Statistical Analysis of Electron-Proton Scattering*

M. W. KIRSON AND J. S. LEVINGER†
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York
(Received 14 January 1963)

We have made a statistical analysis of Stanford measurements of electron-proton scattering. We find that the Rosenbluth formula is valid within the experimental errors in the range $7 \leq q^2 \leq 25\,\mathrm{F}^{-2}$, and in the range of angles covered. We evaluate the form factors G_E, G_M and F_1, F_2, their errors, and correlated errors. Our values for the form factors are in reasonable agreement with the analyses of Hofstadter *et al.*, and of Richard Wilson *et al.* The errors in G_M are particularly small; the errors in F_1, F_2, and G_E are of similar magnitude. The coefficient of correlation between the errors of the form factors F_1 and F_2 is large and negative (-84% to -99%); the coefficient of correlation between the errors of G_E and G_M is almost as large and negative (-74% to -99%). G_E is about 40% larger than $G_M/2.79$ for $16.5 \leq q^2 \leq 25\,\mathrm{F}^{-2}$; the ratio of G_E to $G_M/2.79$ is about two standard deviations from unity in this range of q^2.

I. INTRODUCTION

WE have analyzed the published[1] Stanford differential cross sections for elastic electron-proton scattering in an attempt to answer two questions. First, does the Rosenbluth formula hold in this region? Second, what are the best values for the various form factors as deduced from these experiments, and what are the values of the errors, and the *correlated errors*, in these form factors?

The Stanford measurements, and other measurements[2] of electron-proton scattering, have recently been reviewed by Bishop,[3] and by Hand, Miller, and Wilson.[4] Our emphasis is somewhat different from theirs, and we also differ in some details of the statistical analysis. Specifically, other authors have concentrated on finding the best values of the form factors to fit the experimental data, while we are concentrating here on establishing the validity of the Rosenbluth formula in the region $7 \leq q^2 \leq 25\ \mathrm{F}^{-2}$ and on finding the errors in the form factors. For our purposes it is particularly convenient to deal with a large number of measurements from one laboratory, while others[3,4] are making very useful compilations and comparisons of results from different laboratories.

Of course, one can test any given theoretical expression for the form factors by a direct comparison between the differential cross section calculated with that expression and the experimental value. We[5,6] have recently followed this procedure. We obtained somewhat high, but not completely unreasonable, values of χ^2 for our theoretical expressions. This procedure[5,6] involves two independent theoretical assumptions: First, that the Rosenbluth formula is valid; and, second, that we are using appropriate expressions to calculate the form factors as functions of the four-momentum transfer q^2. In the rest of this paper we are examining only the first assumption, and are adopting no theoretical preconceptions as to the values that the form factors should have. Thus, the two approaches complement each other.

II. VALIDITY OF THE ROSENBLUTH FORMULA

If exchange of a single photon, or of an object of spin unity,[7] is the only mechanism for electron-proton scattering, then the Rosenbluth formula gives us the differential cross section σ as

$$W=\sigma/\sigma_{\mathrm{NS}}=G_E^2/(1+v)+vG_M^2/(1+v)+2G_M^2 v\tan^2(\tfrac{1}{2}\theta). \quad (1)$$

Here σ_{NS} is Hofstadter's "no structure" cross section,[8] and $v=q^2/4M^2$. (q^2 is defined to be positive in the physical region; M is the proton mass.) We are using the electric and magnetic form factors: $G_E=F_1-vKF_2$ and $G_M=F_1+KF_2$, where $K=1.793$. The Dirac and Pauli form factors F_1 and F_2 are normalized to unity for the proton for $q^2=0$. Then $G_E(0)=1$, while $G_M(0)=2.793$. We define $F_E=G_E$ and $F_M=G_M/2.793$ so that $F_E(0)=F_M(0)=1$.

Yount and Pine[2] and Drickey and Hand[2] have each confirmed the validity of the Rosenbluth formula to an accuracy of order 1% in the range $0.28 \leq q^2 \leq 1.6\ \mathrm{F}^{-2}$.

