Gravitational Field Energy and g_{00}*

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It is shown that one of the family of "generalized energy density" definitions being investigated by Komar, one for which the generalized energy density is made positive definite by use of minimal surfaces, leads to a total "generalized energy" which (a) is undefinable for closed universes, (b) is not conserved for some asymptotically flat spaces, and (c) is not the correct total energy for the Oppenheimer-Snyder collapsing star metric where the metric in a neighborhood of infinity is identically the Schwarzschild metric.

I. INTRODUCTION

THE total mass, or energy, of a gravitational system is defined unambiguously whenever space-time is asymptotically flat in the sense that coordinates may be introduced for which $g_{\mu\nu}-\eta_{\mu\nu}$ and all its partial derivatives vanish as fast as $1/r$ on each hypersurface $x^0=\text{const}$. The basic argument which leads to this conclusion was first given by Einstein[1] and Klein,[2] and has been reformulated by Trautman[3] with the necessary careful attention to the way in which $g_{\mu\nu}$ approaches $\eta_{\mu\nu}$ at infinity. Trautman assumes that, on each hypersurface $x^0=\text{const}$, the asymptotic conditions

$$g_{\mu\nu}-\eta_{\mu\nu}=O(1/r), \tag{1}$$

$$g_{\mu\nu,\alpha}=O(1/r^2) \tag{2}$$

hold, where $r^2=(x^1)^2+(x^2)^2+(x^3)^2$, and that these conditions are preserved by (linear) Lorentz transformations. It is next important to realize that conditions (2), holding for fixed x^0, do not exclude radiation, but only require that r be taken larger than the radius of the wave front.[4] In fact the existence of coordinates satisfying Eq. (1) already implies the existence of coordinates for which $g_{\mu\nu,\alpha}$ vanishes faster than $1/r$ at fixed x^0, and the discussion of *total* energy or momentum (including a contribution from all the radiation) can be given on the basis of Eq. (1) alone.[4] An important tool in all recent discussions of energy are the surface integral forms for total energy and momentum introduced by von Freud.[5] Under conditions (1) and (2), for instance, only linear terms survive in these surface integrals as $r\to\infty$ and it is easily seen that the Einstein-von Freud,[5] Landau-Lifshitz,[6] and Papapetrou-Gupta[7,8] formulas all define the same total energy-momentum vector. Infinitely many other "energy" formulas with a formal similarity to those just mentioned can be written down, as Goldberg[9] has shown. The correct energy can be defined by *Gedanken* experiments in asymptotically flat spaces[4] and, under the (unnecessarily stringent) coordinate conditions (1) and (2), can be computed from the surface integral formulas mentioned above, and many others.

If one wishes to exclude the contribution to the total energy from a pulse of outgoing radiation, to obtain a residual total energy, then it is appropriate to think of the energy not as a function of the state of the system on a space-like hypersurface $x^0=\text{const}$, but as a function of a null (or asymptotically null) hypersurface of constant retarded time $u\equiv x^0-r$. An excellent discussion of energy from this viewpoint has been given by Trautman.[3] Since the wave-front theorem[4] should imply that all asymptotically flat metrics have only outgoing radiation in the future (and only incoming radiation in the past), Trautman's coordinate conditions should be no more restrictive than the existence of coordinates where $g_{\mu\nu}-\eta_{\mu\nu}$ and its derivatives are $O(1/r)$.

If radiation is emitted not in the form of massless fields (electromagnetism, gravity) but as particles or in fields with a nonzero rest mass, then Trautman's null cone method will not separate the emitter from the emission. In this case one returns to $x^0=\text{const}$ hypersurfaces, but in an S-matrix limit, $x^0\to\pm\infty$, where the various pieces of the system are either bound together or widely separated. Then the separated bound systems can each be assigned an energy-momentum vector since interaction energies among them should asymptotically vanish. This viewpoint has not been explored in detail, but it is known that the approach to flatness can be a subtle limit.[10,11]

In none of the foregoing summary was any meaningful idea of energy density implied, nor was a total energy

