# Exact Analysis of an Interacting Bose Gas. I. The General Solution and the Ground State

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A gas of one-dimensional Bose particles interacting via a repulsive delta-function potential has been solved exactly. All the eigenfunctions can be found explicitly and the energies are given by the solutions of a transcendental equation. The problem has one nontrivial coupling constant, $\gamma$. When $\gamma$ is small, Bogoliubov's perturbation theory is seen to be valid. In this paper, we explicitly calculate the ground-state energy as a function of $\gamma$ and show that it is analytic for all $\gamma$, except $\gamma=0$. In Part II, we discuss the excitation spectrum and show that it is most convenient to regard it as a double spectrum—not one as is ordinarily supposed.

## I. INTRODUCTION

IN comparatively recent times, as the history of quantum mechanics goes, a vast body of literature has developed on the quantum-mechanical problem of a gas or liquid of particles interacting via a two-body potential. To attack this problem, perturbation theory has been refined and developed in many elegant forms, too numerous and well known to attempt to summarize here. These results have been mainly of two kinds: attempts, based on summing series to all orders, to make exact statements about the true solution of the problem; and attempts to develop approximation schemes to yield quantitative results. It is fair to say, however, that few of these results may claim to be mathematically rigorous. They are accepted because they agree with our intuitive understanding of the problem.

Faced with such a situation it would seem desirable to find a local, time- and velocity-independent, two-body potential (the sort of potential that actually exists in a gas) such that, starting from the Schrödinger equation, and without introducing any approximations, one could derive the energy levels and wave functions of the system. It would also be desirable that such a model problem be three dimensional.

While there do exist several model many-body Hamiltonians in the literature, which are solvable, with one exception, none of them corresponds to having a simple potential of the character mentioned above. The one exception is the work of Girardeau[1] on a gas of impenetrable bosons in one dimension. He showed that the energy spectrum of such a gas is identical with the spectrum of a noninteracting Fermi gas. It is, indeed, unfortunate that, due to the inability of the particles to get past one another, Girardeau's model can only be thought of as an extreme high-density situation—even though the actual density of his gas may be low. By this is meant that if we attempt to draw a parallel between the one-dimensional hard-core model and the three-dimensional hard-core model, then the former resembles the latter only at very high densities when a cell model becomes applicable. In consequence of this fact, the model cannot be used to check any known form of perturbation theory.

Another drawback of Girardeau's model is that it is essentially a zero-parameter model. It might have been hoped that varying the density, $\rho$, or the hard-core radius, $a$, would modify the spectrum in some essential way, but this is not true. If $E_n(p)$ is any energy level of a state having momentum $p$, then $E_n(p)$ is of the form $[\rho/(1-\rho a)]^2 f_n([(1-\rho a)/\rho]p)$. In other words, the hard cores simply play the role of an excluded volume.

We shall propose here a model that in many respects is similar to Girardeau's, but which overcomes the two difficulties mentioned above. We consider a gas of bosons in one dimension interacting via a repulsive $\delta$-function potential, whose Hamiltonian is given by Eq. (2.1) below. As we show, this problem has one nontrivial parameter, namely, $\gamma=\rho^{-1}c$, where $2c$ is the strength of the $\delta$ function. When $\gamma=\infty$ we obviously obtain Girardeau's results since the particles then are impenetrable. When $\gamma=0$ we have the noninteracting Bose gas.

A useful feature of this model is that for small $\gamma$ Bogoliubov's perturbation theory[2] is valid. This is discussed in Sec. IV. The model agrees with all the predictions of that theory except in one important respect: For all values of $\gamma$ the most convenient and natural way to view the spectrum of the gas is in terms of a double elementary boson excitation spectrum—not a single one as previous calculations have suggested. Bogoliubov's single spectrum agrees quite well with one of the spectra we obtain, but the other is totally unaccounted for in his theory. The principal value of this model perhaps lies in this startling fact. The whole question of the excitation spectrum, as well as some heuristic reason that this duplicity of the spectrum might have been anticipated, and why it might also exist in three dimensions, are discussed in the following paper.[3]

We may summarize the results of this paper as follows:

[1] M. Girardeau, J. Math. Phys. **1**, 516 (1960).

[2] See, for example, *The Many Body Problem*, edited by C. De Witt (John Wiley & Sons, Inc., New York, 1958), pp. 347–355.

[3] E. Lieb, following paper [Phys. Rev. **130**, 1616 (1963) (referred to here as II)].

(i) We obtain explicitly the eigenfunctions of the problem and show how they may be derived from the solution of a transcendental algebraic equation. This is the content of Sec. II.

(ii) In Sec. III we find the ground-state energy as a function of $\gamma$ and show that in the limit of a large system it is analytic in $\gamma$ except at $\gamma=0$. This shows that perturbation theory can at best hope to give an asymptotic series for the ground-state energy.

Most quantities of interest, such as the ground-state energy, the velocity of sound, the excitation spectra and others, derived both here and in II, have been calculated explicitly and numerical graphs are given. These quantities are derived from solutions to certain integral equations which cannot be found in closed form. The numerical work thus entailed was performed on an IBM 7090 computer. At the end of Appendix B, an outline of the numerical procedure is given. All of the properties of the solutions claimed above, however, such as their analyticity for $\gamma \neq 0$, have been rigorously proved.

## II. THE PROBLEM AND THE NATURE OF ITS SOLUTIONS

We begin with the Schrödinger equation for $N$ particles in one dimension interacting via a $\delta$-function potential.[4]

$$(-\sum_1^N (\partial^2/\partial x_i^2)+2c \sum_{\langle i,j\rangle} \delta(x_i-x_j))\psi=E\psi, \quad (2.1)$$

where $2c$ is the amplitude of the $\delta$ function. The region of space under consideration is

$$R: \quad \text{all } 0\leq x_i\leq L. \quad (2.2)$$

The wave function $\psi$ satisfies periodic boundary conditions in each variable. We are interested in the repulsive case so that

$$c\geq 0. \quad (2.3)$$

While the attractive case, $c<0$, has a solution, it is not physically meaningful because there is no saturation. It is easy to demonstrate that in this case, for a fixed density, the ground-state energy is proportional to $-N^2$ instead of to $-N$.

It is well known that a $\delta$-function potential is equivalent to the boundary condition

$$\left(\frac{\partial}{\partial x_j}-\frac{\partial}{\partial x_k}\right)\psi\big|_{x_j=x_k+}-\left(\frac{\partial}{\partial x_j}-\frac{\partial}{\partial x_k}\right)\psi\big|_{x_j=x_k-}$$
$$=2c\psi\big|_{x_j=x_k}; \quad (2.4)$$

i.e., $\psi$ is continuous whenever two particles touch, but the jump in the derivative of $\psi$ is $2c$. Moreover, since we are interested in symmetric (Bose) wave functions, the two terms on the left side of Eq. (2.4) are by definition equal.

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[4] $\hbar=1$, $2m=1$. $\sum_{\langle i,j\rangle}$ means summation over pairs.

We can go one step further than this, however, and observe that if we define the region $R_1$ by

$$R_1: \quad 0\leq x_1\leq x_2\leq \cdots \leq x_N\leq L, \quad (2.5)$$

then knowledge of $\psi$ in $R_1$ is equivalent to knowledge of $\psi$ in $R$. Equations (2.1) and (2.4), thus, become

$$-\sum_1^N (\partial^2/\partial x_i^2)\psi=E\psi \text{ inside} R_1, \quad (2.1a)$$

$$(\partial/\partial x_{j+1}-\partial/\partial x_j)\psi\big|_{x_{j+1}=x_j}=c\psi\big|_{x_{j+1}=x_j}. \quad (2.4a)$$

The original Schrödinger equation, (2.1) is thus replaced by a Helmholtz equation (2.1a) together with a mixed boundary condition (2.4a) on the boundary of $R_1$.

