Inequivalent Hamiltonian Formalisms in Electrodynamics*

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In treating the electric dipole approximation of a charged particle undergoing self-interaction, it is shown how, in the cases of a charged free particle and a charged oscillator, the same equations of motion may be derived from two inequivalent Hamiltonian formalisms. The first formalism is the Dirac formalism of Schiller and Schwartz, which must be satisfied by all configuration space solutions generated by the total Hamiltonian. It is pointed out that in order to ensure that the configuration space solutions are consistent with the latter formalism, one must recognize that the constraints impose modifications on the commutation rules satisfied by averaged field variables. While the first formalism is general, the second formalism can be constructed only after one has solutions expressed in terms of a Fourier decomposition of the radiation field. The second formalism is an interaction picture formalism where the radiation field part of the total Hamiltonian is the generator of time translations. The latter formalism is used to give a new derivation of Kramers' integral for the zero-point level shift of an oscillator. We also discuss some asymptotic boundary conditions and comment on a recent paper by Sokolov and Lysov.

I. INTRODUCTION

In two previous papers[1] we have dealt with the equations of motion of a charged particle in an external radiation field and under the influence of an additional external force field. The electric dipole approximation[1,2] was made and self-interaction was taken account of in a standard fashion to put the equations of motion in structure-independent form. In paper I, solutions for the free particle and the harmonic oscillator were discussed. The aforementioned solutions were expressed in terms of solutions to the homogeneous particle equations of motion and the homogeneous wave equation (free-field solution). In I, we raised the question of the existence of constraints among the dynamical variables because the solutions discussed did not reproduce the canonical commutation rules. We also discussed certain ambiguities which arise in connection with solutions based on a plane wave decomposition of the free field.

In paper II, it was shown that the canonical scheme associated with the equations of motion and field equations mentioned above contains relations among the canonical variables. Instead of two canonical vector pairs, there are three pairs with two vector constraint equations between the variables. The constraints turned out to be what Dirac has classified as second-class.[3] It was pointed out how the canonical formalism may be replaced by the Dirac formalism—which reflects the fact that the constraints prevent the solutions from satisfying the full set of canonical commutation rules. Thus, paper II gives a theoretical basis for a structure-independent Hamiltonian formalism in the electric dipole approximation.

In the present note, we return to the discussion of the free-charged particle and the charged harmonic oscillator. We impose the Dirac formalism on the solutions and thereby uncover some previously unrealized effects of the constraints on commutators involving averages of field variables over the charge distribution.

The above remarks pertain to the case where one does not utilize Fourier decompositions of the free field. We will show that if one uses solutions for the dynamical variables which are based on a Fourier decomposition of the free field, and if the Fourier components are treated as oscillator variables, then such solutions are generated by the free-field part of the Hamiltonian. The total Hamiltonian may then be interpreted as an interaction picture Hamiltonian. This offers a Hamiltonian formalism which is different from and not canonically equivalent to that of II. This new formalism presents a suitable framework for the procedure followed by Sokolov and Tumanov[4] in calculating the Lamb shift for the oscillator. The latter situation contrasts with the Kramers,[2] Van Kampen,[5] Steinwedel[6] treatment where one makes an approximation based on embedding the solutions in a large sphere and which has the effect of reducing the Hamiltonian to a sum over decoupled oscillators. The Lamb shift may then be calculated by Kramers[2] technique of quantizing phase-shifted oscillators. However, in using the Sokolov-Tumanov technique, one need not subtract the free-particle shift, while in using the Kramers technique, one must.

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In a discussion of our results, we indicate some reasons for the existence of the different formalisms. We also comment on a recent paper by Sokolov and Lysov[7] and discuss its relation to our present results and those in I.

II. THE HAMILTONIAN AND THE DIRAC FORMALISM

In the section we write the Hamiltonian in terms of the canonical variables appearing in Secs. III of II. We also write down the Dirac commutation rules for those variables since they are the ones which the solutions must satisfy.

The Hamiltonian is (in units where $\hbar=c=1$)

$$\mathcal{H}=\tfrac{1}{2}m\mathbf{V}^2+U(\mathbf{R})+e\mathbf{V}\cdot\mathfrak{A}_1+\frac{1}{2}\int d^3x\,[\mathbf{E}_1^2+\mathbf{H}_1^2]+\mathbf{C}_1\cdot\mathbf{V}+\mathbf{C}_2\cdot\ddot{\mathbf{R}},\quad(1)$$

where $\mathbf{C}_1$ and $\mathbf{C}_2$ are constraints to be defined below, $U(\mathbf{R})$ is an external potential, and

$$\mathbf{E}_1(\mathbf{x},t)=-\partial_t\mathbf{A}_1(\mathbf{x},t)-\frac{e}{4\pi}T\frac{\ddot{\mathbf{R}}(t)}{r};\qquad \mathbf{H}_1(\mathbf{x},t)=\nabla\times\mathbf{A}_1(\mathbf{x},t).\quad(2)$$

$\mathbf{A}_1(\mathbf{x},t)$ is a transverse vector potential so that

$$\nabla\cdot\mathbf{A}_1=\nabla\cdot\mathbf{E}_1=0.\quad(3)$$

The letter T preceding any vector means "transverse part of." $\mathfrak{A}_1=\mathbf{A}_1(0,t)$ and represents an average over a point charge distribution. In fact, hereafter, all field variables averaged over a point-charge distribution are denoted by a boldfaced German capital.

The canonically conjugate pairs are

$$\begin{aligned}\mathbf{R}(t)&\leftrightarrow\mathbf{P}_R=m\mathbf{V}+e\mathfrak{A}_1,\\ \mathbf{V}(t)&\leftrightarrow\mathbf{P}_v=-e\mathfrak{Z}_1,\\ \mathbf{A}_1(\mathbf{x},t)&\leftrightarrow\mathbf{P}_{A_1}=-\mathbf{E}_1,\end{aligned}\quad(4)$$

where

$$\mathfrak{Z}_1=\mathbf{Z}_1(0,t)$$

and

$$\mathbf{Z}_1(\mathbf{x},t)=\frac{1}{4\pi}\int d^3x'\,T\frac{E_1(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|},\quad(5)$$

is a Hertz potential satisfying $\partial_t\mathbf{Z}_1=\mathbf{A}_1$. The coordinates are represented by the variables on the left of the double arrows in Eqs. (4), while the momenta are represented by the variables immediately on the right. $\mathbf{R}(t)$ is the particle position variable and $\mathbf{V}(t)$ is its velocity operator.

[7] A. A. Sokolov and B. A. Lysov, Phys. Rev. **128**, 2422 (1962).

The equation of motion for $\mathbf{R}(t)$ is

$$m\ddot{\mathbf{R}}+e\partial_t\mathfrak{A}_1=-\nabla U,\quad(6)$$

while $\mathbf{A}_1$ satisfies the field equation,

$$\Box\mathbf{A}_1=(\nabla^2-\partial_t^2)\mathbf{A}_1=\frac{e}{4\pi}T\left(\frac{1}{r}\frac{d^3\mathbf{R}}{dt^3}\right).\quad(7)$$

In all the above equations, self-interaction has been taken into account so that m is the observed mass and $\mathbf{A}_1$ plays the role of an external field.