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[3] A. Trautman, *Lectures on General Relativity* (mimeographed notes) (King's College, London, 1958); and *Gravitation: An Introduction to Current Research*, edited by L. Witten (John Wiley & Sons, Inc., New York, 1962), Chap. 8.
[4] R. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. **122**, 997 (1961); see also the review by C. W. Misner, in Proceedings of the Warsaw-Jabłonna Conference on Relativistic Theories of Gravitation, 1962 [PWN-Pergamon Press, New York, (to be published)].
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defined for a closed universe, or for any other system which could not be inserted into a region of previously flat empty space, nor was the total energy-momentum idea related formally to the translation symmetry of the asymptotically flat boundary conditions. These topics have been a major concern in the work of Komar[12–14] and that of Bergmann[15] and Møller[16,17] from which Komar starts. Møller's energy depends not only on the metric, but also on the choice of a family of hypersurfaces ($x^0=$ const) and a congruence of curves ($x^i=$ const, $i=1, 2, 3$). Komar's modification requires only one of these additional elements. The curve family may be defined by its tangent vector field ξ^μ, in terms of which Komar's "generalized energy" formulas are written (units: $16\pi\gamma=1=c$)

$$m_\xi=\int_V 2(\xi^{\mu;\nu}-\xi^{\nu;\mu})_{;\nu}(-g)^{1/2}d\sigma_\mu. \tag{3}$$

If it is desired instead to use a family of hypersurfaces $t(x^\mu)=$ const, then Komar takes

$$\xi_\mu=(t_{;\alpha}t^{;\alpha})^{-1}t_{;\mu}. \tag{4}$$

Pirani[18] has attempted an interpretation of Eq. (3) as giving a "relative energy" depending on the 4-velocities ξ^μ of a network of observers. Komar has been investigating ways[13] in which ξ^μ might be related to the time-translation symmetry of the flat space boundary conditions (and the other symmetries, to give momentum and angular momentum integrals). One particular choice[14] where ξ^μ is orthogonal [Eq. (4)] to a family of minimal surfaces gives the desirable result $m>0$. It is this particular method of defining energy that will be considered in the present paper. We find that it is defective in that the resulting "energy" can be incorrect even for space-times which reduce to the exterior Schwarzschild solution for r larger than some r_0; and that for other physically reasonable solutions, including Brill's time-symmetric wave pulse,[19] it is not conserved. Thus positive definiteness of the total energy of gravitational systems is not established except in some special cases.[19–21] The difficulty with the Møller-Komar generalized energy formulas as *energy* formulas is that in surface integral form they involve predominantly g^{00}, as we will see in Sec. III, and g^{00} does not possess the asymptotic coordinate invariance properties of the Newtonian field[4] h^T formed from the g_{ij} which appears in the standard surface integrals [cf. Eq. (23) and reference 4]. Thus, exceptional care in choosing t or ξ^μ is required before Eq. (3) provides a method of computing the total energy. Of course, one of the motivations for investigating the Møller-Komar energies is the desire to avoid the coordinate conditions (1), and thus, for instance, to be able to define the energy of a closed universe. This last topic is discussed briefly in the next section.

II. ENERGY OF CLOSED UNIVERSES

A closed surface cannot be covered by a single set of singularity-free coordinates; thus to define an integral over closed surfaces always involves considering changes of coordinates and naturally suggests the use of tensor quantities. The various pseudotensor definitions of energy could not, therefore, be used to define the total energy of a closed universe, although the possibility of reducing the energy to a surface integral form suggested that closed universes should be assigned total energy zero. The Møller-Komar energy formulas allowed this argument to be carried through,[12,16] since Eq. (3) implies that

$$m_\xi=\oint_{\partial V} 2(\xi^{\mu;\nu}-\xi^{\nu;\mu})(-g)^{1/2}dS_{\mu\nu} \tag{5}$$

by Stokes' theorem. If the hypersurface V in Eq. (3) is closed, i.e., if $\partial V\equiv bdy V=0$, then from Eq. (5) evidently $m_\xi=0$. Consequently, *no restriction on the vector field* ξ^μ *which implies that* $m_\xi>0$ *in Eq. (3) can be satisfied in a closed universe.* In particular, then, the assumption that ξ^μ is orthogonal via Eq. (4) to a family of minimal surfaces, which Komar has shown[14] gives $m_\xi>0$ in Eq. (3) (apart from the case of flat space) is impossible in a closed universe. More precisely, Komar has shown[22] that there exist no families of closed space-like minimal hypersurfaces in nonflat space-times satisfying

$$R_{**}\equiv n^\mu R_{\mu\nu}n^\nu \geqslant 0. \tag{6}$$

(Here n^μ is the unit normal to a space-like surface.) This last condition, via the Einstein equations, is always satisfied; it states that negative pressures may not exceed, in magnitude, one-third of the matter energy density [cf. Eq. (26)].