The last step is to interpret the periodic boundary conditions on $R$ in terms of $\psi$ defined in $R_1$. The periodic boundary condition reads in part

$$\psi(0,x_2,\cdots,x_N)=\psi(L,x_2,\cdots,x_N), \quad (2.6)$$

with a similar condition for the derivatives. The argument of the right-hand side of Eq. (2.6) is not in $R_1$, but by definition

$$\psi(L,x_2,\cdots,x_N)\equiv\psi(x_2,x_3,\cdots,x_N,L). \quad (2.7)$$

Hence, the original periodic condition is equivalent to the following boundary condition on $R_1$:

$$\psi(0,x_2,\cdots,x_N)=\psi(x_2,\cdots,x_N,L), \quad (2.8a)$$

$$\frac{\partial}{\partial x}\psi(x,x_2,\cdots,x_N)\big|_{x=0}=\frac{\partial}{\partial x}\psi(x_2,\cdots,x_N,x)\big|_{x=L}. \quad (2.8b)$$

Equations (2.8), together with Eq. (2.4a), now completely cover the entire boundary of $R_1$.

We now make the following *ansatz* for $\psi$: Let $\{k\}=k_1, \cdots, k_N$ be an ordered set of $N$ numbers and define

$$\psi(x_1,\cdots,x_N)=\sum_P a(P)P \exp(i\sum_{j=1}^N k_j x_j), \quad (2.9)$$

where the summation extends over all permutations of $\{k\}$, and $a(P)$ are certain coefficients depending on $P$. The wave function $\psi$ is thus a generalization of the one-component Fermi function for which $a(P)$ would be $(-)^P$. It is to be understood that Eq. (2.9) defines $\psi$ *only* in $R_1$, the extension to the rest of $R$ following from the requirement of total symmetry under all particle permutations.

Can we choose $a(P)$ so that $\psi$ satisfies the Schrödinger equation in $R_1$? Obviously Eq. (2.1a) is satisfied with

$$E=\sum_1^N k_j^2. \quad (2.10)$$

Let us now examine Eq. (2.4a) when $x_1=x_2$. If all the $k$'s are distinct (as will prove to be the case), then the $N!$ terms in Eq. (2.9) are linearly independent. Let $P$ be the permutation that takes $\{k\}$ into $p$, $q$, $k_{\alpha_3}, \cdots, k_{\alpha_n}$ and $Q$ be the permutation that takes $\{k\}$ into $q$, $p$, $k_{\alpha_3}, \cdots, k_{\alpha_n}$, where $q=k_{\alpha_1}$ and $p=k_{\alpha_2}$. If Eq. (2.4a) is to be satisfied, there must be a relationship between $a(P)$ and $a(Q)$ depending on $p$ and $q$, and not involving the other $a(P)$'s. Denote by $y$ the common value of $x_1$

and $x_2$ and substitute the two terms of $\psi$ corresponding to $P$ and $Q$ into Eq. (2.4a). We must be able to satisfy

$$i(q-p)[a(P)-a(Q)]\exp(i(p+q)y+\sum_{j=3}^{N}k_{\alpha_j}x_j)$$
$$=c\exp(i(p+q)y+i\sum_{j=3}^{N}k_{\alpha_j}x_j)$$
$$\times[a(P)+a(Q)]\quad(2.11)$$

for all values of $y, x_3, \cdots, x_N$. This is, indeed, possible if

$$a(Q)=-a(P)\frac{c-i(q-p)}{c+i(q-p)}=-a(P)\exp(i\theta_{\alpha_1\alpha_2}),\quad(2.12)$$

where

$$\theta_{ij}=\theta(k_i-k_j),\quad(2.13a)$$

and

$$\theta(r)=-2\tan^{-1}(r/c).\quad(2.13b)$$

Assuming $r$ to be real we have

$$\pi\geq\theta(r)\geq-\pi.\quad(2.13c)$$

Thus, by regarding $\psi$ as the sum of $\frac{1}{2}(N!)$ pairs of terms corresponding to permutations such as $P$ and $Q$ above, we can satisfy Eq. (2.4a) for $x_1=x_2$ for each pair separately when the coefficients are given by Eq. (2.12).

The reader will easily verify that we can satisfy Eq. (2.4a) simultaneously for all $N-1$ possibilities $x_j=x_{j+1}$ if we choose the $a(P)$ in the following way: Let

$$a(I)=1.\quad(2.14)$$

*Rule.* If $P$ takes $\{k\}$ into $\{k\}'=k_{\alpha_1}, \cdots, k_{\alpha_N}$, then rearrange $\{k\}'$ into $\{k\}$ by the process of transposing only adjacent $k$'s. For each transposition, write down the factor $-e^{i\theta_{st}}$ where $k_s$ and $k_t$ are the transposed $k$'s in question, $k_s$ lying to the left of $k_t$ before the transposition. The product of all such factors thus obtained is $a(P)$. There are, in general, many different ways to get $\{k\}'$ into $\{k\}$ but they all give rise to the same set of factors. For example, if

$$P=\begin{pmatrix}123\\321\end{pmatrix},$$

then for both the paths $(321)\rightarrow(231)\rightarrow(213)\rightarrow(123)$ and $(321)\rightarrow(312)\rightarrow(132)\rightarrow(123)$ we obtain the result $a(P)=-\exp i(\theta_{32}+\theta_{31}+\theta_{21})$.

An alternative definition of the above rule which makes it evident that $a(P)$ is indeed independent of the path is the following: Write down $k_1, \cdots, k_N$ in a line. Under it write down $k_{\alpha_1}, \cdots, k_{\alpha_N}$. For each $k_j$ draw a straight line between the two points at which it occurs in the two rows. For each crossing point of two lines corresponding to $k_\alpha$ and $k_\beta$, $a(P)$ contains the factor $-e^{i\theta_{\alpha\beta}}$ where $k_\alpha$ precedes $k_\beta$ in the set $\{k\}'$.

We conclude, therefore, that for *any* set $\{k\}$, $\psi$ defined by Eq. (2.9) will satisfy Eqs. (2.1a) and (2.4a) when $a(P)$ is given by the above rule. There is one exception: All the $k$'s in $\{k\}$ must be distinct, otherwise $\psi$ will vanish identically. Condition (2.4a) determines only the form of $\psi$. The allowed values of the $k$'s will be determined by Eq. (2.8). It is easily verified that

these equations are equivalent to the $N$ equations.