The constraints $\mathbf{C}_1$ and $\mathbf{C}_2$ are

$$\mathbf{C}_1=\mathbf{P}_R-m\mathbf{V}-e\mathfrak{A}_1=0,\quad(8)$$

$$\mathbf{C}_2=\mathbf{P}_v+e\mathfrak{Z}_1.\quad(9)$$

The equal-time commutators between the canonical pairs are

$$[\mathbf{R}(t),\mathbf{P}_R(t)]=[\mathbf{V}(t),\mathbf{P}_v(t)]=i\mathbf{I},\quad(10)$$

$$[\mathbf{A}_1(x,t),\mathbf{P}_{A_1}(\mathbf{x}',t)]=i\left[\delta(\mathbf{x}-\mathbf{x}')\mathbf{I}-\frac{1}{4\pi}\nabla\nabla'\left(\frac{1}{|x-x'|}\right)\right]\quad(11)$$

all others being zero. Here $\mathbf{I}$ is the unit dyadic.

The equal-time commutator between the constraints is

$$[\mathbf{C}_1,\mathbf{C}_2]=-im_0\mathbf{I},\quad(12)$$

so that the constraints are second class.[3] $m_0=m-\delta m$ is the unrenormalized mass and $\delta m=\lim_{r\to0}e^2/(6\pi r)$ is the electromagnetic mass.

As developed in paper II, the modified variables in the Dirac formalism are

$$\begin{aligned}&\mathbf{R}^*=\mathbf{R}-\frac{1}{m_0}\mathbf{C}_2,\quad \mathbf{P}_R^*=\mathbf{P}_R,\\ &\mathbf{V}^*=\mathbf{V}+\frac{1}{m_0}\mathbf{C}_1,\quad \mathbf{P}_v^*=\mathbf{P}_v-\frac{m}{m_0}\mathbf{C}_2,\\ &\mathbf{A}_1^*=\mathbf{A}_1-\frac{1}{m_0}\mathfrak{G}(x)\cdot\mathbf{C}_1,\\ &\mathbf{P}_{A_1}^*=\mathbf{P}_{A_1}-\frac{e}{m_0}\left[\delta(\mathbf{x})+\frac{1}{4\pi}\nabla\nabla\frac{1}{r}\right]\cdot\mathbf{C}_2,\end{aligned}\quad(13)$$

where $\mathfrak{G}(\mathbf{x})=(e/4\pi)T(\mathbf{I}/r)$ with $\mathbf{I}$ being the unit dyadic, and the dot following the dyadic $\mathfrak{G}$ means contract $\mathfrak{G}$ with $\mathbf{C}_1$. The equal-time commutation rules satisfied by the modified variables are the same as those satisfied by the solutions of the equations of motion and the field equations. These commutation rules are

$$[\mathbf{R}^*,\mathbf{P}_R^*]=i\mathbf{I}, \qquad [\mathbf{R}^*,\mathbf{V}^*]=(i/m_0)\mathbf{I},$$

$$[\mathbf{R}^*,\mathbf{A}_1^*]=-(i/m_0)\mathfrak{G}, \quad [\mathbf{A}_1^*,\mathbf{P}_v^*]=i(m/m_0)\mathfrak{G},$$

$$[\mathbf{A}_1^*(\mathbf{x},t),\mathbf{P}_{A_1}^*(\mathbf{x}',t)]=i\left[\delta(\mathbf{x}-\mathbf{x}')\mathbf{I}-\frac{1}{4\pi}\nabla\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\right]+\frac{ei}{m_0}\mathfrak{G}(\mathbf{x})\cdot\left[\delta(\mathbf{x}')+\frac{1}{4\pi}\nabla'\nabla'\frac{1}{r'}\right], \tag{14}$$

$$[\mathbf{V}^*,\mathbf{P}_{A_1}^*]=-\frac{ei}{m_0}\left[\delta(\mathbf{x})\mathbf{I}+\frac{1}{4\pi}\nabla\nabla\frac{1}{r}\right], \quad [\mathbf{V}^*,\mathbf{P}_v^*]=-(\delta m/m_0)i\mathbf{I}.$$

Those equal-time commutators not appearing in Eqs. (14), such as for example commutators with $\mathbf{P}_R^{*\prime}$ are zero.

III. CONFIGURATION SPACE COMMUTATORS

The solution of Eq. (7) for $\mathbf{A}$ will depend on the boundary conditions imposed on the self-field. The discussion below is based on the following solution for $\mathbf{A}_1$:

$$\mathbf{A}_1(\mathbf{x},t)=\mathbf{A}(\mathbf{x},t)-\mathfrak{G}(\mathbf{x})\cdot\mathbf{V}(t), \tag{15}$$

where $\mathfrak{G}(\mathbf{x})$ has been defined below Eq. (13) and $\mathbf{A}(\mathbf{x},t)$ satisfies

$$\Box\mathbf{A}(x,t)=-eT\delta(x)\mathbf{V}(t). \tag{16}$$

For $\mathbf{A}(\mathbf{x},t)$ we write

$$\mathbf{A}(\mathbf{x},t)=\mathbf{A}_h(\mathbf{x},t)+\tfrac{1}{2}\mathfrak{G}(\mathbf{x})\cdot[\mathbf{V}(t+r)+\mathbf{V}(t-r)], \tag{17}$$

where $\mathbf{A}_h(x,t)$ is a solution of the homogeneous wave equation, $\Box\mathbf{A}_h=0$. In particular, we write $\mathbf{A}_h(x,t)$ in the form

$$\mathbf{A}_h(\mathbf{x},t)=\mathbf{A}_{0h}(\mathbf{x},t)+\tfrac{1}{2}\mathfrak{G}(\mathbf{x})\cdot[\mathbf{V}(t-r)-\mathbf{V}(t+r)], \tag{18}$$

where $\mathbf{A}_{0h}$ is an arbitrary solution of the homogeneous wave equation. Thus, we have used the Dirac separation in Eq. (17).[8] It then follows from Eqs. (15) and (17) that

$$\mathfrak{A}_1=\mathfrak{A}_h. \tag{19}$$

A. Oscillator

For the oscillator, $U=\frac{1}{2}K\mathbf{R}^2$, so that Eq. (6) becomes

$$\ddot{\mathbf{R}}+k_0^2\mathbf{R}=-(e/m)\partial_t\mathfrak{A}_h, \tag{20}$$

where $k_0^2=K/m$. We write the characteristic function of the homogeneous part of Eq. (20) in the form

$$D(\omega)=\omega^2+k_0^2, \tag{21}$$

and introduce the Green's function

$$G(\tau)=\epsilon(\tau)\frac{\sin k_0\tau}{2k_0}, \tag{22}$$

where $\tau=t-t'$, and

$$\epsilon(\tau)=1, \quad \tau>0; \qquad =-1, \quad \tau<0. \tag{23}$$

$G(t,t')$ satisfies

$$d_t^2G+k_0^2G=\delta(\tau). \tag{24}$$

The general solution of Eq. (20) will now be written

$$\mathbf{R}(t)=(2mk_0)^{-\frac{1}{2}}[\mathbf{R}_0e^{-k_0t}+\mathbf{R}_0^\dagger e^{ik_0t}]+\mathfrak{R}(t), \tag{25}$$

where

$$\mathfrak{R}(t)=-\frac{e}{2mk_0}\int_{-\infty}^{\infty}dt'\;\epsilon(t-t')\sin k_0(t-t')\partial_{t'}\mathfrak{A}_h(t'). \tag{26}$$

Note, however, that when Eq. (18) is taken account of, Eq. (25) becomes an integrodifferential equation. The solution of the latter equation will not concern us here. Our main concern is to find a configuration space representation of different time commutation rules consistent with Eqs. (14). As indicated below, Eq. (25) suffices for the latter purpose.