We have reviewed these known facts to emphasize that the following three properties for an energy formula are incompatible: (a) a surface integral or Gaussian flux formulation, (b) general covariance sufficient for applicability to closed spaces, (c) positive definiteness.

III. CONSERVATION OF ENERGY

All the Komar energies satisfy a formal conservation law; the integrand in Eq. (3) has a vanishing divergence.

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[14] A. Komar, Phys. Rev. **129**, 1873 (1963).
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[19] D. Brill, Ann. Phys. (N. Y.) 7, 466 (1959).
[20] H. Araki, Ann. Phys. (N. Y.) 7, 456 (1959).
[21] R. Arnowitt, S. Deser, and C. W. Misner, Ann. Phys. (N. Y.) **11**, 116 (1960).
[22] Arthur B. Komar, Ph.D. thesis, Princeton University, 1956, (unpublished), Eq. (3.21)ff.

The actual independence of m_ξ from the hypersurface V will, in the case of asymptotically flat spaces, depend on whether corresponding integrals at spatial infinity vanish. Møller found that his energy proposal was not Lorentz invariant due to the fact that such ("Poynting flux") integrals did not vanish properly,[17] and the questions of conservation and coordinate invariance are, of course, closely related.[1,2] Instead of evaluating integrals on time-like hypersurfaces to check the conservation of m_ξ, we will reduce the integral (3) to a form in which its time derivative can be computed in some examples.

Let us restrict ourselves to the subclass of Komar's generalized energies defined in terms of a preferred family of space-like hypersurfaces $t=\text{const}$. Then we may for convenience choose as the time coordinate x^0 just $x^0=t$, so

$$t_{;\alpha}t^{;\alpha}=(\text{grad}x^0)^2=g^{00}\equiv -1/N^2, \tag{7}$$

where we introduce $-N^{-2}$ as a convenient notational replacement for g^{00}. By Eq. (4) we find

$$\xi_\mu=-N^2(\text{grad}x^0)=-N^2(1;0,0,0). \tag{8}$$

Forming the integral (3) for a $t=\text{const}$ surface gives

$$m_t=\int 2[(\xi^{0;k}-\xi^{k;0})(-g)^{1/2}]_{,k}d^3x. \tag{9}$$

By introducing the contravariant components ${}^3g^{ij}$ of the metric g_{ij} induced on the hypersurface $x^0=\text{const}$, namely,

$${}^3g^{ij}=g^{ij}-g^{0i}g^{0j}(g^{00})^{-1} \tag{10}$$

and by using the identity

$$(-g)^{1/2}=N({}^3g)^{1/2}, \tag{11}$$

where ${}^3g=\det g_{ij}$, we find that (8) inserted in (9) reduces to

$$m_t=\int 4[{}^3g^{kl}({}^3g)^{1/2}N_{,k}]_{,l}d^3x$$
$$=\int 4N_{|k}{}^{|k}({}^3g)^{1/2}d^3x. \tag{12}$$

In the last line here, there appear covariant derivatives with respect to the spatial metric[23] g_{ij}. Now the Einstein equations do not specify the time dependence of N [ADM[23] 7–(3.15)]; this is determined by the coordinate condition defining t. The Einstein equation for the time derivative of the trace K of the second fundamental form K_{ij} of a $t=\text{const}$ hypersurface is

$$\partial K/\partial t=-N^{|i}{}_{|i}+N^iK_{,i}+N(K_{ij}K^{ij}+R_{**}), \tag{13}$$

where $N_i=g_{0i}$ and indices are raised with ${}^3g^{ij}$. [This equation is easily derived from the form of the Einstein equations given in ADM 7–(3.15), using $\pi^{ij}=-({}^3g)^{1/2}(K^{ij}-{}^3g^{ij}K)$. It has been given by Peres,[24] and, in nearly this form, by Komar.[22]] This formula allows us to rewrite Eq. (12) as

$$m_t=4\int [N(K_{ij}K^{ij}+R_{**}) +N^iK_{,i}-(\partial K/\partial t)]({}^3g)^{1/2}d^3x. \tag{14}$$