$$(-)^{N-1}e^{-ik_jL}=\exp(i\sum_{s=1}^{N}\theta_{sj}),\quad(\text{all }j),\quad(2.15)$$

where we have defined

$$\theta_{kk}=0.\quad(2.16)$$

Although Eqs. (2.15) are $N$ equations in $N$ unknowns, there will be many sets of solutions. The problem we have here is quite similar to the one-dimensional Heisenberg model of ferromagnetism with nearest neighbor interactions, first solved by Bethe.[5] Let us agree to order the $k$'s (assuming they are real) so that

$$k_1<k_2<\cdots<k_N.\quad(2.17)$$

By observing that

$$\theta_{ij}=-\theta_{ji},\quad(2.18)$$

and taking the product of all $N$ equations in (2.15), we find that

$$\sum_1^N k_j=2(\pi/L)n,\quad(2.19)$$

where $n$ is an integer. We also find, by virtue of Eq. (2.13a), that if $\{k\}$ is a solution to Eq. (2.15), then the set $\{k'\}$ defined by

$$k_i'=k_i+2\pi n_0/L\quad(2.20)$$

is a solution for any integer $n_0$. The significance of Eq. (2.20) is the following: Since Eq. (2.9) defines $\psi$ only in $R_1$, the $k$'s cannot be regarded as true wave vectors. But the sum of the $k$'s *is* a true wave vector and is, in fact, the total momentum of $\psi$; i.e.,

$$P_{op}\psi=(\sum_1^N(-i\partial/\partial x_j))\psi=(\sum_1^N k_j)\psi=p\psi\quad(\text{in}R).\quad(2.21a)$$

Equation (2.19) tells us that the total momentum must be an integral multiple of $2\pi/L$—an expected result. Equation (2.20) tells us that for any state with momentum $p$ there is a state with momentum

$$p'=p+2\pi n_0\rho,\quad(2.21b)$$

where

$$\rho=N/L,\quad(2.22)$$

obtained by a simple shift of the $k$'s. Conversely for any state with momentum $|p|\geq\pi\rho$ there is a corresponding state having a momentum in the range

$$-\pi\rho<p\leq\pi\rho.\quad(2.23)$$

We have, thus, to consider only states in the interval (2.23). All other states are obtained by the simple "umklapp," Eq. (2.20), which has the physical meaning of imparting a total momentum to the system while preserving the "internal" state of the system. The integer $n_0$ in Eq. (2.20) is thus one of the quantum numbers

[5] See R. Orbach, Phys. Rev. **112**, 309 (1958). This contains references to earlier papers.

describing a state. The energy of the new state is

$$E' = \sum_1^N (k_j')^2 = E + \frac{4\pi n_0 \rho}{N} p + \frac{1}{N}(2\pi n_0 \rho)^2, \quad (2.24)$$

a well-known consequence of Galilean relativity.

To elucidate the nature of Eq. (2.15) we must study the two-body problem. To avoid loss of continuity we have relegated this to Appendix A. The principal results are as follows: (a) The *ansatz*, Eq. (2.9), appears to give all eigenstates of the problem; (b) for $\gamma > 0$, the case in which we are interested, the $k$'s are always real; (c) for $\gamma = 0$ we obtain the noninteracting solution, while for $\gamma = \infty$ we obtain Girardeau's solution[6] with a continuous transition in between.

These same considerations apply to the $N$-particle case. Since the $k$'s are real, we may order them according to Eq. (2.17). For real $\{k\}$ we can say something about the solution to Eq. (2.15). Dividing two successive equations and equating exponents, we have

$$\delta_j \equiv (k_{j+1} - k_j)L = \sum_{s=1}^N (\theta_{s,j} - \theta_{s,j+1}) + 2\pi n_j$$
$$(j = 1, 2, \cdots, N-1), \quad (2.25)$$

where $n_j$ is an integer depending upon $j$, defined for $j = 1, 2, \cdots, N-1$. These $n_j$'s are the generalization of $n_1$ in Eq. (A3). Since $\theta_{kj}$ is a monotonically increasing function of $j$ by Eqs. (2.13), the sum in Eq. (2.25) is negative. Therefore,

$$n_j \geq 1 \quad \text{(all } j\text{)}, \quad (2.26)$$

if there is to be a solution. Since

$$k_j = k_1 + (1/L)\sum_{s=1}^{j-1} \delta_s, \quad (2.27a)$$

and

$$k_\alpha - k_\beta = (1/L)\sum_{s=\beta}^{\alpha-1} \delta_s \quad (\alpha > \beta), \quad (2.27b)$$

the right side of Eq. (2.25) involves only the $\delta$'s. Equation (2.25) is thus a self-contained set determining the $\delta$'s. If any set $\{n\}$ is chosen satisfying (2.26), there is presumably exactly one solution for the $\delta$'s. At least that was the case for two particles and will prove to be true for the cases we shall investigate involving a large number of particles. We also note that

$$\delta_j < 2\pi n_j. \quad (2.28)$$

The remaining step is to determine the individual $k$'s so as to satisfy Eqs. (2.15) and (2.23). If the $j=1$ equation of (2.15) is satisfied, the remainder will automatically be by virtue of Eq. (2.25). Thus, if we choose

$k_1$ such that

$$k_1 = -\frac{1}{L}\sum_{k=1}^N \theta_{k1} - \frac{2\pi m}{L} + \frac{\epsilon(N)}{L}$$

$$= -\frac{1}{L}\sum_{k=1}^N \theta\left(\frac{1}{L}\sum_{j=1}^{k-1} \delta_j\right) - \frac{2\pi m}{L} + \frac{\epsilon(N)}{L}, \quad (2.29)$$

where

$$\epsilon(N) = \pi \quad \text{for } N \text{ even,}$$
$$= 0 \quad \text{for } N \text{ odd,} \quad (2.30)$$

and $m$ is some integer, and use Eq. (2.27a) to determine the other $k$'s, Eq. (2.15) will be satisfied. To determine the integer $m$, which is not arbitrary, we observe that

$$p = \sum_1^N k_j = Nk_1 + \frac{1}{L}\sum_{j=1}^{N-1} (N-j)\delta_j$$

$$= \frac{1}{L}\sum_{j=1}^{N-1} (N-j)\delta_j - \rho \sum_{j=1}^N \theta_{j1} - 2\pi m\rho + \epsilon(N)\rho. \quad (2.31)$$

Equation (2.23) then fixes $m$.

As further support for the assertion that Eq. (2.25) has a unique solution for a given $\{n\}$, we shall consider the two limiting cases $c=0$ and $c=\infty$.

$c = \infty$: In this case all the $\theta$'s are zero, whence

$$\lim_{c\to\infty} \delta_j = 2\pi n_j, \quad (2.32)$$

so that not only is the solution of (2.25) unique, but (2.28) becomes an equality. It is clear that we recover all Girardeau's wave functions and energies; the term $\epsilon(N)$ just gives the extra $\pi$ mentioned in footnote 6 for the even $N$ case.

$c = 0$: By Eq. (2.13)

$$\lim_{c\to 0} \theta(x) = -\pi \quad \text{for } x > 0 \quad (2.33)$$
$$= \pi \quad \text{for } x < 0,$$

and, hence, Eq. (2.25) yields

$$\lim_{c\to 0} \delta_j = 2\pi(n_j - 1). \quad (2.34)$$

There is, however, a *caveat*: if some $n_j = 1$ then $\delta_j \to 0$, but it is then not clear that we can use the limiting form (2.33). We must ask for $\lim_{c\to 0}(\delta/c)$. Only if it is infinity can we use Eq. (2.33) and thereby obtain Eq. (2.34). But if $\delta \not\to 0$ Eq. (2.33) is certainly correct. Hence, by *reductio ad absurdum*, the only consistent conclusion is that Eq. (2.34) is always correct. The solution to Eq. (2.25) is again unique. We have, of course, recovered the well-known Bose functions for free particles. For $N$ even, the $\epsilon(N)$ term in Eq. (2.31), which in this case is unwanted, is compensated by the term $\sum \theta_{j1}$.