* Supported in part by the Office of Naval Research.
† AVCO Visiting Professor.
1 F. Bumiller, M. Croissiaux, E. Dally, and R. Hofstadter, Phys. Rev. **124**, 1623 (1961). Denoted by BCDH.
2 R. M. Littauer, H. F. Schopper, and R. R. Wilson, Phys. Rev. Letters **7**, 141 (1961); P. Lehmann, R. Taylor, and Richard Wilson, Phys. Rev. **126**, 1183 (1962); P. Lehmann and R. Dudelzak, in *1962 International Conference on High-Energy Physics at CERN* (CERN, Scientific Information Service, Geneva, 1962), p. 194; D. Yount and J. Pine, Phys. Rev. **128**, 1842 (1962); K. Berkelman, R. M. Littauer, and J. Rouse, in *1962 International Conference on High-Energy Physics at CERN* (CERN, Scientific Information Service, Geneva, 1962); D. J. Drickey and L. N. Hand, Phys. Rev. Letters **9**, 521 (1962); T. Janssens, R. Hofstadter, E. B. Hughes, and M. Yearian, Bull. Am. Phys. Soc. **7**, 488 (1962); P. A. M. Gram and E. B. Dally, *ibid.* **7**, 489 (1962).
3 G. R. Bishop, in *1962 International Conference on High-Energy Physics at CERN* (CERN, Scientific Information Service, Geneva, 1962), p. 753.
4 L. N. Hand, D. G. Miller, and Richard Wilson, Rev. Mod. Phys. **35**, 332 (1963). Denoted by HMW.
5 J. S. Levinger, Nuovo Cimento **26**, 813 (1962).
6 J. S. Levinger and M. W. Kirson, in *Proceedings of the Eastern Theoretical Physics Conference, 1962* (Gordon and Breach Publishers, Inc., New York, 1963).
7 S. Fubini, in *1962 International Conference on High-Energy Physics at CERN* (CERN, Scientific Information Service, Geneva, 1962), p. 767.
8 R. Herman and R. Hofstadter, *High Energy Electron Scattering Tables* (Stanford University Press, Stanford, California, 1960).

TABLE I. Data on electron-proton scattering.[a]

Electron energy in MeV	Scattering angle θ in deg	q^2 in F^{-2}	$W=\sigma/\sigma_{NS}$	% error	Shifted q_s^2	Shifted W_s
500	75	6.82	0.558	10.0	7	0.546
600	60	7.00	0.511	6.1	7	0.511
750	45	6.86	0.438	7.1	7	0.432
440	135	9.44	1.871	10.0	9	1.891
600	75	9.30	0.420	10.0	9	0.430
700	60	9.17	0.392	4.4	9	0.400
875	45	9.05	0.384	5.8	9	0.385
500	135	11.5	1.697	10.0	11.5	1.697
600	90	11.3	0.536	7.5	11.5	0.529
700	75	12.0	0.359	6.6	11.5	0.372
800	60	11.5	0.304	3.9	11.5	0.304
550	135	13.3	1.657	10.0	13	1.677
750	75	13.4	0.317	5.8	13	0.327
850	60	12.8	0.284	7.1	13	0.280
600	135	15.1	1.536	10.8	15	1.544
800	75	14.9	0.344	6.6	15	0.343
900	60	14.1	0.264	4.2	15	0.253
650	135	17.0	1.239	5.0	16.5	1.283
750	90	16.1	0.430	7.7	16.5	0.417
850	75	16.5	0.264	7.1	16.5	0.264
1000	60	16.8	0.236	8.5	16.5	0.237
675	135	17.9	1.194	7.3	18	1.187
800	90	17.7	0.317	4.3	18	0.310
900	75	18.0	0.275	3.6	18	0.275
750	145	21.4	1.568	6.1	21.5	1.567
775	135	21.8	0.871	17.3	21.5	0.894
900	90	21.2	0.280	5.5	21.5	0.274
835	145	24.9	1.437	5.8	24.9	1.437
850	135	24.9	0.830	6.6	24.9	0.830
975	95	24.9	0.252	8.0	24.9	0.252

[a] The first 3 columns and the fifth column are copied from Table I of reference 1. The fourth column gives the ratio W of the measured differential cross section to the "no structure" cross section. The last two columns give the value W_s for a common shifted value of q_s^2, the four-momentum transfer.