When we take as a condition to determine the function t the requirement that each $t=\text{const}$ hypersurface be a minimal surface, i.e., require

$$K=0, \tag{15}$$

then m_t is clearly positive [cf. Eq. (6)]:

$$m_t=4\int (K_{ij}K^{ij}+R_{**})N({}^3g)^{1/2}d^3x. \tag{16}$$

Thus we have rederived Komar's result[14] in a different notation.[25] This last equation will also allow us to compute the time derivatives of m_t in some examples.

The simplest examples to work with are empty space-times, so we set $R_{**}=0$ in Eq. (16). As an additional simplifying assumption we will require that the hypersurface $t=0$ be a hypersurface of time symmetry, i.e., that

$$K_{ij}|_{t=0}=0 \tag{17}$$

These assumptions evidently imply, from Eq. (16), that

$$m_t|_{t=0}=0. \tag{18}$$

This shows that m_t, defined using a family of minimal hypersurfaces, is not the correct total energy for the Brill time-symmetric gravitational wave pulses,[19] whose total energy is known to be well defined and strictly positive. When we inquire how Eq. (18) is compatible with Komar's correct proof[14] that $m_t=0$ implies that space is flat, we find that the proof of flatness[26] involves not only $K_{ij}|_0=0$, but also $(\partial K_{ij}/\partial t)_0=0$, i.e., that m_t both vanish for $t=0$ and be conserved. Thus, m_t cannot be conserved for the nonflat Brill time-symmetric waves, or for any nonflat empty space-time where $t=0$ is a surface of time-inversion symmetry.

Let us see the failure of m_t to be conserved in empty, time-symmetric, space-times in a little more detail. From Eq. (16) we readily compute, using $R_{**}=0$ and Eq. (17), that $(\partial m_t/\partial t)_0=0$ and

$$\left(\frac{\partial^2 m_t}{\partial t^2}_{t=0}\right)=4\left[\int \dot{K}_{ij}\dot{K}_{lm}\,{}^3g^{il}\,{}^3g^{jm}N({}^3g)^{1/2}d^3x\right]_{t=0},$$

[23] All (3+1)-dimensional notations follow R. Arnowitt, S. Deser, and C. W. Misner in *Gravitation: An Introduction to Current Research*, edited by L. Witten, (John Wiley & Sons, Inc., New York, 1962), Sec. 7-3.2 referred to as ADM.

[24] A. Peres, Bull. Res. Council Israel **8F**, 179 (1960), Eq. (46).

[25] This formula also establishes the relationship between Komar's work in reference 14 and that of Peres in Nuovo Cimento **26**, 53 (1962).

[26] This proof is contained in the next paragraph here, with only notational changes from Komar, reference 14.

where $\dot{K}_{ij}=\partial K_{ij}/\partial t$. From Eq. (13), taken at $t=0$, namely, $N^{|i}{}_{|i}=0$, we have $(N)_{t=0}=1$, and then the Einstein equation for $\dot{K}_{ij}$ [ADM 7-(3.15b)] gives

$$\dot{K}_{ij}|_{t=0}={}^3R_{ij}|_{t=0}, \qquad (20)$$

and, therefore,

$$(\partial^2 m_t/\partial t^2)_{t=0}=4\int {}^3R_{ij}\,{}^3R^{ij}({}^3g)^{1/2}d^3x\bigg|_{t=0}. \qquad (21)$$

Thus m_t can be conserved only if $({}^3R_{ij})_0=0$, i.e., only if (with an appropriate choice of spatial coordinates) both $(K_{ij})_0=0$ and $(g_{ij})_0=\delta_{ij}$, which implies that space-time is flat.