We, thus, may conclude that the $n_j$ are a complete set of "internal" quantum numbers for the system. Together with $n_0$ [Eq. (2.20)] they specify all states. By virtue of Eq. (2.28) moreover, the $n_j$ may be thought of heuristically as the spacing between the $k$'s.

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[6] Girardeau obtained the solution to the hard-core problem only for $N$ odd. For $N$ even, however, there is an equally simple solution. One simply uses the Fermi function satisfying *antiperiodic* boundary conditions, i.e., $Lk = \pi + 2n\pi$, where $n$ is an integer. This exactly compensates the unwanted change of sign of his $A$ function. For two particles the ground state has $k_2 = -k_1 = \pi/L$.

The exact spacing depends upon $c$ to be sure, and also upon the relative position of $\delta_j$ in the set $\{\delta\}$. The fact that even for $n_j=$const the spacing is not uniform (except for $c=\infty$) gives rise to the principal difficulty in the analysis of the many-particle case. One other peculiarity of the problem should be remarked upon. The limiting case $c=\infty$ proved above to be particularly simple, but for $c\rightarrow0$ the problem became pathological. Indeed, the latter limit will prove to be surprisingly delicate as we shall see.

## III. GROUND-STATE ENERGY AND WAVE FUNCTION

We are interested in passing to the limit of a large system. This means $N$, $L\rightarrow\infty$ such that $\rho=$fixed constant. From dimensional considerations the ground-state energy can always be written

$$E_0=N\rho^2l(N,cL), \tag{3.1}$$

where $l$ is a dimensionless function of its arguments. But if $E_0$ is to be an extensive variable (as will, indeed, prove to be the case), $l$ can depend only on intensive variables. The only dimensionless intensive variable in the problem is

$$\gamma=c/\rho. \tag{3.2}$$

Therefore,

$$E_0=N\rho^2e(\gamma). \tag{3.3}$$

It is easy to show directly from Eq. (2.1) that $e(\gamma)$ is a monotonically increasing function of $\gamma$. We also expect that $e(0)=0$ (free particles) and $e(\infty)=\pi^2/3$ (Girardeau's solution). We can easily get an upper bound for $e(\gamma)$ by a variational calculation. Using the unperturbed Bose function, $\psi=1$, we obtain

$$e(\gamma)\leq\gamma. \tag{3.4}$$

Using Girardeau's function,[1] we have

$$e(\gamma)\leq\pi^2/3. \tag{3.5}$$

Returning to Eq. (2.25), it is clear that the choice

$$n_j=1 \tag{3.6}$$

gives the ground state, for this choice minimizes the $\delta$'s and hence allows the $k$'s to be as close to zero as possible. We also see from the symmetry of the equations that if $k$ is in $\{k\}$, then so is $-k$. Indeed this is always so for $n_j=$const and implies that $p=0$ for such states. Equation (2.31) determines $k_1$ in this case:

$$-k_1=\frac{1}{NL}\sum_1^{N-1}(N-j)\delta_j=k_N\equiv K(\gamma). \tag{3.7}$$

From Eq. (2.28) *et seq.* we see that

$$K(0)=0, \quad K(\infty)=\pi\rho, \tag{3.8}$$

with presumably a continuous monotonic transition in between.

We are, thus, led to expect that $K(\gamma)$ is an intensive variable and that the process of passing to an infinite system consists in "filling in" more and more points in the set $\{k\}$ between $-K$ and $K$ with the understanding that the spacing between $k$'s may not be uniform. This may be shown as follows: The inequality $k_{j+1}-k_j<2\pi/L$, Eq. (2.28), permits us for a large system to use a Taylor expansion in Eq. (2.25), i.e.,

$$\theta(k_s-k_j)-\theta(k_s-k_{j+1})$$
$$=-2c(k_{j+1}-k_j)\frac{1}{c^2+(k_s-k_j)^2}+O(1/L^2), \tag{3.9}$$

and, hence,

$$(k_{j+1}-k_j)=-2c(k_{j+1}-k_j)\frac{1}{L}\sum_{s=1}^N[c^2+(k_s-k_j)^2]^{-1}$$
$$+\frac{2\pi}{L}+O(1/L^2). \tag{3.10}$$

Define

$$k_{j+1}-k_j=1/Lf(k_j), \tag{3.11}$$

by means of which the sum in Eq. (3.10) may be approximated by an integral by Poisson's formula to the required accuracy in $L^{-1}$. Equation (3.10) becomes

$$2c\int_{-K}^K\frac{f(p)}{c^2+(p-k)^2}dp=2\pi f(k)-1. \tag{3.12}$$

The meaning of $f(k)$ is that for a large system

$$Lf(k)dk=\text{number of }k\text{'s in }(k,k+dk). \tag{3.13}$$

The subsidiary condition determining the number of particles is

$$\int_{-K}^K f(k)dk=\rho, \tag{3.14}$$

while the ground-state energy is given by

$$E_0=\sum_1^N k_j^2=\frac{N}{\rho}\int_{-K}^K f(k)k^2dk. \tag{3.15}$$

One final condition implied by Eqs. (3.11) and (3.13) is

$$f(k)\geq0. \tag{3.16}$$

Let us change variables as follows: Define

$$k\equiv Kx; \quad c\equiv K\lambda; \quad f(Kx)\equiv g(x), \tag{3.17}$$

in terms of which Eqs. (3.12), (3.14), and (3.15) become, respectively,

$$1+2\lambda\int_{-1}^1\frac{g(x)dx}{\lambda^2+(x-y)^2}=2\pi g(y), \tag{3.18}$$

$$e(\gamma)=\frac{\gamma^3}{\lambda^3}\int_{-1}^1 g(x)x^2dx, \tag{3.19}$$

$$\gamma\int_{-1}^1 g(x)dx=\lambda. \tag{3.20}$$

The program, therefore, consists of the following steps: (i) Solve Eq. (3.18) for a fixed $\lambda$; (ii) use Eq. (3.20) to determine $\lambda$ as a function of $\gamma$; (iii) Eq. (3.19) then gives $e(\gamma)$; (iv) Eq. (3.17) gives $K(\gamma)$, i.e.,

$$K = \rho\gamma\lambda^{-1} = \rho\left[\int_{-1}^{1} g(x)dx\right]^{-1}. \quad (3.21)$$

Equation (3.18) is an inhomogeneous Fredholm equation of the second kind with an inhomogeneous term that is positive definite (i.e., $+1$). In Appendix B we discuss this equation in considerable detail and prove the following:

(a) For *any* inhomogeneous term there is exactly one solution $g(y)$.

(b) $g(y)$ is an infinitely differentiable (analytic) function of $\lambda$ for $\lambda > 0$.

(c) If the inhomogeneous term is positive definite [as in Eq. (3.18)], $g(y) > 0$ for all $y$.

(d) If the inhomogeneous term is bounded above (below), then $g(y)$ is bounded above (below).