Now the average of a field function is a function of the particle variables. Therefore, we may question if we are properly taking account of the constraints if we assume that $\mathbf{A}_h(\mathbf{x},t)$ satisfies free-field commutation rules and simply interchange the order of integration and commutation in the calculation of such commutators as $[\mathfrak{R}(t),\mathbf{A}_h(\mathbf{x},t')]$ and $[\mathfrak{R}(t),\mathbf{A}(\mathbf{x},t')]$. In fact, if such an interchange were made, we would not be able to reproduce all of Eqs. (14). For example, the above-mentioned prescription leads to

$$e[\mathfrak{A}_h(t),\mathfrak{R}(t')]=\frac{ie^2}{3\pi m}[\delta(\tau)-\tfrac{1}{2}\epsilon(\tau)k_0\sin k_0\tau]\mathbf{I}. \tag{27}$$

Now

$$\delta m=\lim_{r\to 0}\frac{e^2}{6\pi r}=\frac{e^2}{3\pi^2}\int_0^\infty dk=\lim_{\tau\to 0}\frac{e^2}{3\pi}\delta(\tau), \tag{28}$$

so that $e[\mathfrak{A}_h(t),\mathfrak{R}(t)]=i\delta m/m$. However, from Eqs. (14), we may use either $[\mathbf{R}(t),\mathbf{P}(t)]=i$ or $[\mathbf{R}(t),\mathbf{A}(\mathbf{x},t)]=0$ [the latter commutator arising from Eq. (15)] to conclude that $[\mathfrak{A}_h(t),\mathbf{R}(t)]=i\delta m/m_0$. Moreover, Eq. (27) is consistent with

$$(d_{t'}^2+k_0^2)[\mathfrak{A}_h(t),\mathfrak{R}(t')]=-(e/m)[\mathfrak{A}_h(t),\partial_{t'}\mathfrak{A}_h(t')]$$

[8] P. A. M. Dirac, Proc. Roy. Soc. (London) A167, 148 (1938).

and

$$(d_{t'}^2+k_0^2)[\mathfrak{A}_h(t),\,\mathbf{R}(t')-\mathfrak{R}(t')]=0$$

so that we may write

$$e[\mathfrak{A}_h(t),\mathbf{R}(t')]=i\delta m\left[\frac{1}{m_0}-\frac{1}{m}\right]\cos k_0\tau\mathbf{I}+e[\mathfrak{A}_h(t),\mathfrak{R}(t')]. \quad (29)$$

A term proportional to $\sin k_0\tau$ is excluded from Eq. (29) because that would lead to $[\partial_t\mathfrak{A}_h(t),\mathbf{R}(t)]\neq 0$, while $e\partial_t\mathfrak{A}=e\partial_t\mathfrak{A}_h+\delta m\mathbf{R}$, Eq. (20), and $[\partial_t\mathfrak{A}(t),\mathbf{R}(t)]=0$ imply that $[\partial_t\mathfrak{A}_h(t),\mathbf{R}(t)]=0$. By utilizing Eq. (29) in the commutation rule $[\mathbf{R}(t),\dot{\mathbf{R}}(t)]=i/m_0$, we may calculate $[\mathbf{R}_0,\mathbf{R}_0^\dagger]$. The latter commutator enables us to calculate $[\mathbf{R}(t),\,\mathbf{V}(t'\pm r)]$. $[\mathbf{A}_h(\mathbf{x},t),\mathbf{R}(t')]$ is calculated in the same fashion as Eqs. (27) and (29) consistent with the requirement that $\Box[\mathbf{A}_h(\mathbf{x,t}),R(t')]=0$ and $\lim_{r\to 0}[\mathbf{A}_h(\mathbf{x},t),\mathbf{R}(t')]=[\mathfrak{A}_h(t),\mathbf{R}(t')]$. Finally, we find

$$\begin{aligned}[\mathbf{R}(t),\mathbf{A}(\mathbf{x},t')]=\frac{i}{2}\mathfrak{G}(\mathbf{x})\Big\{&\frac{\delta m^2}{e^2 m m_0 k_0}[\sin k_0(\tau+r)-\sin k_0(\tau-r)]+\frac{1}{m}[\epsilon(\tau+r)\cos k_0(\tau+r)-\epsilon(\tau-r)\cos k_0(\tau-r)]\\&+\left(\frac{1}{m_0}-\frac{\delta m}{m^2}\right)[\cos k_0(\tau+r)+\cos k_0(\tau-r)]-\frac{\delta m^2}{2m^2m_0}[k_0(\tau+r)\sin k_0(\tau+r)+k_0(\tau-r)\sin k_0(\tau-r)]\Big\}\\&+\frac{e^2 i}{6\pi m^2}\mathfrak{G}(\mathbf{x})\{\delta(\tau+r)+\delta(\tau-r)-\tfrac{1}{2}k_0\epsilon(\tau+r)[\tfrac{3}{2}\sin k_0(\tau+r)+\tfrac{1}{2}k_0(\tau+r)\cos k_0(\tau+r)]\\&-\tfrac{1}{2}k_0\epsilon(\tau-r)[\tfrac{3}{2}\sin k_0(\tau-r)+\tfrac{1}{2}k_0(\tau-r)\cos k_0(\tau-r)]\}. \quad (30)\end{aligned}$$

By letting $\tau\to 0$ in Eq. (30), we arrive at a contradiction with Eqs. (14). As pointed out above, we must have $[\mathbf{R}(t),\mathbf{A}(x,t)]=0$. But the right-hand side of Eq. (30) involves an even function of τ. Hence, we can get zero only by cancellation of terms. As may be seen from inspection, however, the cancellation of terms comes about only if we also allow $r\to 0$. Thus, in the above case, the order of performing the processes of commutation and averaging is not interchangeable.

With the above remarks in mind, we will now proceed to find different-time commutation rules subject to the following requirements: (1) The commutators shall be invariant under time translation. (2) The different-time commutators shall be solutions of a set of differential equations to be specified below, with Eqs. (14) to be imposed as boundary conditions.

The above-mentioned differential equations are

$$\Box[\mathbf{A}_h(\mathbf{x},t),\mathbf{R}(t')]=0, \quad (31)$$

$$(d_{t'}^2+k_0^2)[\mathbf{A}_h(\mathbf{x},t),\mathbf{R}(t')]=-\frac{e}{m}[\mathbf{A}_h(\mathbf{x},t),\partial_{t'}\mathfrak{A}_h(t')], \quad (32)$$

$$(d_t^2+k_0^2)[\mathbf{R}(t),\mathbf{R}(t')]=-\frac{e}{m}[\partial_t\mathfrak{A}_h(t),\mathbf{R}(t')], \quad (33)$$

$$\begin{aligned}(d_{t'}^2+k_0^2)[\mathbf{A}(\mathbf{x},t),\mathbf{R}(t')]&=-\frac{e}{m}[\mathbf{A}(\mathbf{x},t),\partial_{t'}\mathfrak{A}_h(t')]\\&=-\frac{e}{m}[\mathbf{A}_h(\mathbf{x},t),\partial_{t'}\mathfrak{A}_h(t')], \quad (34)\end{aligned}$$

$$\Box[\mathbf{A}(\mathbf{x},t),\mathbf{R}(t')]=-\tfrac{1}{2}eT\delta(x)[(\mathbf{V}(t+r)+\mathbf{V}(t-r)),\,\mathbf{R}(t')], \quad (35)$$

$$(d_t^2+k_0^2)[\mathbf{R}(t),\,\mathfrak{G}(\mathbf{x})\cdot(\mathbf{V}(t'+r)+\mathbf{V}(t'-r))]=0, \quad (36)$$

$$\Box[\mathbf{A}(\mathbf{x},t),\mathbf{A}(\mathbf{x}',t')]=-\tfrac{1}{2}[Te\delta(x)(\mathbf{V}(t+r)+\mathbf{V}(t-r)),\,\mathbf{A}(\mathbf{x}',\,t')]. \quad (37)$$

In writing down Eqs. (34)–(37), we have taken account of the difficulty which arose in connection with Eq. (30). In other words, we are not assuming that the average of a commutator involving field functions is necessarily equal to the corresponding commutator of the averaged field functions. In particular, it should be noted in connection with Eqs. (34)–(37) that we are treating $\mathfrak{G}(\mathbf{x})\cdot[\mathbf{V}(t+r)+\mathbf{V}(t-r)]$ as a field function. In the treatment leading to Eq. (30), the latter function was not treated as a field function.