IV. GAUSSIAN FLUX INTEGRALS

One of the advantages of a surface integral formula for total energy is that it allows us to define the mass of the sun without first developing a theory of stellar structure and stellar evolution. For instance, the Landau-Lifshitz formula[6] which reads[27]

$$\begin{aligned} m&\equiv\int(-g)(T^{00}+t^{00})d^3x\\ &=\int[(-g)(g^{00}g^{ij}-g^{0i}g^{0j})]_{,ij}d^3x\\ &=\int-[{}^3g^{ij}({}^3g)]_{,ij}d^3x \end{aligned} \qquad (22)$$

gives the surface integral form

$$m=-\oint[{}^3g^{ij}({}^3g)]_{,j}dS_i, \qquad (23)$$

which depends explicitly only on the spatial components g_{ij} of the metric in the neighborhood of infinity. Implicitly of course, this formula depends also on the choice of coordinates, but the coordinate conditions (1) and (2) are more than sufficient to guarantee that Eq. (23) yields the correct total energy,[4] independent of the choice of coordinates within these asymptotically rectangular restrictions. An energy computation is, thus, possible without knowing the matter distribution T^{00}, or the metric $g_{\mu\nu}$, in the strong-field central regions because there are surface integral formulas where *both* the integrand *and* the coordinate conditions depend only on the asymptotic metric.

The Komar generalized energy formulas also provide surface integral forms, as we have seen in Eq. (5); or from Eq. (12) we have

$$m_t=4\oint N^{|i}({}^3g)^{1/2}dS_i. \qquad (24)$$

[27] A. Peres, Nuovo Cimento **15**, 351 (1960), Eq. (25)ff.

When the function t is chosen by the requirement that each hypersurface $t=$ const be a minimal surface, however, we will see that a computation of m_t requires more than the asymptotic metric, and can depend, in fact, on some details of the stellar interior. If the interior of a spherically symmetric star is assumed to be static, then, as Komar has shown, m_t is equal to the Schwarzschild constant m. As a different, but simple, description of the stellar interior, let us assume that it consists of pressureless fluid (dust) of uniform proper rest density, collapsing radially, maintaining its spherical symmetry. This is described by the metric of Oppenheimer and Snyder.[28] As a further simplification, we compute m_t only for the hypersurface of time inversion symmetry, $t=0$, at which instant the radial velocity of the matter is zero, and $K_{ij}=0$. To determine N we use the coordinate condition $K=0=\partial K/\partial t$ in Eq. (13), always now at $t=0$, to get

$$-N^{|i}{}_{|i}+NR_{**}=0. \qquad (25)$$

The Einstein equations give [cf. Eq. (6) and $g_{**}\equiv n^\mu g_{\mu\nu}n^\nu=-1$]

$$R_{**}=\tfrac{1}{2}(T_{**}+\tfrac{1}{2}T)=\tfrac{1}{4}(T_{**}+{}^3g^{ij}T_{ij})=\tfrac{1}{4}(\rho+3p) \qquad (26)$$

with the stress-energy tensor

$$T^{\mu\nu}=(\rho+p)u^\mu u^\nu+pg^{\mu\nu}, \qquad (27)$$

and the initial condition $u^\mu=n^\mu$. It is then already evident that for the same initial conditions on ρ, u^μ, g_{ij}, and $K_{ij}=0$, the collapsing star, $p=0$, and the static solution $p\neq0$ will give different functions N at $t=0$. We complete the computation of N to show that, not surprisingly, these different functions N, in fact, yield different values of m_t in Eq. (24).

The solution of Eq. (25) or $N^{|i}{}_{|i}=0$ in the exterior region where

$$\begin{aligned} dl^2&=[1-(m/8\pi r)]^{-1}dr^2+r^2d\Omega^2,\\ d\Omega^2&=d\theta^2+\sin^2\theta d\phi^2, \end{aligned} \qquad (28)$$

is

$$N=1-\epsilon+\epsilon[1-(m/8\pi r)]^{1/2} \qquad (29)$$

with the boundary condition $N=1$ for $r=\infty$. From the surface integral (24) we see that the constant

$$\epsilon=m_t/m \qquad (30)$$

is the total generalized energy m_t in units of the total energy m. The value of ϵ will be determined by matching Eq. (29) to a solution of Eq. (25) in the interior region. The metric here is the Friedman metric,[29] which at $t=0$