The former group measured the ratio of positron-proton to electron-proton scattering. The latter group analyzed their 4 quite accurate measurements of electron-proton scattering at different angles for $q^2=1.6$ F^{-2}, and showed that the Rosenbluth formula agreed with their angular distribution. Our analysis is like that of Drickey and Hand, applied to data at higher q^2.

Since G_E and G_M are each functions only of q^2, we want to fit Eq. (1) to data taken at the same q^2. We must then make small shifts in the published data[1] to obtain values of W at the same q^2, but at different scattering angles θ. We want the shifts to be very small so as not to introduce appreciable additional errors which would be hard to calculate. We have, therefore, looked for "natural accumulation points" where with small shifts we could have several measurements of W at the same q^2. The shifting was done using plots of W vs q^2 at fixed angle. The slope S' of a smooth curve was used for the shifting operation, but the measured point W was used without smoothing. That is, $W_s(q^2+\Delta) = W(q^2)+S'\Delta$. Here the value of the shift in q^2 is designated by Δ, and the subscript s shows that we have shifted.

Our procedure of making only small shifts (usually 0.2 F^{-2} or less) is different from that of HMW,[4] who shifted all the Stanford data to even integral values of q^2. Our procedure also differs from the smoothing procedure of BCDH.[1]

The data used are given in Table I. We give the experimental values of q^2, W, and the percent error, and also the values of the shifted W_s with the same percent error. We assume that the errors are random. We then make a least-squares fit of Eq. (1) to the shifted data for each q^2 value, with results given in Table II. We define

$$x=\tan^2(\tfrac{1}{2}\theta)+\frac{1}{2(1+v)}. \tag{2}$$

Then Eq. (1) becomes

$$W=I+Sx, \tag{3}$$

where the intercept

$$I=G_E^2/(1+v), \tag{4}$$

and the slope

$$S=2vG_M^2=2v(2.793)^2F_M^2. \tag{5}$$

The χ^2 values given in the third column are seen by inspection to be generally satisfactory. We can obtain better statistics by treating each set of W_s at a given q^2 as independent of the other sets of W_s. We can then add together all the χ^2 values, giving a total of 14.0. We have fitted 30 data points, using 2 adjustable parameters for each of the 9 values of q^2. Thus we have 12 degrees of freedom, so the χ^2 value of 14.0 is quite reasonable. (Note that HMW[4] quote a χ^2 value of 13 for 6 points at $q^2=10$ F^{-2}.) We conclude that the BCDH data are consistent with the validity of the Rosenbluth formula in the region $7\leq q^2\leq 25$ F^{-2} and roughly $60°\leq\theta\leq 145°$. (The limits on θ depend on the q^2 value; the lower limit varies from 45° for the lowest q^2 to 95° for the highest q^2 considered.) Deviations from the Rosenbluth formula which would cause a failure of linearity between W and $\tan^2(\frac{1}{2}\theta)$ in this range should be less than a typical error of a value of W, namely, about 5%. [We cannot make a precise statistical statement without making a

TABLE II. Least-squares fit to data of Table I.[a]

q_s^2 in F^{-2}	No. of points	χ^2	Slope S	Error $(\Delta S/S)$	Intercept I	Error $(\Delta I/I)$	Correlation coefficient
7	3	0.8	0.300	49.3%	0.253	45.5%	−0.98
9	4	2.0	0.256	13.2%	0.205	14.0%	−0.86
11.5	4	1.4	0.269	10.3%	0.098	27.1%	−0.92
13	3	0.4	0.250	12.1%	0.079	40.1%	−0.90
15	3	1.1	0.244	11.9%	0.072	36.1%	−0.95
16.5	4	4.0	0.194	6.2%	0.088	21.6%	−0.76
18	3	4.1	0.162	9.9%	0.101	19.9%	−0.93
21.5	3	0.03	0.141	7.1%	0.075	29.2%	−0.79
24.9	3	0.2	0.131	6.1%	0.040	66.8%	−0.74

