)^3E^{-1} \times\exp(-\epsilon_0/k_BT). \quad (14)$$

Evaluation, with $\epsilon_0=1.34\times10^{-3}$ eV, yields

$$W_N=4.45\times10^9T^{1/2}\exp(-15.5/T) \times(E+\epsilon_0)^3/E\epsilon_0^2\ \text{sec}^{-1}. \quad (15)$$

At 1.44°K and $E=\epsilon_0$ corresponding to the Schrieffer and Ginsberg calculation, we obtain $[W_N(1.44°\text{K})]^{-1}=1.2\times10^{-6}$ sec.

IV. THE UMKLAPP PROCESS LIFETIME

The rate of recombination via umklapp processes involving a particular reciprocal lattice vector $\mathbf{K}$ and phonons of polarization λ is

$$W_U(\mathbf{K}\lambda)=\frac{2\pi}{\hbar}\frac{\hbar N}{2Ms_\lambda}\int_{|\mathbf{k}-\mathbf{k}'+\mathbf{K}|<q_D}\frac{d^3k'}{(2\pi)^3}\frac{1}{|\mathbf{k}-\mathbf{k}'+\mathbf{K}|} \times\left(\mathbf{e}_{q\lambda}\cdot\frac{\mathbf{k}-\mathbf{k}'}{|\mathbf{k}-\mathbf{k}'|}\right)^2I^2(|\mathbf{k}-\mathbf{k}'|)\frac{1}{2}\left[1+\frac{\epsilon_0^2-\epsilon\epsilon'}{EE'}\right] \times f'\delta(E+E'-\hbar s_\lambda|\mathbf{k}-\mathbf{k}'+\mathbf{K}|), \quad (16)$$

where an electron-phonon matrix element of the form (5) has been assumed. If the injection energy E is close to ϵ_0, and $T<T_c$, the magnitude of the phonon wave vector $\mathbf{q}=\mathbf{k}-\mathbf{k}'+\mathbf{K}$ is close to $q_{\min}(\lambda)=2\epsilon_0/\hbar s_\lambda$ which has the value 3.4×10^7 cm^{-1} for transverse phonons and 1.8×10^7 cm^{-1} for longitudinal phonons. This is small compared with q_D or K; accordingly we can say that $\mathbf{k}-\mathbf{k}'$ is nearly equal to $-\mathbf{K}$, and the matrix element I^2 can be treated as constant. Since energy and momentum conservation restrict K to the 14 shortest reciprocal lattice vectors, we can set I^2 equal to the previously evaluated I_U^2. The polarization factor leads to extremely messy geometrical considerations, and we shall replace it by $\frac{1}{3}$ with, we believe, negligible error. If we write $d^3k'=k'^2\sin\theta'd\theta'd\varphi'dk'$, where θ' is measured from the polar axis $\mathbf{k}+\mathbf{K}$, the angular integrations are

easily done, removing the delta function. We get a nonzero answer only if

$$k_0-q_{\min}(\lambda)<|\mathbf{k}+\mathbf{K}|<k_0+q_{\min}(\lambda). \quad (17)$$

Otherwise the argument of the delta function never vanishes. Strictly, the condition (17) holds only when the injection energy E is equal to ϵ_0, and in other cases $q_{\min}$ should be replaced by $(E+\epsilon_0)/\hbar s_\lambda$; we assume throughout that $T<T_c$ which implies $E'\sim\epsilon_0$. Integrating over dk', we obtain

$$W_U(\mathbf{K}\lambda)=\frac{1}{12(2\pi)^{1/2}}\frac{I_U^2}{s_\lambda^2}\frac{1}{|\mathbf{k}+\mathbf{K}|}\frac{N}{M}\frac{m}{\hbar^3}\left(1+\frac{\epsilon_0}{E}\right)\times(\epsilon_0 k_B T)^{1/2}\exp(-\epsilon_0/k_B T) \quad (18)$$

provided (17) is satisfied; otherwise $W_U(\mathbf{K}\lambda)=0$.