In solving Eqs. (31)–(37) we use Eq. (22) and the $\frac{1}{2}$[advanced+retarded] Green's functions for solving the inhomogeneous equations. The solutions which reproduce Eqs. (14) may now be written

$$e[\mathbf{A}_h(\mathbf{x},t),\mathbf{R}(t')]=\frac{ie}{2m_0}\mathfrak{G}(\mathbf{x})[\epsilon(\tau+r)\cos k_0(\tau+r)-\epsilon(\tau-r)\cos k_0(\tau-r)], \tag{38}$$

$$e[\mathfrak{A}_h(t),\mathbf{R}(t')]=\frac{ie^2}{3\pi m_0}[\delta(\tau)-\tfrac{1}{2}\epsilon(\tau)k_0\sin k_0\tau]\mathbf{I}, \tag{39}$$

$$[\mathbf{R}(t),\mathbf{R}(t')]=-\frac{i\sin k_0\tau}{k_0 m}-\frac{ie^2}{6\pi m m_0}\epsilon(\tau)[\cos k_0\tau-\tfrac{1}{2}k_0\tau\sin k_0\tau]\mathbf{I}, \tag{40}$$

$$[\mathbf{A}(\mathbf{x},t),\mathbf{R}(t')]=\frac{i}{2m_0}\mathfrak{G}(\mathbf{x})[\epsilon(\tau+r)\cos k_0(\tau+r)-\epsilon(\tau-r)\cos k_0(\tau-r)-\cos k_0(\tau+r)-\cos k_0(\tau-r)], \tag{41}$$

$$[\mathbf{A}(\mathbf{x},t),\mathbf{A}(\mathbf{x}',t')]=iT\frac{\mathbf{I}}{4\pi|\mathbf{x}-\mathbf{x}'|}[\delta(\tau+|\mathbf{x}-\mathbf{x}'|)-\delta(\tau-|\mathbf{x}-\mathbf{x}'|)]+\frac{ik_0}{2m_0}\mathfrak{G}(\mathbf{x})\cdot\mathfrak{G}(\mathbf{x}')[\epsilon(\tau+r+r')\sin k_0(\tau+r+r')$$
$$-\epsilon(\tau-r-r')\sin k_0(\tau-r-r')-\sin k_0(\tau+r+r')-\sin k_0(\tau-r-r')], \tag{42}$$

where in Eq. (42) we have set

$$[\mathbf{A}_h(\mathbf{x},t),\mathbf{A}_h(\mathbf{x}',t')]=iT\frac{\mathbf{I}}{4\pi|\mathbf{x}-\mathbf{x}'|}[\delta(\tau+|\mathbf{x}-\mathbf{x}'|-\delta(\tau-|\mathbf{x}-\mathbf{x}'|)], \tag{43}$$

$$[\mathbf{A}_h(\mathbf{x},t),\mathfrak{G}(\mathbf{x}')\cdot(\mathbf{V}(t'+r')+\mathbf{V}(t'-r'))]=\frac{ik_0}{4m_0}\mathfrak{G}(\mathbf{x})\cdot\mathfrak{G}(\mathbf{x}')[\epsilon(\tau+r-r')\sin k_0(\tau+r-r')$$
$$-\epsilon(\tau-r-r')\sin k_0(\tau-r-r')+\epsilon(t+r+r')\sin k_0(\tau+r+r')$$
$$-\epsilon(\tau-r+r')\sin k_0(\tau-r+r')], \tag{44}$$

and

$$\tfrac{1}{4}[\mathfrak{G}(\mathbf{x})\cdot(\mathbf{V}(t+r)+\mathbf{V}(t-r)),\mathfrak{G}(\mathbf{x}')\cdot(\mathbf{V}(t'+r')+\mathbf{V}(t'-r'))]$$
$$=\frac{ik_0}{4m_0}\mathfrak{G}(\mathbf{x})\cdot\mathfrak{G}(\mathbf{x}')[\sin k_0(\tau+r-r')-\sin k_0(\tau-r-r')+\sin k_0(\tau-r+r')-\sin k_0(\tau+r+r')-\sin k_0(\tau+r-r')$$
$$-\sin k_0(\tau-r-r')-\sin k_0(\tau-r+r')-\sin k_0(\tau+r+r')]. \tag{45}$$

The first four terms of Eq. (45) represent a solution of the homogeneous wave equation while the latter four are a solution of the inhomogeneous wave equation. Equation (45) represents a solution of the homogeneous wave equation with respect to $(\mathbf{x},t)$ and the inhomogeneous wave equation with respect to $(\mathbf{x}',t')$. The reverse of the latter case is true for $[\mathfrak{G}(\mathbf{x})\cdot(\mathbf{V}(t+r)+\mathbf{V}(t-r)),\mathbf{A}_h(\mathbf{x}',t')]$. The latter commutator is added to Eqs. (43)–(45) to produce Eq. (42).

In solving Eq. (33) for Eq. (40), or alternatively, in using the derivative of Eq. (39) in Eq. (25), we have implied that

$$[\mathbf{A}_h(\mathbf{x},t),\mathbf{R}_0]=[\mathfrak{A}_h(t),\mathbf{R}_0]=0. \tag{46}$$

Moreover, in calculating the particular solution of Eq. (33), one encounters a term of the form

$$\sin k_0\tau\lim_{|s|\to\infty}|s|,$$

which is a solution of the homogeneous equation and where $\pm|s|$ represent the limits of a time integration. The latter term is cancelled by adding an appropriate solution of the homogeneous equation. We then have

$$[\mathbf{R}_0,\mathbf{R}_0^\dagger]=i\left(1-\frac{e^2k_0^2}{12\pi m_0}\lim_{|s|\to\infty}|s|\right)\mathbf{I}. \tag{47}$$

Upon comparing Eq. (38) with Eq. (32), one finds that we must set

$$[\mathbf{A}_h(\mathbf{x},t),\mathfrak{A}_h(t')]=i\frac{m}{m_0}\mathfrak{G}(\mathbf{x})[\delta(\tau+r)-\delta(\tau-r)]. \tag{48}$$

Note that Eq. (43) does not reduce to Eq. (48) upon setting $\mathbf{x}'=0$ so that in this case the order in which averaging and commutation are performed is not interchangeable.