[28] J. R. Oppenheimer and H. Snyder, Phys. Rev. **56**, 455 (1939). For a clear geometrical presentation of this metric by matching together pieces of the Schwarzschild-Kruskal metric and the Friedman cosmological models, see D. L. Beckedorff, senior thesis, Mathematics Department, Princeton University, 1962 (unpublished). Only the initial conditions, not the complete solution, are actually used in the computations which follow, and we include a verification of the initial value equations.

[29] L. Landau and E. Lifshitz, reference 6, Secs. 11-13 and 11-14.

has the same g_{ij} as the interior Schwarzschild solution,[30] namely,

$$dl^2=4a_0^2[d\chi^2+\sin^2\chi d\Omega^2]. \tag{31}$$

Matching this metric at χ_0 to the exterior Schwarzschild metric (28) at

$$r=r_0\equiv 2a_0\sin\chi_0 \tag{32}$$

gives, as a matching condition,

$$m=16\pi a_0\sin^3\chi_0. \tag{33}$$

The Einstein (initial value) equation for T_{**} gives ${}^3R=T_{**}=\rho$ or[29] $\rho=6/(2a_0)^2$. Then by Eq. (26) we have $R_{**}=\frac{1}{4}\rho=3/(8a_0^2)$, and Eq. (25) for N becomes

$$-\frac{1}{\sin^2\chi}\frac{\partial}{\partial\chi}\sin^2\chi\frac{\partial N}{\partial\chi}+\tfrac{3}{2}N=0 \tag{34}$$

with the solution

$$N=N_0\frac{\sqrt{2}}{\sin\chi}\sinh\left(\frac{\chi}{\sqrt{2}}\right). \tag{35}$$

Since $N^{|i}{}_{|i}=0$ in the exterior region, the integral (24) can also be evaluated on the boundary surface (32) and gives, from Eq. (35),

$$\epsilon=m_t/m=2N_0(\sin\chi_0)^{-3}[\sin\chi_0\cosh(\chi_0/\sqrt{2}) -\sqrt{2}\cos\chi_0\sinh(\chi_0/\sqrt{2})]. \tag{36}$$

This serves as the matching condition on the normal derivative of N. Matching the values of N from Eqs. (35) and (29) at the boundary (32) gives another relation; upon eliminating N_0 between these two, we find

$$\epsilon(1-\epsilon+\epsilon\cos\chi_0) =\frac{2\sqrt{2}\sinh(\chi_0/\sqrt{2})}{\sin^3\chi_0}\left[\cosh\left(\frac{\chi_0}{\sqrt{2}}\right)-\sqrt{2}\sinh\left(\frac{\chi_0}{\sqrt{2}}\right)\cot\chi_0\right].$$

This equation allows $\epsilon=1$ (i.e., $m_t=m$) only in the limit $\chi_0\rightarrow 0$, which from

$$\sin^2\chi_0=m/8\pi r_0\equiv 2\gamma m/c^2r_0, \tag{37}$$

we see corresponds to the weak field (dilute matter) limit. For nonzero m/r_0 we see that the generalized energy m_t defined by minimal hypersurfaces, and the Schwarzschild mass m of the exterior metric, disagree at $t=0$.

In both Secs. III and IV the computations of m_t are based on the assumption that the initial minimal hypersurface, $t=0$, can be imbedded in a family of minimal surfaces $t=$const. This assumption is supported by the fact that Eq. (25), which is actually a linearized form of the equation for a minimal hypersurface, can be solved in each case. However, Bergmann and Komar[31] have pointed out that this by no means guarantees that the true nonlinear equation for the family of minimal hypersurfaces has a solution, even for small finite t. Thus, a possible alternative to the conclusions reached in Secs. III and IV (that m_t is not conserved and is not the total energy) is that m_t may not be defined at all.

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[30] C. Møller, *The Theory of Relativity* (Oxford University Press, New York, 1952), Sec. 124.

[31] P. G. Bergmann and A. Komar (private communication).