The total rate W_U is obtained by summing (18) over polarizations and over all (if any) reciprocal lattice vectors $\mathbf{K}$ which satisfy (17). Most of the contribution comes from transverse phonons for the following reasons: (1) $s_T\sim\frac{1}{2}s_L$, and (18) varies as s_λ^{-2}; (2) the condition (17) is more restrictive for longitudinal phonons than for transverse phonons; (3) there are two transverse polarization vectors. Accordingly, we neglect the longitudinal polarization. We evaluate W_U numerically, for the case when the injected electrons have energy $E=\epsilon_0$ ($|\mathbf{k}|=k_0$) and a wide spread in angles, by averaging (18) over the Fermi surface and summing over reciprocal lattice vectors. Only the 14 shortest reciprocal lattice vectors can satisfy (17) with $\mathbf{k}$ on the Fermi surface. We are interested in the average over the Fermi surface of a function which equals $|\mathbf{k}+\mathbf{K}|^{-1}$ when (17) is satisfied and is zero otherwise. An elementary integration shows that this average is equal to $q_{\min}/k_0K$ (provided $K-k_0<k_0-q_{\min}$ and $K+k_0>k_0+q_{\min}$, which is the case of interest to us). Taking $q_{\min}=3.4\times10^7$ cm^{-1}, and $K=2.20\times10^8$ cm^{-1} (eight times) and $K=2.54\times10^8$ cm^{-1} (six times), we obtain the total rate W_U by replacing the factor $|\mathbf{k}+\mathbf{K}|^{-1}$ in (18) by $k_0^{-1}[8(3.4\times10^7/2.20\times10^8)+6(3.4\times10^7/2.54\times10^8)]=2.04k_0^{-1}$, setting $s=s_T$, and multiplying by 2 for polarizations. The result is

$$W_U=3.3\times10^{12}T^{1/2}\exp(-15.5/T)\ \text{sec}^{-1}. \quad (19)$$

For $T=1.44$°K we obtain

$$[W_U]^{-1}=1.67\times10^{-8}\ \text{sec}.$$

V. DISCUSSION

Despite the difference in kinematics between the normal and umklapp processes, we obtain a lifetime not very different from the Schrieffer-Ginsberg value 4.3×10^{-8} sec. Most of the effect of the kinematics "cancels out" if one uses the high-temperature resistivity to determine the electron-phonon coupling.

Schrieffer and Ginsberg assume equal Bloch constants (essentially the average over directions of $q^{-2}|\mathbf{e}\cdot\mathbf{I}|^2$) for longitudinal and transverse phonons in normal processes, while we take the Bloch constant for transverse phonons as zero [from Eq. (5)]. It is hard to estimate quantitatively how wrong the polarization factors might be; but the assumptions that the unit cell is spherical and that phonons are strictly longitudinal or transverse seem reasonably accurate in a cubic lattice.

We have taken Z equal to the valency of lead, namely, 4, and $k_0=(3\pi^2NZ)^{1/3}=1.57\times10^8$ cm^{-1}. Schrieffer and Ginsberg take $Z=1.24$ and $k_0=1.08\times10^8$ cm^{-1}, the value of Z being determined by measurements of the anomalous skin effect.[14] If we were to adopt the latter value of k_0, the rate of recombination via umklapp processes would be greatly diminished. The inequality (17) could not be satisfied when $K=2.54\times10^8$ cm^{-1} and would restrict $\mathbf{k}$ to a very small portion of the Fermi surface when $K=2.20\times10^8$ cm^{-1}. However, neutron scattering[15] provides a rather direct measurement of k_0, and indicates a value very close to the one we have adopted. At this point we are probably encountering the limitations of the free-electron picture, and it is doubtful whether a set of parameters can be found which will account for all experiments.

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