It is seen from Eqs. (34), (38), and (41) that

$$\tfrac{1}{2}[\mathfrak{G}(\mathbf{x})\cdot(\mathbf{V}(t+r)+\mathbf{V}(t-r)),\mathbf{R}(t')]$$
$$=-\frac{i}{2m_0}\mathfrak{G}(\mathbf{x})[\cos k_0(\tau+r)+\cos k_0(\tau-r)] \quad (49)$$

and

$$[\mathfrak{G}(\mathbf{x})\cdot(\mathbf{V}(t+r)+\mathbf{V}(t-r)),\partial_{t'}\mathfrak{A}_h(t')]=0. \quad (50)$$

If we set $\mathfrak{G}(x')=1$ in Eq. (45) and compare the remainder with the time derivative of Eq. (49), we see that in this case, the order in which averaging with respect to $\mathbf{x}'$ and commutation are performed is interchangeable. However, if we average with respect to $\mathbf{x}$ in Eq. (49) and compare the result with the time derivative of Eq. (40), we see that the above type of interchange is no longer possible. If we average Eq. (50) with respect to $\mathbf{x}$ and compare the result with the second time derivative of Eq. (39), we see that here too, the order in which averaging and commutation are performed is not interchangeable. We may also compare Eq. (50) with the time derivative of Eq. (44) and again conclude that the above-mentioned interchange of order is not allowed.

Equations (38)–(50) represent the basic set of commutation rules from which all others may be derived. A similar set of equations could have been developed utilizing the average of Eq. (18) in Eq. (20). The latter procedure would yield the usual third-order equation,

$$d_t^3\mathbf{R}-\alpha d_t^2\mathbf{R}-\alpha k_0^2\mathbf{R}=(e\alpha/m)\partial_t\mathfrak{A}_{0h}, \quad (51)$$

where $\alpha=6\pi m/e^2$.

We could again follow the procedure which led to Eq. (30) and we would again arrive at the same problem with regard to interchangeability of the order of averaging and commutation. There is also the additional problem that one must use solutions of the homogeneous part of Eq. (51) in following the procedure leading to Eq. (30). In fact, we have not been able to arrive at a set of different-time commutation rules satisfying equations such as (16) and (51) without utilizing solutions of the homogeneous equation. Thus, we cannot, in the latter circumstance, impose subsidiary conditions to eliminate the "run-away" terms in solutions. However, if we relax the requirement that equations such as both (16) and (51) are satisfied by the different-time commutation rules, it is possible to write down a set of commutation rules which will reproduce Eqs. (14) and which do not utilize solutions of the homogeneous part of Eq. (51); but such rules will also relax the requirement of interchanging the order of averaging and commutation. Equation (51) will be discussed further in Sec. IV.

B. Free Particle

For the free particle, $U(\mathbf{R})=0$. If the free particle is treated in the same manner that we treated the oscillator, one need only set $k_0=0$ and Eqs. (38)–(46) would be taken over as free-particle commutators. $(\mathbf{R}_0+\mathbf{R}_0^\dagger)$ coalesces into a single constant operator, so that Eq. (47) would be dispensed with. However, Eqs. (48)–(50) would be retained with $k_0=0$.

An alternative approach may be based on utilizing the fact that

$$\mathbf{P}=m\dot{\mathbf{R}}+e\mathfrak{A}_h=\text{const}, \quad (52)$$

when $k_0=0$. Consider the following soluton of Eq. (52):

$$\mathbf{R}(t)=\mathbf{R}_0-\frac{e}{2m}\int_{-\infty}^{\infty}dt'\,\epsilon(t-t')\mathfrak{A}_h(t')+(\mathbf{P}/m)t. \quad (53)$$

If we set $k_0=0$ in Eq. (25) and integrate parts, then Eqs. (25) and (53) will be equivalent if we absorb the resulting infinite constant of integration into $\mathbf{R}_0$ and set

$$\mathbf{P}=\frac{e}{2}[\mathfrak{A}_h(\infty)+\mathfrak{A}_h(-\infty)]. \quad (54)$$

Equation (54) also arises from using Eq. (25) to calculate $\dot{\mathbf{R}}(t)$ and comparing the result with Eq. (52). Since $[\mathbf{P},\dot{\mathbf{R}}]=[\mathbf{P},\mathfrak{A}_h]=0$, it follows from Eqs. (52)–(53) that

$$[\mathbf{R}_0,\mathbf{P}]=i\mathbf{I}. \quad (55)$$

In concluding this section, we remark that if we set $k_0=0$ and follow the procedure which led to Eq. (30), we again come to the conclusion that the order in which averaging and commutation are performed is not interchangeable.

IV. FOURIER-DECOMPOSED SOLUTIONS

In this section, we discuss other solutions of Eqs. (24) and (51) which lead to commutation rules that are inconsistent with Eqs. (14). In each case, a Fourier decomposition of the free field is introduced and in each case, the Fourier components will be assumed to satisfy the usual oscillator commutation rules. In the next section we show that the above Fourier-decomposed solutions are associated with a canonical formalism which is not equivalent to the one in paper II via a canonical transformation.

Note that in each case considered, we do not impose the requirement that the inhomogeneous solution vanishes in the limit $t\rightarrow\pm\infty$. The latter point is discussed further in Sec. VI. The Fourier decompositions will be based on plane waves and dipole waves. We treat the plane waves first.

A. Plane Waves

Suppose we introduce a plane wave decomposition for $\mathbf{A}_h(\mathbf{x},t)$,

$$\mathbf{A}_h(\mathbf{x},t)=\frac{1}{(2\pi)^{3/2}}\times\int\frac{d^3k}{(2k)^{1/2}}T[\mathbf{a}_k e^{i(\mathbf{k}\cdot\mathbf{x}-kt)}+\mathbf{a}_k^\dagger e^{-i(\mathbf{k}\cdot\mathbf{x}-kt)}], \quad (56)$$

where $\mathbf{a}_k$ and $\mathbf{a}_k^\dagger$ satisfy the usual commutation rules. Insert Eq. (56) into Eq. (26) and interchange the order of integration. The result is

$$\mathfrak{R}(t)=-\frac{e}{m}\frac{i}{(2\pi)^{3/2}}P\times\int\frac{d^3k}{k^2-k_0^2}\left(\frac{k}{2}\right)^{1/2}[T\mathbf{a}_k e^{-ikt}-T\mathbf{a}_k^\dagger e^{ikt}], \quad (57)$$

where $P\int d^3k\, f(k)$ denotes Cauchy principal value integration and $T\mathbf{a}_k=\lim_{\mathbf{x}\to 0}T\mathbf{a}_k e^{i\mathbf{k}\cdot\mathbf{x}}=\mathbf{k}\times(\mathbf{a}_k\times\mathbf{k})/k^2$. Using Eq. (57) to calculate commutators is equivalent to using Eq. (26) and interchanging the order of commutation and integration. Therefore, the discussion leading to Eq. (30) is applicable here.

With the above remarks in mind, we turn to solving Eq. (51). The characteristic function of the homogeneous equation is

$$D(\omega)=\omega^3-\alpha\omega^2-\alpha k_0^2. \quad (58)$$

Following I, we introduce the Green's function

$$\mathcal{G}(\tau)=\frac{\theta(-\tau)e^{\omega_0\tau}}{|\nu|^2}+\frac{\theta(\tau)}{2i\omega_2}\left[\frac{e^{-\beta^*\tau}}{\nu^*}-\frac{e^{-\beta\tau}}{\nu}\right], \quad (59)$$

where $\theta(\tau)=\frac{1}{2}(1+\epsilon(\tau))$, $\tau=t-t'$, $\nu=\omega_0+\beta$, $\beta=\omega_1+i\omega_2$ and the asterisk denotes complex conjugation; ω_0, $-\beta$, and $-\beta^*$ (with $\omega_0>0$ and $\omega_1>0$) are the zeros of $D(\omega)$. $\mathcal{G}(\tau)$ satisfies

$$d_\tau{}^3\mathcal{G}-\alpha d_\tau{}^2\mathcal{G}-\alpha k_0^2\mathcal{G}=-\delta(\tau). \quad (60)$$