[a] A least-squares fit to the data of Table I at fixed q^2 is made using Eq. (2), plotting W vs $x=\tan^2(\frac{1}{2}\theta)+\frac{1}{2(1+q^2/4M^2)}$. The correlation coefficient is $\langle\Delta S\Delta I\rangle/(\Delta S)(\Delta I)$.

TABLE III. Proton form factors.[a]

q^2 in F^{-2}	F_E	$\Delta F_E/F_E$	F_M	$\Delta F_M/F_M$	F_1	$\Delta F_1/F_1$	F_2	$\Delta F_2/F_2$	Corr. coeff.	F_E/F_M	$\Delta(F_E/F_M)$
7	0.522	22.8%	0.498	24.7%	0.585	14.7%	0.450	53.0%	−0.98	1.048	0.49
9	0.475	7.0%	0.405	6.6%	0.535	4.5%	0.332	16.0%	−0.87	1.173	0.15
11.5	0.333	13.6%	0.367	5.1%	0.411	9.5%	0.343	13.8%	−0.91	0.906	0.17
13	0.300	20.1%	0.333	6.0%	0.379	12.1%	0.308	18.0%	−0.96	0.899	0.23
15	0.290	18.0%	0.306	5.9%	0.371	11.9%	0.270	18.0%	−0.84	0.946	0.22
16.5	0.323	10.8%	0.260	3.1%	0.385	7.0%	0.191	13.5%	−0.93	1.240	0.17
18	0.347	9.9%	0.228	5.0%	0.395	6.1%	0.135	22.3%	−0.94	1.520	0.22
21.5	0.305	14.6%	0.195	3.5%	0.351	9.4%	0.108	25.4%	−0.99	1.564	0.27
24.9	0.227	33.4%	0.175	3.1%	0.284	20.1%	0.116	33.3%	−0.99	1.293	0.46

[a] The proton form factors, errors, and correlated errors. The last two columns give the ratio F_E/F_M and its statistical error. The correlation coefficient for F_E and F_M has the same value as the correlation coefficient for I and S given in Table II. The correlation coefficient for F_1 and F_2 is $\langle \Delta F_1 \Delta F_2 \rangle / (\Delta F_1)(\Delta F_2)$.

specific hypothesis concerning the shape $f(\theta)$ of the nonlinear terms.] Our conclusions agree with the estimate of Drell and Fubini[9] that two-photon exchange is expected to contribute less than 1% to the cross section in this energy region.

Note the large negative correlation coefficient between the error in the intercept I and the error in the slope S.

III. THE FORM FACTORS

From the values of the slope and intercept given in Table II, we determine the values of the electric and magnetic form factors F_E and F_M. These values are given in Table III. We shall not attempt a detailed comparison with the form factor values of BCDH or of HMW, particularly since our values, in general, do not lie at their values of q^2. A visual comparison is given in Fig. 1, where we plot all 3 sets of values of $G_E(q)^2$ and $G_M(q^2)$. The smooth curves are taken from reference 5, Eqs. (7) and (8). This fit to G_E and G_M assumed measured masses for ρ and ω, with adjustable coupling parameters, and also a "soft core" at roughly 1200 MeV as well as small hard cores. Other curves of the Clementel-Villi form have been used by several other authors to obtain similar good fits to the BCDH data.