These statements enable us to claim that $g(y,\gamma)$, $e(\gamma)$, and $K(\gamma)$ are analytic functions of $\gamma$ (except for $\gamma = 0$). Since there are then no unusual kinks or points of discontinuous derivatives, it becomes a straightforward matter to evaluate all quantities numerically with confidence. The proof is as follows: (b) and (c) above, together within Eq. (3.20), imply that $\gamma$ is an analytic function of $\lambda$ for $\lambda$ in $(0,\infty)$. Equation (3.19) then implies that $e$ is analytic in $\lambda$; Eq. (3.21) implies that $K$ is analytic in $\lambda$. The problem is then to prove that (i) $\lambda$ is an analytic function of $\gamma$ and that (ii) the range $0 < \lambda < \infty$, in fact, covers the required range $0 < \gamma < \infty$. By the implicit function theorem, statement (i) will be true if we can show that

$$0 \neq \frac{\partial\gamma}{\partial\lambda} = \left[\int_{-1}^{1} g(x)dx\right]^{-2}$$

$$\times \left\{\int_{-1}^{1} g(x)dx - \lambda\int_{-1}^{1}\frac{\partial}{\partial\lambda}g(x)dx\right\}, \quad (3.22)$$

for all $\lambda$. By (b) above, $\left[\int_{-1}^{1} g(x)dx\right]^{-2} \neq 0$. As for the two terms in { }, we observe that if we define

$$h(x,\lambda) \equiv \left(\frac{\partial}{\partial\lambda} + \frac{x}{\lambda}\frac{\partial}{\partial x}\right)g(x,\lambda), \quad (3.23)$$

and integrate Eq. (3.18) by parts several times, we easily obtain an equation for $h$, viz.,

$$2\pi h(y,\lambda) = 2\lambda\int_{-1}^{1} dx\,\frac{h(x,\lambda)}{\lambda^2 + (x-y)^2}$$

$$- 2g(1,\lambda)\left(\frac{1}{\lambda^2 + (y+1)^2} + \frac{1}{\lambda^2 + (y-1)^2}\right). \quad (3.24)$$

By (c) above, the inhomogeneous term in Eq. (3.24) is negative definite and, consequently, $h(x,\lambda) < 0$ [by then applying (c) to Eq. (3.24)]. We have, therefore,

$$0 > \lambda\int_{-1}^{1} h(x,\lambda)dx$$

$$= \int_{-1}^{1}\left(\lambda\frac{\partial g(x,\lambda)}{\partial\lambda} - g(x,\lambda)\right)dx + 2g(1,\lambda) \quad (3.25)$$

by using the definition (3.23) and integrating by parts. Since $g(1,\lambda) > 0$, the integral in Eq. (3.25) is negative definite. But this integral is just the negative of the term in { } in (3.22). Hence, statement (i) above is proved. We see, incidently, that $\lambda$ and $\gamma$ are monotonically increasing functions of each other.

As for (ii) above, we see from Eq. (3.18) that

$$\lim_{\lambda\to\infty} g(x,\lambda) = 1/2\pi, \quad (3.26)$$

and, consequently, from Eq. (3.20)

$$\lim_{\lambda\to\infty}\gamma = \infty. \quad (3.27a)$$

As $\lambda \to 0$, on the other hand, the bounds mentioned in (d) above guarantee that $g(x,\lambda) \nrightarrow 0$. [On the contrary $g(x,\lambda) \to \infty$ in this limit.] Equation (3.20) then implies that

$$\lim_{\lambda\to 0}\gamma = 0. \quad (3.27b)$$

Equations (3.27) establish (ii) above.

To recapitulate, we have shown that all quantities are analytic functions of $\gamma$ in $(0,\infty)$. As the above arguments tend to suggest, however, and as we, in fact, show later, there is a very serious singularity at $\gamma = 0$. The physical meaning of this singularity is that any interaction, however weak (but volume-independent), leads in the limit of a large system to a system that is in some way basically different from the noninteracting case. If one tries to find a series for $e(\gamma)$ for small $\gamma$, such a series can at best be asymptotic—a result already indi-
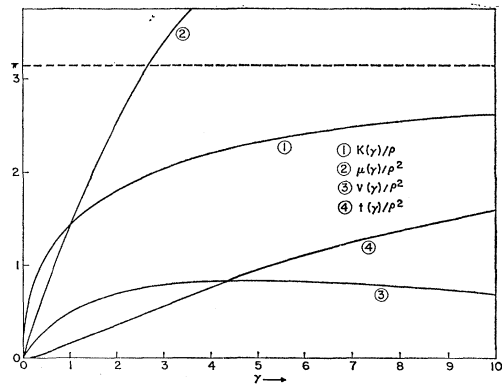


FIG. 1. Various numerically derived properties of the ground state plotted as functions of $\gamma = c/\rho$. $K$ = cutoff momentum, $\mu$ = chemical potential, $v$ = potential energy per particle, and $t$ = kinetic energy per particle. As $\gamma \to \infty$: $K \to \pi\rho$; $\mu \to \pi^2\rho^2$, $v \to 0$, and $t \to \frac{1}{3}\pi^2\rho^2$.

cated, but not proved, by the conventional perturbation theoretic treatments.

In Appendix B an outline is given of the numerical solution of Eqs. (3.18–20). While nonpathologic behavior is guaranteed by the above analyticity considerations, the singularity at $\gamma=0$ makes the numerical calculation more and more difficult as this limit is approached.

In plotting results we use units in which $\rho=1$, so that $K/\rho$ varies between 0 and $\pi$. We have also eliminated the dependent parameter $\lambda$ in favor of the independent parameter $\gamma$ by using Eq. (3.20). [Numerical values of $\lambda$ as a function of $\gamma$ can be obtained, if desired, from the relation $\lambda=\gamma(K/\rho)^{-1}$, Eq. (3.21).] Figure 1
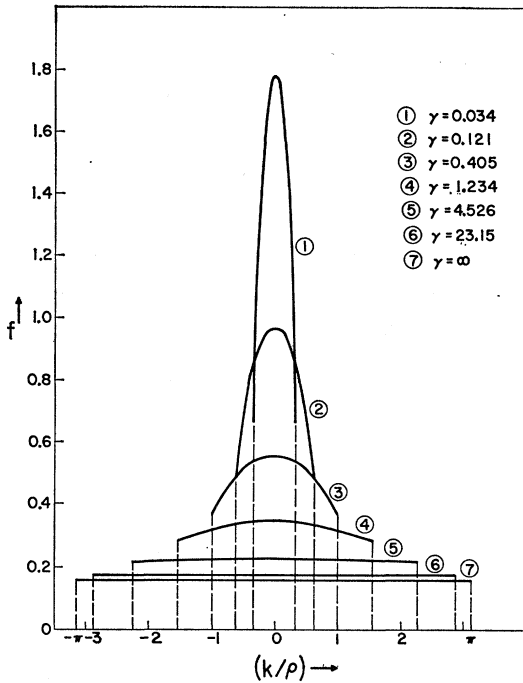


FIG. 2. The distribution function of "quasi-momenta" in the ground state for various values of $\gamma=c/\rho$. The vertical dashed lines are the cutoff momenta $K$ (cf. Fig. 1). When $\gamma=\infty$, $f=(2\pi)^{-1}$. For all $\gamma$, $\int_{-K}^{K} f(k)dk=\rho$.

shows $K(\gamma)$ while Fig. 2 shows the density function $f(k,\gamma)$ for several values of $\gamma$. It will be noticed that as $\gamma\to 0$ small values of $k$ become relatively more important. Figure 3 shows the end result of the calculation, $e(\gamma)$. The straight dashed line is the zeroth-order perturbation theory result $e=\gamma$, which also coincides with the primitive upper bound obtained before [Eq. (3.4)]. The other dashed curve, $e=\gamma[1-(4/3\pi)\sqrt{\gamma}]$, is the result of Bogoliubov's perturbation theory which we shall discuss presently. It is in fair agreement with the exact answer up to $\gamma=2$. We may, therefore, regard $0\le\gamma<2$ as the weak coupling region. On the other hand, it is interesting to note that even for $\gamma=10$, $e(\gamma)$ is
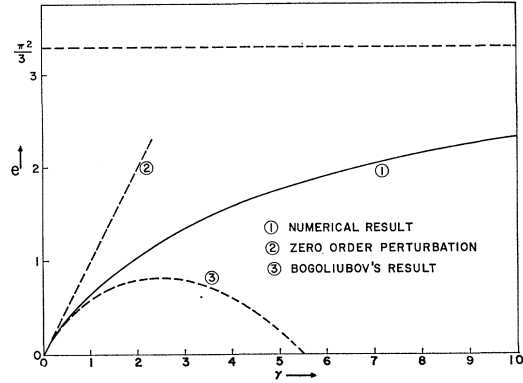