Omitting the runaway exponentials which arise from the homogeneous part of Eq. (51), we write as a particular solution of Eq. (51)

$$\mathbf{R}(t)=-\frac{e\alpha}{m}\int_{-\infty}^{\infty}dt'\,\mathcal{G}(t-t')\partial_{t'}\mathfrak{A}_{0h}(t'). \quad (61)$$

Upon introducing the plane wave decomposition and interchanging orders of integration, Eq. (61) becomes

$$\mathbf{R}(t)=-\frac{e\alpha}{m}\frac{i}{(2\pi)^{3/2}}\times\int d^3k\left(\frac{k}{2}\right)^{1/2}\left[T\frac{\mathbf{a}_k e^{-ikt}}{D(-ik)}-T\frac{\mathbf{a}_k^\dagger e^{ikt}}{D(ik)}\right]. \quad (62)$$

If one now uses Eq. (62) to calculate $[\mathbf{R}(t),\mathbf{V}(t)]$, the resulting integral is convergent and since m_0 appears nowhere, we find $[\mathbf{R}(t),\mathbf{V}(t)]\neq i/m_0$. The latter inequality contradicts Eqs. (14).

B. Dipole Waves

Dipole wave expansions may be either of two types. One type arises from expanding the plane waves in terms of multipole waves and cutting off after the dipole term. The other type arises from embedding the system of charge plus radiation field in a perfectly reflecting sphere. The latter type involves a normalization different from the former type. Because of the electric dipole approximation, the former type will reproduce the results of the plane wave expansions. Consequently, we turn directly to the second type of dipole wave expansion. The latter type of expansion has previously been used by Kramers,[2] Van Kampen,[5] and Steinwedel.[6] The afore-mentioned authors have assumed that the radius of the embedding sphere is of sufficient magnitude to impose an approximate boundary condition on the dipole wave.[9] As a result of the approximation, it becomes possible to decompose the Hamiltonian into a sum of decoupled oscillator Hamiltonians. We will dicuss Kramers solutions first.

In writing down solutions, Kramers utilizes only the solutions of the inhomogeneous equations of motion of the particle. Kramers solution for $\mathbf{R}(t)$ is of the form

$$\mathbf{R}(t)=-\frac{1}{e}\sum_k\left(\frac{3}{k^3L}\right)^{1/2}(\sin\eta)\mathbf{q}_k(t), \quad (63)$$

where L is the radius of the embedding sphere. For a free particle,

$$\sin\eta=k/(k^2+\beta^2)^{1/2}, \quad (64)$$

where $\beta=\alpha/4\pi$. For the oscillator,

$$\sin\eta=k^3/[k^6+\beta^2(k^2-k_0^2)^2]^{1/2}, \quad (65)$$

where as above, $k_0^2=k/m$.

$$\dot{\mathbf{R}}(t)=-\frac{1}{e}\sum_k\left(\frac{3}{k^3L}\right)^{1/2}k(\sin\eta)\mathbf{p}_k(t), \quad (66)$$

where we have used $d\mathbf{q}_k/dt=k\mathbf{p}_k$ in Eq. (66). Since $[\mathbf{q}_k,\mathbf{p}_k]=i$, it follows from Eqs. (63)–(66) that $i[\mathbf{R}(t),\dot{\mathbf{R}}(t)]<0$. The latter inequality contradicts Eqs. (14) since $\delta m>0$ and $m_0<0$. Therefore, according to Eqs. (14), $i[\mathbf{R}(t),\dot{\mathbf{R}}(t)]>0$.

The Van Kampen-Steinwedel form of the solution utilizes solutions of the homogeneous particle equations of motion. For the free particle we have[10]

$$\mathbf{R}(t)=\frac{\mathbf{R}'}{(m)^{1/2}}-\frac{1}{e}\sum_k\left(\frac{3}{k^3L}\right)^{1/2}(\sin\eta)\mathbf{q}_k \quad (67)$$

and

$$\dot{\mathbf{R}}(t)=\frac{\mathbf{P}'}{(m)^{1/2}}-\frac{1}{e}\sum_k\left(\frac{3}{k^3L}\right)^{1/2}k(\sin\eta)\mathbf{p}_k, \quad (68)$$

[9] The requirement that the tangential component of the electric field must vanish on the surface of the embedding sphere imposes certain conditions on the dipole wave functions. However, if the radius of the sphere is sufficiently large it becomes approximately true that the trigonometric part of the dipole wave function vanishes on the surface of the sphere. In the latter approximation, the normal component of the electric field will vanish also.

[10] In writing Eqs. (68)–(69), we used Kramers' oscillator variables, which are related to Steinwedel's by the transformation $q_k\to(k)^{1/2}a_n$, $p_k\to p_n/(k)^{1/2}$.

where $\mathbf{R}'=(m)^{1/2}\mathbf{R}_0$ and $\mathbf{P}'=P/(m)^{1/2}$. Since $\mathbf{R}'$ and $\mathbf{P}'$ commute with $\mathbf{q}_k$ and $\mathbf{p}_k$ and $[\mathbf{R}',\mathbf{P}']=i$, we again arrive at the contradiction with Eqs. (14) which we have discussed above in connection with the calculation of $i[\mathbf{R}(t),\dot{\mathbf{R}}(t)]$ using Kramers solutions.

For the oscillator we have

$$\mathbf{R}(t)=\left(\frac{3}{L'}\right)^{1/2}\frac{\mathbf{R}'}{ek_0}-\frac{1}{e}\sum_k\left(\frac{3}{k^3L}\right)^{1/2}(\sin\eta)\mathbf{q}_k, \quad (69)$$

and

$$\dot{\mathbf{R}}(t)=\left(\frac{3}{L'}\right)^{1/2}\frac{\mathbf{P}'}{ek_0}-\frac{1}{e}\sum_k\left(\frac{3}{k^3L}\right)^{1/2}k(\sin\eta)\mathbf{p}_k, \quad (70)$$

where $L'=L+3m/e^2k_0^2$ and R' and P' are oscillator variables associated with the Hamiltonian, $\frac{1}{2}(\mathbf{P}'^2+k_0^2\mathbf{R}'^2)$. The conditions of the preceding paragraph obtain well here as well so that we again arrive at a contradiction with Eqs. (14) when calculating $i[\mathbf{R}(t),\dot{\mathbf{R}}(t)]$.

V. ALTERNATIVE CANONICAL FORMALISM

Suppose we consider as canonical variables the oscillator variables introduced by Fourier decomposing the free field. Then the results of I and Sec. IV show that the Fourier-decomposed solutions discussed in that section cannot be considered as a canonical transformation from one set of canonical variables to another. The generator of time translations [and of Eqs. (6) and (7)] may now be taken to be the free-field Hamiltonian. It, therefore, becomes possible to introduce an interaction picture as follows. We insert the Fourier decomposed particular solutions of Sec. IV into the Hamiltonian defined by Eq. (1). The Hamiltonian may then be separated into a sum of two parts, $\mathcal{H}=\mathcal{H}_0+\mathcal{H}_1$. $\mathcal{H}_0$ is the free-field Hamiltonian and $\mathcal{H}_1$ is the remainder. The time derivatives of all operators and in particular, Eqs. (6) and (7) are determined by forming commutators with $\mathcal{H}_0$. If we now consider $\mathcal{H}_1$, as an interaction Hamiltonian, the above-mentioned interaction picture is defined. It is essential to remember, however, that before one can define such an interaction picture, one must first know solutions of Eqs. (6) and (7) which are functionals of $\mathfrak{A}_h(t)$ and $\partial_t\mathfrak{A}_h(t)$.