We also compare form factors for $q^2 = 18$ F^{-2} in Table IV. We see that the values of the different form factors, as determined by 3 different analyses of the same experimental data, are in good agreement with each other—that is, well within the quoted errors. (Much of the difference between our form factors and those of HMW is due to their approximately 5% increase in the BCDH cross sections to account for Tsai's radiative corrections, while we have used the BCDH published cross sections.) Our interest in this paper lies mainly in the errors in our present knowledge of the form factors. Table IV shows that our estimate of the errors is somewhat greater than that given in the other analyses. BCDH have already remarked on the importance of correlated errors between F_1 and F_2.

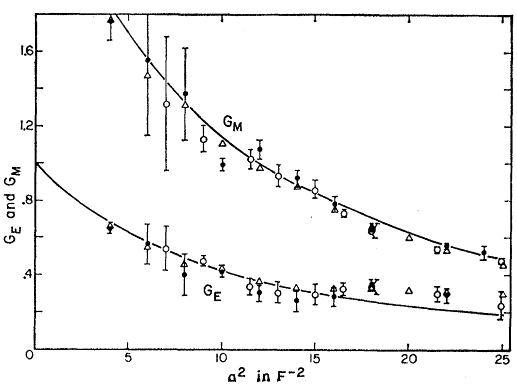

FIG. 1. The form factors G_M (above) and G_E (below) plotted against momentum transfer q^2 in F^{-2}. The triangles are from BCDH, reference 1; the solid circles and error bars from HMW, reference 4; the open circles and error bars are from Table III of this paper. The smooth curves are taken from Eqs. (7) and (8) of reference 5.

TABLE IV. Comparison of analyses of proton form factors, $q^2 = 18F^{-2}$.

Form factor	BCDH[a]	HMW[b]	From Table III
F_E	0.327	0.350±0.024	0.347±0.034
F_M	0.236	0.237±0.006	0.228±0.011
F_1	0.382±0.012	0.402	0.395±0.024
F_2	0.154±0.010	0.145	0.135±0.030

[a] See reference 1; note that the errors quoted here (0.012 and 0.010) may, according to BCDH, be approximately twice as large as given, provided they are correlated.
[b] See reference 4. References a and b do not give errors for $F_{E,M}$ and for $F_{1,2}$, respectively.

We determine the errors of the various form factors from the errors ΔS and ΔI and correlation coefficient r from Table II. From Eqs. (4) and (5), we have the simple results: $\Delta F_E/F_E = \frac{1}{2}\Delta I/I$; $\Delta F_M/F_M = \frac{1}{2}\Delta S/S$; and the correlation coefficient r between F_E and F_M equals the correlation coefficient between I and S. We must take the correlation of errors into account in determining the errors in F_1 and F_2. We find

$$\Delta F_1 = (1+v)^{-1}[(\Delta G_E)^2 + v^2(\Delta G_M)^2 + 2vr(\Delta G_E)(\Delta G_M)]^{1/2}, \quad (6)$$

[9] See S. D. Drell and F. Zachariasen, *Electromagnetic Structure of Nucleons* (Oxford University Press, Oxford, 1961).

and

$$\Delta F_2 = (1.793)^{-1}(1+v)^{-1}[(\Delta G_M)^2 + (\Delta G_E)^2 - 2r(\Delta G_E)(\Delta G_M)]^{1/2}. \quad (7)$$

For $q^2 \approx 10\text{F}^{-2}$, $\Delta G_M \approx \Delta G_E$ from Table III, $r \approx -1.0$ and $v = q^2/90 \ll 1$. Equation (6) gives $\Delta F_1 < \Delta G_E$ while Eq. (7) gives $\Delta F_2 > \Delta G_M$. For $q^2 \approx 25\ \text{F}^{-2}$, ΔF_M is much smaller than any of the other 3 errors, ΔF_1, ΔF_2, or ΔF_E.