FIG. 3. The ground-state energy $E_0=N\rho^2 e(\gamma)$. The full curve gives $e(\gamma)$ obtained numerically. As $\gamma\to\infty$, $e\to\frac{1}{3}\pi^2$. Curve 2 is the zero-order perturbation theory result, i.e., the expectation value of the potential in the noninteracting ground state. Curve 3 is the result of Bogoliubov's perturbation theory, which is seen to be good up to $\gamma\sim 2$.

still quite far from its asymptotic limit, $\frac{1}{3}\pi^2$. In fact, in order to get within 10% of this asymptotic limit $\gamma$ must be about 37. The intermediate coupling region, by one definition at least, is, therefore, $2<\gamma<37$.

There are other quantities of interest in connection with the ground state; these are also plotted in Fig. 1. There is the chemical potential $\mu$, or energy necessary to add one particle to the system, defined by

$$\mu=\frac{\partial E_0}{\partial N}=\rho^2\left(3e-\gamma\frac{de}{d\gamma}\right). \tag{3.28}$$

Also interesting is the potential energy per particle $v$, which by a well-known theorem is

$$v=\frac{c}{N}\frac{\partial}{\partial c}E_0=\rho^2\gamma\frac{de}{d\gamma}, \tag{3.29}$$

and the kinetic energy per particle

$$t=\frac{1}{N}E_0-v=\rho^2\left(e-\gamma\frac{de}{d\gamma}\right). \tag{3.30}$$

We conclude this section with a list of the asymptotic forms of the various quantities for large and small $\gamma$.

*Large $\gamma$.* This is the easiest case. The denominator in the integral of Eq. (3.18) may be effectively replaced by $\lambda^2$. The error thus introduced in the integral is small, but by Eq. (3.26) the integral itself is small compared to 1, so that the resultant error in $g$ will be quite small indeed. This substitution leads to $g(x)=$ const which may be readily evaluated as

$$g(x)=\lambda(2\pi\lambda-4)^{-1}. \tag{3.31}$$

Substitution of Eq. (3.31) into Eqs. (3.19) and (3.20)

leads (for large $\gamma$) to

$$\lambda = \frac{1}{\pi}(\gamma+2), \quad e = \frac{1}{3}\pi^2\left(\frac{\gamma}{\gamma+2}\right)^2, \quad K = \pi\frac{\gamma}{\gamma+2}\rho,$$

$$\mu = \frac{3\gamma+2}{\gamma+2}\rho^2 e, \quad v = \frac{4}{\gamma+2}\rho^2 e, \quad t = \frac{\gamma-2}{\gamma+2}\rho^2 e. \tag{3.32}$$

These results are accurate to 1% for $\gamma$ as small as 10.

*Small $\gamma$.* As we show in Appendix B, as $\lambda \to 0$ the factor $2\pi$ on the right side of Eq. (3.18) becomes an eigenvalue of the integral equation. Consequently, $g(x,\lambda)$ has a singularity at $\lambda=0$. In addition to this difficulty, the kernel of the integral equation also becomes quite pathological. It will be recognized that as $\lambda \to 0$ the kernel becomes a well-known representation for $2\pi\delta(x-y)$ so that in this limit Eq. (3.18) reads: $2\pi g(x) = 2\pi g(x)+1$. It is clear, therefore, that in this limit $g$ becomes unbounded—a statement borne out by the limits on $g$ given in Appendix B.

There does not seem to be any simple way to get a systematic, reliable expansion of $g$ as $\lambda \to 0$. We can, however, guess the zeroth order form for $g$:

$$g(x,\lambda) \sim \frac{1}{2\pi\lambda}(1-x^2)^{1/2}, \quad (\lambda \to 0) \tag{3.33}$$

and one can then show, using bounding arguments similar to those in Appendix B, that the correction to (3.33) is of higher order in $\lambda$ for all $x$, and is positive definite. Beyond this, we have been unable to obtain an unambiguous correction to (3.33). One of the major difficulties is deciding what happens at the end points, $|x|=1$.

Equation (3.33) allows us to find only the leading term for the quantities mentioned in (3.32). These turn out to be (for small $\gamma$)

$$\lambda = \frac{1}{2}\sqrt{\gamma}, \quad e = \gamma, \quad K = 2\rho\sqrt{\gamma},$$

$$\mu = 2\rho^2\gamma, \quad v = \gamma, \quad t = 0. \tag{3.34}$$

The equation $e=\gamma$ is plotted in Fig. 3. We see that the upper bound for $e$ [Eq. (3.4)] is, in fact, its asymptotic form—a result predicted by elementary perturbation theory.

It will be noticed from Fig. 1, as well as from Eqs. (3.32) and (3.34) ,that for small $\gamma$ the potential energy dominates the kinetic. The reverse is true for large $\gamma$; in fact $v \to 0$ as $\gamma \to \infty$. This behavior is exactly the same as for three dimensions—it is often said that a large potential behaves like a kinetic energy barrier. This fact is supposed by some to be tied up with the ability of particles to "go around" each other in three dimensions. But as we can clearly see, it is also present in one dimension. Thus, the difference between one and three dimensions does not lie here—it is apparently immaterial to the particles whether they can "get around" each other or merely "through" each other.

## IV. PERTURBATION THEORY

The well-known perturbation theory of Bogoliubov[2] was at first assumed to be an expansion in the density. That this is incorrect was realized when it was found that the correct parameter for a low-density expansion is the scattering length (in three dimensions) and not $\int v(x)d^3x$ as appears in Bogoliubov's theory. But Bogoliubov's theory *is* asymptotically correct if we regard it as an expansion in the potential. In other words, it may be expected to give the first two terms in the energy correctly for any density if the potential is so weak that it may be treated by the Born approximation. In our case, therefore, the Bogoliubov theory should be correct for small $\gamma$. On the other hand, small $\gamma$ may be thought of as high density—the reverse of the usual (incorrect) assumption mentioned above.

This last observation leads to a second reason for believing in the Bogoliubov result for small $\gamma$. One can show (we shall not do so here) that if the Fourier transform of the potential is everywhere nonnegative, then Bogoliubov's theory is correct—at least as far as the ground-state energy is concerned—in the limit of *high* density. The $\delta$-function potential certainly satisfies the above criterion.

As stated in the introduction, one of the uses of an exact model is as a check for approximation schemes. It is indeed fortunate that the present model has a definite range of coupling constant over which the Bogoliubov theory may be unambiguously expected to be correct. Our model, therefore, should serve not only to establish the validity of the Bogoliubov theory but should also serve as an indicator (taking appropriate account of the difference between one and three dimensions) of the range over which the theory is reasonably accurate.