As an illustration of the above scheme, we will apply it to the oscillator solution represented by Eq. (62). We introduce a Fourier decomposition for $e^{\pm ikr}/4\pi r$ and $1/4\pi r$ and write

$$\mathbf{A}_1(\mathbf{x},t)=\mathbf{A}_{0h}(\mathbf{x},t)+\frac{1}{(2\pi)^{3/2}}\int\frac{d^3kd^3k'}{(2k)^{1/2}}\{F_R(k',k)[\hat{k}'\times(\mathbf{a}_k\times\hat{k}')]e^{i\mathbf{k}'\cdot\mathbf{x}}e^{-ikt}+F_R^*(k',k)[\hat{k}'\times(\mathbf{a}_k^\dagger\times\hat{k}')]e^{-i\mathbf{k}'\cdot\mathbf{x}}e^{ikt}\}, \quad (71)$$

where $\hat{k}=\mathbf{k}/|\mathbf{k}|$,

$$-\mathbf{E}_1(\mathbf{x},t)=\partial_t\mathbf{A}_{0h}(\mathbf{x},t)-\frac{i}{(2\pi)^{3/2}}\int d^3kd^3k'\,(k/2)^{1/2}\{T_R(k',k)[\hat{k}'\times(\mathbf{a}_k\times\hat{k}')]e^{i\mathbf{k}'\cdot\mathbf{x}}e^{-ikt}-T_R^*(k',k)[\hat{k}'\times(\mathbf{a}_k^\dagger\times\hat{k}')]e^{-i\mathbf{k}'\cdot\mathbf{x}}e^{ikt}\}, \quad (72)$$

where

$$F_R(k',k)=\frac{k^2}{k'^2}T_R(k',k),$$
$$T_R(k',k)=-\frac{3k^2}{(2\pi)^2}\frac{1}{(k'^2-k^2-i\epsilon)D_R(-ik)}. \quad (73)$$

Therefore, when we use Eqs. (71)–(73) to re-express $\mathcal{H}$ in Eq. (1) and use the Sokolov-Tumanov technique[4] to calculate the zero-point level shift of the charged oscillator, we find

$$(\Delta\epsilon)_0=-\frac{3}{2}\frac{\alpha}{\pi}\int\frac{dk\,k^3(k^2-3k_0^2)}{D_R(ik)D_R(-ik)}-\frac{3}{2}. \quad (74)$$

Note that the zero-point energy of the charged oscillator is $\frac{3}{2}$ so that we have subtracted it from the integral in Eq. (74) in defining $(\Delta\epsilon)_0$. Note also that the integral in Eq. (74) is the same one obtained by Kramers.[2,11]

Now, apart from the zero-point energy sums,

$$\mathcal{H}_0=\int d^3k\,k(\hat{k}\times\mathbf{a}_k^\dagger)\cdot(\hat{k}\times\mathbf{a}_k) \quad (75)$$

will be the only single sum over $\mathbf{k}$ in $\mathcal{H}$. The other sums will be double sums of the form $\int d^3kd^3k'$. We treat $\mathcal{H}_0$ as the free-field Hamiltonian so that the remaining terms may be interpreted as the time-dependent interaction Hamiltonian in the interaction picture. If we now transform to a Schrödinger picture at time $t=0$, the Schrödinger operators are identical with the interaction picture operators evaluated at time $t=0$.

VI. DISCUSSION

Reference 1 raised the question of the relationship between the assumption of certain asymptotic boundary conditions, the assumption that the free field may be decomposed into uncoupled field oscillators and the

[11] The above integral is obtained from Kramers' (reference 2) Eq. (32) after integrating by parts and going to the limit of large cutoff. The difference between the above integral and that obtained by Sokolov and Tumanov (reference 4) results from their substituting solutions into a standard oscillator Hamiltonian rather than Eq. (1).

possible existence of constraints among the canonical variables. Reference 2 showed how constraints are introduced as a result of mass renormalization. A Hamiltonian formalism was then developed in II which is general and which must be satisfied by all configuration space solutions generated by the total Hamiltonian. In the present paper, we utilize the formalism of II to return to the above-mentioned question raised in I.

In order to understand the results obtained in the preceding sections, it is necessary to make two points. The first point is that once the dynamical variables are identified and their commutation rules are established, any expressions of the dynamical variables in terms of other operators must be made to conform to the established commutation rules. If the expressions for the dynamical variables are determined by solving differential equations with a linear homogeneous part, then the commutation rules for the operator solutions of the homogeneous equations must be determined so that dynamical variables satisfy the commutation rules established for them. Eqs. (14) represent the established commutation rules to be satisfied in Secs. III and IV. In the latter sections, we showed that for the examples discussed, the assumption that the averages of free-field commutators are equal to the commutators of free-field averages is inconsistent with Eqs. (14). In Sec. III, we exhibited commutation rules for the averaged free field which removed the inconsistency.

The second point referred to above is that the particular solutions of the inhomogeneous differential equations are linear in the free field. Thus, if one assumes that the free-field oscillator variables satisfy canonical commutation rules, the time derivatives of the particular solutions may be generated by the free-field Hamiltonian. The latter was the situation described in Sec. V. There, the field oscillators are chosen as the basic dynamical variables and all other variables are expressed as functions of the field oscillators.

As a final item, we comment on a recent paper by Sokolov and Lysov[7] (hereafter referred to as SL). As pointed out above, the use of Eq. (61) is equivalent to the use of Eq. (62) in calculating commutators if one can interchange the order of commutation and integration. In SL, the latter interchange was made, while in I, such an interchange was not performed in the calculation of certain commutators. In particular, it was claimed in I that $[\mathbf{R}(t),\mathbf{V}(t)]=0$ if one uses Eq. (61). SL has questioned the latter result as well as the interpretation on which it is based. Therefore, it is hoped that the following remarks will serve to clarify the situation.

The discussion which follows will be concerned only with the particular solution of Eq. (51) as represented by Eq. (61). Equation (2) of SL assumes the asymptotic condition

$$\lim_{t\to-\infty}\mathbf{R}(t)=\lim_{t\to-\infty}\mathbf{V}(t)=0. \tag{76}$$

Classically, we may apply L'Hospital's rule[12] to Eq. (61) and find

$$\mathbf{R}(-\infty)=-\frac{e\alpha}{m\omega_0|\beta|^2}\partial_t\mathfrak{A}_{0h}(-\infty), \tag{77}$$

$$V(-\infty)=0. \tag{78}$$

Equations (76) and (77) then imply that $\partial_t\mathfrak{A}_{0h}(-\infty)=0$. Moreover, the first of Eqs. (17) in SL is obtained from Eq. (61) by integrating by parts. The result is

$$\mathbf{R}(t)=-\frac{e\alpha}{m}\left\{\left[\frac{\theta(-t+\infty)}{|\nu|^2}e^{\omega_0(t-\infty)}+\frac{\theta(t-\infty)}{2i\omega_2}\left(\frac{e^{-\beta^*(t-\infty)}}{\nu^*}-\frac{e^{-\beta(t-\infty)}}{\nu}\right)\right]\mathfrak{A}_{0h}(\infty)\right.$$
$$\left.-\left[\frac{\theta(-t-\infty)}{|\nu|^2}e^{\omega_0(t+\infty)}+\frac{\theta(t+\infty)}{2i\omega_2}\left(\frac{e^{-\beta^*(t+\infty)}}{\nu^*}-\frac{e^{-\beta(t+\infty)}}{\nu}\right)\right]\mathfrak{A}_{0h}(-\infty)\right\}+\frac{e\alpha}{m}\int_{-\infty}^{\infty}dt'\,\partial_{t'}\mathcal{G}(t-t')\mathfrak{A}_{0h}(t'). \tag{79}$$