Bishop[3] and HMW[4] have recently discussed the approximate equality of the form factors F_E and F_M. (A historical note: It was observed[10] in 1960 that F_2 decreased more rapidly with q^2 than does F_1. Then new form factors F_E and F_M were defined, with the result that F_E decreases more rapidly than F_1 and F_M decreases less rapidly than F_2. This might make it possible to reunite the split form factors.) This equality does seem valid[2] in the range $0 < q^2 \leq 2.98\ \text{F}^{-2}$ and by definition must hold precisely for $q^2 = 0$. However, according to our analysis in Table III, they appear unequal in the range $7 \leq q^2 \leq 25\ \text{F}^{-2}$. The last two columns give the ratio F_E/F_M, with its standard error. (In calculating the standard error of F_E/F_M it is important to use the strong negative correlation between the errors in F_E and F_M.) The four highest values of q^2 (16.5, 18, 21.5, and 24.9 F^{-2}) give a ratio F_E/F_M which is larger than unity by an amount which is statistically significant, namely, 1.4, 2.4, 2.1, and 0.6 standard errors, respectively. In no case is F_E/F_M significantly less than unity.

[10] R. Hofstadter, *Nuclear and Nucleon Structure*, (W. A. Benjamin, Inc., New York, 1963), p. 61–63.

ACKNOWLEDGMENTS

We are very grateful to many associates in the Laboratory of Nuclear Studies for helpful discussions: K. Berkelman, H. A. Bethe, J. Orear, R. F. Peierls, and R. R. Wilson. We wish to thank Richard Wilson for sending us preprints and useful comments.

PHYSICAL REVIEW VOLUME 130, NUMBER 4 15 MAY 1963

p-p Higher Partial Waves and the 3-Meson Model

WILLIAM RAMSAY*
University of California, San Diego, La Jolla, California
(Received 30 November 1962)

The pion, the ρ–ω vector pair, and a scalar 2π resonance are taken to explain the long-range nuclear forces. The calculated values are fitted to *p–p* phase-shift solutions for various values of the unknown mass and coupling constants.

1. INTRODUCTION

If the one-meson exchange forces from the pion, the ρ and ω vector particles, and a scalar meson or resonance are combined with unitarity, one can get satisfactory fits to the nucleon-nucleon scattering phase shifts.[1,2] Bryan, Dismukes, and Ramsay[1] have done a *p-p* calculation of this kind, solving the Schrödinger equation by iteration, while Scotti and Wong[2] have fitted *p-p* and *n-p* data with a dispersion-theoretical approach. These fits have been made to the lower partial waves or to the experimental observables; it is of some interest, however, to examine the model's predictions for the higher partial waves. True, the higher angular momentum waves are comparatively ill-determined by a phase-shift analysis; on the other hand, they depend only on the relatively long-range forces and should therefore be especially amenable to a Born approximation, that is, first-order calculation. In terms of elastic unitarity, which relates the scattering matrix to its absolute value squared, the high-l waves are quite small, therefore leading one to hope that their squares can be neglected. If the model *does* represent the higher waves accurately, it implies both that the inelastic unitarity contributions are small, and that forces of the range of $\frac{1}{2}$ or $\frac{1}{3}$ of a pion Compton wavelength are adequately represented by the π, ρ, ω, and scalar mesons. In this connection, it should be noted that although there is no experimental agreement on indications that a scalar resonance exists,[1] such a "particle" may be a reasonable method of approximating the cross-channel S-wave contributions from two-pion exchange, just as the ρ meson provides the main part of the cross-channel P-wave forces. Whether such an approximation is sensible, of course, depends on the unknown details of the dynamics. The approximation of cuts by poles is a standard technique in S-matrix theory and may or may not work. The hypothesis of a scalar meson is presented here as just that, a hypothesis. A successful use of the scalar meson in phenomenology would tend, of course, to indicate that it is a useful approximation for certain pion-exchange forces.

2. ANALYSIS

In this note, the parameters of the three-meson model[1] are adjusted to fit the Livermore group phase-shift

* National Science Foundation Postdoctoral Fellow.

[1] R. Bryan, C. Dismukes, and W. Ramsay, Nucl. Phys. (to be published).

[2] A. Scotti and D. Wong, Phys. Rev. Letters **10**, 142 (1963).