The Bogoliubov prescription easily yields the following results: The spectrum of elementary excitations is given by

$$\epsilon(p) = \rho^2|p/\rho|[(p/\rho)^2+4\gamma]^{1/2}. \tag{4.1}$$

We have purposely written $\epsilon(p)$ in terms of the dimensionless variable $p/\rho$. The ground-state energy is given by

$$E_0 = \frac{1}{2}N^2(2c/L) + \frac{1}{2}\sum_p [\epsilon(p)-p^2-N(2c/V)]$$
$$= N\rho^2\gamma[1-(4/3\pi)\sqrt{\gamma}]. \tag{4.2}$$

The expression for $e(\gamma)$, (4.2) is plotted in Fig. 3; it is adequate up to approximately $\gamma=2$. Equation (3.34) shows the leading term is exact. While we have not found an analytic expression for the second term, $(4/3\pi)\gamma^{3/2}$, the numerical results of Fig. 3 indicate that it too is correct. It is interesting to note that the exact equation for $e(\gamma)$ is so pathological at $\gamma=0$ that it was an effort to find even the zeroth-order term for $e(\gamma)$, while perturbation theory gives the first two terms by elementary quadrature.

As for the elementary excitations, we shall show in II that Eq. (4.1) is also fairly accurate up to about $\gamma=2$, but that there is another type of elementary excitation unaccounted for by Bogoliubov's theory.

## ACKNOWLEDGMENTS

## APPENDIX A. THE TWO-BODY PROBLEM

To elucidate the nature of Eqs. (2.15), we shall study the two-body problem. To highlight the basic difference between the repulsive and attractive cases, we shall consider both. For the two-body problem, (2.15) becomes

$$-e^{ik_2L}=e^{i\theta_{21}}=-e^{-ik_1L}. \qquad (A1)$$

There are several ways to solve this problem and we shall choose one which may be generalized to the $N$-body case. Define

$$(k_2-k_1)L=\delta, \quad \gamma=\tfrac{1}{2}cL, \qquad (A2)$$

then, multiplying the two equations (A1), we obtain

$$\delta=2\theta(k_2-k_1)+2\pi n_1 \qquad (A3)$$
$$=-4\tan^{-1}(\delta/2\gamma)+2\pi n_1,$$

where $n_1$ is an integer. If we invert Eq. (A3) we obtain

$$\delta/2\gamma=-\tan(\delta/4), \quad n_1 \text{ even} \qquad (A4a)$$

$$=[\tan(\delta/4)]^{-1}, \quad n_1 \text{ odd}. \qquad (A4b)$$

It must be remembered that Eqs. (A4) have far more solutions than Eq. (A3) (because $|\theta|<\pi$ by definition). Now if $z\tan z=$ real, then $z$ is either real or imaginary, not complex. The same applies if $(1/z)\tan z=$ real. Hence, the roots of Eq. (A3) are either real or imaginary. For the real roots we choose $\delta$ positive in accordance with (2.17). For $\gamma>0$ (repulsive case), there are, in fact, only real roots as Eqs. (A4) show. For every solution to either Eq. (A4a) or (A4b) there exists an integer $n_1$ such that Eq. (A3) is satisfied, and conversely for every integer $n_1\geq 1$ there is a unique positive solution to Eq. (A3) given by either (A4a) or (A4b). The single exception is $n_1=0$ for which there can be no solution because the first term on the left side of Eq. (A3) is negative ($\delta=0$ is not allowed). We see that

$$2\pi(n_1-1)<\delta<2\pi n_1. \qquad (A5)$$

In the limiting cases the roots corresponding to $n_1=1$,

2, 3, etc., are, respectively,

$$\delta=0, 2\pi, 4\pi, \text{ etc.} \quad (\gamma=0) \qquad (A6a)$$

$$\delta=2\pi, 4\pi, 6\pi, \text{ etc.} \quad (\gamma=\infty). \qquad (A6b)$$

We have now to determine $k_1$ and $k_2$ separately so as to satisfy Eqs. (A1). Substituting the value of $\theta_{21}$ from Eq. (A3) into Eq. (A1) we see that for every root, $\delta$, there is in fact a unique solution satisfying (2.23) given by

$$Lk_2=-Lk_1=\tfrac{1}{2}\delta; \quad p=0, \quad (n_1 \text{ odd}) \qquad (A7a)$$

$$Lk_2=\tfrac{1}{2}\delta+\pi; \quad Lk_1=-\tfrac{1}{2}\delta+\pi;$$
$$p=2\pi/L=\pi\rho, \quad (n_1 \text{ even}). \qquad (A7b)$$

Thus, we see that for every integral value of $n_1\geq 1$ there corresponds a unique state having a definite energy and a momentum in the range (2.23). The integer $n_1$ may, therefore, be considered the second or "internal," quantum number of the problem which, together with $n_0$, specifies a state. The ground state corresponds to $n_1=1$ and has zero momentum as expected.

For $\gamma<0$ (attractive case), the situation is more complicated and rather surprising. The real roots are much the same as before except that here $n_1=0$ becomes permissible if $\gamma>-2$. Equation (A5) is changed to

$$2\pi n_1<\delta<2\pi(n_1+1), \qquad (A5a)$$

while Eqs. (A7) remain unaltered. For the imaginary roots, it is wise to return to the original definition of $e^{i\theta}$ [Eq. (2.12)] in order to avoid ambiguity in the definition of $\tan^{-1}$. One finds that for $n_1=1$ there is always a pair of equal and opposite imaginary roots given by Eq. (A4b). Only one of them need be considered, for the other corresponds to interchanging $k_1$ and $k_2$. If we write $\delta=i\alpha$ $(\alpha>0)$, then

$$\alpha>-2\gamma. \qquad (A8)$$

By Eq. (A7a) this state has zero momentum and its energy is

$$E_0=-(1/2L^2)\alpha^2<-\tfrac{1}{2}c^2. \qquad (A9)$$

This state, which is the ground state of the problem, is the analog of the single bound state for an attractive delta function in infinite space for which $E_0=-\tfrac{1}{2}c^2$, $\psi=\exp(-\tfrac{1}{2}c|x_1-x_2|)$. Putting the particles in a box with infinite walls (zero boundary conditions) would reduce the binding energy, however, contrary to Eq. (A9). The periodic boundary conditions are responsible for this anomalous result.

Even more surprising is the appearance of a second bound state. We saw above that for $\gamma>-2$ there is an $n_1=0$ real solution. In this case there is no other bound state. But if $\gamma<-2$ (strong coupling) the real solution for $n_1=0$ disappears in favor of an imaginary solution

to Eq. (A4a). For this solution

$$\alpha' < -2\gamma. \tag{A10}$$

By Eq. (A7b) this state has a momentum $p=2\pi/L$ and an energy

$$E_0' = -\frac{1}{2L^2}(\alpha')^2 + \frac{2}{L^2}\pi^2 > -\tfrac{1}{2}c^2. \tag{A11}$$

For $\gamma = -2$ the $n_1 = 0$ root is $\delta = 0$ which must be discarded. On the other hand, the state cannot disappear for a single value of $\gamma$. The paradox is resolved by taking the limit $\gamma \to -2$, whereupon we find

$$\psi' = [2 + c(x_2 - x_1)] \exp\left[ i\frac{\pi}{L}(x_1 + x_2) \right], \quad (\gamma = -2), \tag{A12}$$

in $R_1$. Two facts should be noticed about this second bound state. Firstly, it has a nonzero momentum, but it does *not* correspond to the translation of a bound state with $p=0$. Translated states of two particles with zero momentum must have momenta which are integral multiples of $4\pi/L$. Secondly, this state does not disappear in the limit $L \to \infty$ as might have been hoped. In fact, for *any* fixed $c$ we can always choose $L$ large enough such that $\gamma < -2$ and this second state will make its appearance. As $L \to \infty$ both inequalities (A8) and (A10) approach equalities so that $E_0$ and $E_0'$ becomes asymptotically degenerate and equal to the infinite space binding energy.