For finite t, the integrated part vanishes in Eq. (79). But if we first take the limit as $t\to-\infty$ and apply L'Hospital's rule to the integral in Eq. (79), we find

$$\mathbf{R}(-\infty)=\frac{e\alpha}{m|\nu|^2}\mathfrak{A}_{0h}(-\infty). \tag{80}$$

Equations (76) and (80) then imply that $\mathfrak{A}_{0h}(-\infty)=0$. If we differentiate Eq. (79), or equivalently, integrate the derivative of Eq. (61) by parts, and take the limit as $t\to-\infty$, we find

$$\mathbf{V}(-\infty)=-\frac{e\alpha}{m}\frac{\omega_0}{|\nu|^2}\mathfrak{A}_{0h}(-\infty). \tag{81}$$

Comparison of Eqs. (78) and (81) again leads us to conclude that $\mathfrak{A}_{0h}(-\infty)=0$. The point to be remembered in connection with Eqs. (80)–(81) is that the limit $t\to-\infty$ is taken in Eq. (61) and its derivative before integrating by parts.

[12] For example, a typical limiting process would involve

$$\lim_{t\to\pm\infty}e^{\omega_0 t}\int_t^\infty dt'\,e^{-\omega_0 t'}\partial_{t'}\mathfrak{A}_{0h}(t')=\lim_{t\to\pm\infty}\left[\int_t^\infty dt'\,e^{-\omega_0 t'}\partial_{t'}\mathfrak{A}_{0h}(t')\right]\Big/e^{-\omega_0 t}=\lim_{t\to\pm\infty}\frac{e^{-\omega_0 t}\partial_t\mathfrak{A}_{0h}(t)}{\omega_0 e^{-\omega_0 t}}=\partial_t\mathfrak{A}_{0h}(\pm\infty)/\omega.$$

The relationships between the zeros of Eq. (58) and the constant coefficients therein must also be employed to obtain the final answers.

If the dynamical variables mentioned above are considered as operators in a Hilbert space, then the limiting processes leading to Eqs. (76)–(81) may be applied to matrix elements.

Another argument leading to $\partial_t\mathfrak{A}_{0h}(-\infty)=0$ is that it follows from the requirement that the solution of Eq. (51) must reduce to a solution of the homogeneous equation in the limit $t\to-\infty$. It reflects the vanishing of the coupling with the external radiation field in the limit $t\to-\infty$.

However, $\lim_{t\to-\infty}\mathbf{A}_{0h}(\mathbf{x},t)\neq 0$ and

$$\lim_{t\to-\infty}\partial_t\mathbf{A}_{0h}(\mathbf{x},t)\neq 0.$$

Moreover, if one introduces a plane wave decomposition of $\mathbf{A}_{0h}$ in terms of creation and annihilation operators, then $\lim_{t\to-\infty}\mathfrak{A}_{0h}(t)\neq 0$ and $\lim_{t\to-\infty}\partial_t\mathfrak{A}_{0h}(t)\neq 0$ if one assumes that the average of the Fourier transform of $\mathbf{A}_{0h}$ is equal to the Fourier transform of $\mathfrak{A}_{0h}$. The latter result contradicts Eqs. (76). Therefore, if one insists on requiring Eqs. (76) to be satisfied, one may conclude that Eq. (62) does not represent the Fourier transform of Eq. (61). Under such circumstances, the use of Eq. (62) is not necessarily equivalent to the use of Eq. (61) in calculating commutators.

Another argument leading to the above conclusion runs as follows. Suppose one assumes that Eqs. (61) and (62) are equivalent and calculates $[\mathbf{R}(t),\mathbf{V}(t)]$ using Eq. (61). The resulting double integral contains $(\partial_t\mathfrak{A}_{0h}(t),\partial_{t'}\mathfrak{A}_{0h}(t')]$ in the integrand. Under the assumptions made, the preceding commutator reduces to $-(i/3\pi)\partial_\tau{}^3\delta(\tau)$, which is zero inside the light cone and at the vertex of the light cone. But, the commutator is multiplied by a function of τ possessing a nonzero third derivative so that the integral is nonvanishing. As noted in Sec. IV, the result contradicts Eqs. (14). However, upon calculating the different-time commutator, one finds

$$\lim_{t\to-\infty}[\mathbf{R}(t),\mathbf{V}(t')]=\lim_{t'\to-\infty}[\mathbf{R}(t),V(t')]=0$$

if one of the variables is held fixed while the limit is taken with the other. A nonvanishing result is obtained only when t, $t'\to-\infty$ simultaneously.[13] Moreover, if one requires $\mathbf{R}(-\infty)=\mathbf{V}(-\infty)=0$, one expects $[\mathbf{R}(-\infty),\mathbf{V}(-\infty)]=0$. Of course, the latter results prevent the possibility of obtaining $\mathbf{R}(t)$ and $\mathbf{V}(t)$ from a unitary transformation with the Hamiltonian as generator of time translations. In fact, if $[\mathbf{R}(t),\mathbf{V}(t)]$ is continuous, it must vanish for $t>-\infty$ if it vanishes for $t\to-\infty$ unless it violates time-translation invariance and is a function of time. In any case, it seems that the plane wave decomposition of $\mathbf{A}_{0h}$ as used above, leads to a contradiction of Eqs. (76).

In I, the position was taken that the vanishing of an operator when $t\to\pm\infty$ requires equal-time commutators involving the operator to vanish or be time dependent. Since $\mathbf{V}(-\infty)$ was assumed to vanish, the above requirement was imposed on $[\mathbf{R}(t),\mathbf{V}(t)]$. In particular, $[\partial_t\mathfrak{A}_{0h}(t),\partial_{t'}\mathfrak{A}_{0h}(t')]$ was not replaced by $-(i/3\pi)\partial_t{}^3\delta(\tau)$ in the integrand of the resulting double integral and was considered to vanish everywhere in the region of integration. However, it was not assumed that $\mathbf{R}(-\infty)=\mathbf{P}(-\infty)=0$. Since $\ddot{\mathbf{R}}(-\infty)=0$,

$$\mathbf{P}(-\infty)=e\mathfrak{A}_{0h}(-\infty). \tag{82}$$

Therefore, in view of Eqs. (77) and (82), we are also not assuming that $\partial_t\mathfrak{A}_{0h}(-\infty)=\mathfrak{A}_{0h}(-\infty)=0$. Consequently, $[\mathbf{R}(t),\mathbf{P}(t)]$ was calculated from $[\mathbf{R}(t), e\mathfrak{A}_{0h}(t)]$. But as with $[\mathbf{R}(t),\mathbf{V}(t')]$, we have

$$\lim_{t\to-\infty}[\mathbf{R}(t),e\mathfrak{A}_{0h}(t')]=\lim_{t'\to-\infty}[\mathbf{R}(t),e\mathfrak{A}_{0h}(t')]=0.$$

Regardless of how one performs limiting processes, the main point we have tried to emphasize in the above discussion is that the calculation of commutators should be consistent with the boundary conditions assumed for the operators.

In Sec. III, we did not assume the asymptotic vanishing of Eq. (26) or of $\mathfrak{A}_h$ and its derivatives. Therefore, since $\mathbf{A}_h$ contains the particle's own radiation field, the particle is never decoupled from it.

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[13] The situation is similar to the one involving the integrated part in Eq. (79).