Having obtained these wave functions we may ask if the *ansatz* Eq. (2.9) [with the single exception, Eq. (A12)] exhausts all the solutions of the Schrödinger equation. It seems clear that it, in fact, does so, at least for the repulsive case, although we cannot prove that this is true. We are led to this view because in the repulsive case we obtain the well-known wave functions for $\gamma = 0$ (free particles) and we obtain Girardeau's solution for $\gamma = \infty$ (hard cores), with a continuous transition from one to the other. For $\gamma < 0$, on the other hand, the situation is not quite so clear. As $\gamma \to 0$ we recover the free particle solutions. For $\gamma \neq 0$ we can only observe that as $L \to \infty$ we obtain the well-known bound state and scattering solutions for the attractive $\delta$-function potential.

These same considerations apply to the $N$-particle problem and we shall therefore suppose that Eqs. (2.9) and (2.15) yield all the wave functions of the problem for the repulsive case.

From consideration of the two-body problem we are led to the following hypothesis: For the repulsive case every solution $\{k\}$ contains only real $k$'s, which we may, therefore, order according to (2.17). For the attractive case, which we shall not consider further, complex $k$'s may appear. One should expect many types of nonreal solutions for $\gamma < 0$ corresponding to two-body, three-body, etc., bound states.

## APPENDIX B. SOLUTION OF THE BASIC INTEGRAL EQUATION

### I. Existence, Uniqueness, and Analyticity of the Solution

(i) The integral equation (3.18) is a special case of

$$r(y) + \int_{-1}^{1} K(y-x)g(x)dx = \sigma g(y), \tag{B1}$$

where

$$K(y) = 2\lambda/(\lambda^2 + y^2)$$

and $\lambda > 0$. Equation (B1) can be brought into the form

$$u(s) - \mu \int_{-a}^{a} R(s-t)u(t)dt = v(s), \tag{B2}$$

by letting $y = \lambda s$, $x = \lambda t$, $a = 1/\lambda < \infty$, $\mu = 2/\sigma$, $u(s) = g(y)$, $v(s) = r(y)/\sigma$, and

$$R(s,t) = R(s-t) = 1/[1 + (s-t)^2]. \tag{B3}$$

By definition,[7] $R$ is positive definite if

$$J_a = \int_{-a}^{a} \int_{-a}^{a} R(s,t)u(s)u(t)dsdt > 0, \tag{B4}$$

for any square integrable $u(s) \not\equiv 0$. For $R$ to be positive definite with respect to the interval $(-a, a)$ it is sufficient that $R$ be positive definite with respect to $(-\infty, \infty)$ because the square integrable functions with respect to $(-a, a)$ can be thought of as a subset of square integrable functions with respect to $(-\infty, \infty)$ which vanish identically for $|s| > a$. Let

$$\eta(s) = \int_{-\infty}^{\infty} R(s-t)u(t)dt \quad \text{and} \quad u^*(p), \eta^*(p)$$

be the Fourier transforms of $u(s)$ and $\eta(s)$, respectively. Then, according to Parseval's equation,[8] and because $\eta(s)$ is real,

$$J_\infty = \int_{-\infty}^{\infty} u(s)\eta(s)ds = \int_{-\infty}^{\infty} u^*(p)\bar{\eta}^*(p)dp.$$

As $\eta(s)$ is defined by a convolution, one has

$$\eta^*(s) = (2\pi)^{1/2}R^*(p)u^*(p),$$

and, thus

$$J_\infty = (2\pi)^{1/2} \int_{-\infty}^{\infty} \bar{R}^*(p)|u^*(p)|^2 dp.$$

[7] R. Courant and D. Hilbert, *Methoden der Mathematischen Physik* (Springer-Verlag, Berlin, 1931), Vol. I, p. 105.
[8] W. Schmeidler, *Integralgleichungen mit Anwendungen in Physik und Technik* (Akademische Verlagsgesellschaft, Leipzig, 1955), Vol. 1, pp. 74, 75.

It is well known[9] that

$$R^*(p) = \bar{R}^*(p) = (\pi/2)^{1/2} e^{-|p|} > 0,$$

and, thus, $J_\infty > 0$ except for $u(s) \equiv 0$, that is (B4) is proved. Since $R$ is positive definite, all eigenvalues of (B1) or (B2) are positive.[10]

(ii) Consider the kernel $R$ and its iterates[11]

$$R^{(1)}(s,t) = R(s,t), \qquad (B5a)$$

$$R^{(i+1)}(s,t) = \int_{-a}^{a} R^{(i)}(s,\tau) R(\tau,t) d\tau, \quad (i \geq 1). \quad (B5b)$$

From the definition (B3) of $R$ it is evident that there exist constants $C_j > 0$ such that $|\partial^j R^{(1)}/\partial s^j| \leq C_j$ for any $j \geq 0$. One has

$$\int_{-a}^{a} R(\tau,t) d\tau = T(a,t) = \arctan(a-t) + \arctan(a+t). \quad (B6)$$

It is easy to see that

$$\pi/2 < \arctan 2a \leq T(a,t) \leq 2 \arctan a = \pi - \epsilon, \quad \epsilon > 0. \quad (B7)$$

Assume now that, for a certain $i \geq 1$ and any $j \geq 0$,

$$|\partial^j R^{(i)}(s,t)/\partial s^j| \leq C_j (\pi - \epsilon)^{i-1}.$$

Then, because $\partial^j R^{(i)}(s,t)/\partial s^j$ is continuous in $s$ for all $j$, one can differentiate under the integral sign and, using (B7), one gets

$$\left| \frac{\partial^j R^{(i+1)}(s,t)}{\partial s^j} \right| = \left| \int_{-a}^{a} \frac{\partial^j R^{(i)}(s,\tau)}{\partial s^j} \right.$$
$$\left. \times R(\tau,t) d\tau \right| \leq C_j (\pi - \epsilon)^i. \quad (B8)$$

As (B8) holds for $i=0$ it follows by induction that (B8) holds for all $i \geq 0$. Therefore, the Neumann series and its derivatives with respect to $s$ taken term by term, that is,

$$\frac{\partial^j \rho(s,t)}{\partial s^j} = \sum_{i=0}^{\infty} \mu^i \frac{\partial^j R^{(i+1)}(s,t)}{\partial s^j} \quad (B9)$$

for $j \geq 0$, converge absolutely and uniformly in both variables $s$ and $t$ for all $\mu < 1/(\pi - \epsilon)$. The same result holds for the derivatives with respect to $t$ as can be proved in a similar way.

From the uniform convergence of the Neumann series for $j=0$ follows first the existence of the resolvent kernel, $\rho(s,t)$, in particular for $\mu = 1/\pi$, and the validity of the representation[12]

$$u(s) = v(s) + \mu \int_{-a}^{a} \rho(s,t) v(t) dt, \quad (B10)$$

[9] G. A. Campbell and R. M. Foster, *Fourier Integrals for Practical Applications* (American Telephone and Telegraph Company, New York, 1942), p. 45.
[10] Reference 7, p. 112.
[11] Reference 8, p. 270.
[12] Reference 7, p. 119.

or of the equivalent representation

$$g(y) = \frac{1}{\sigma} r(y) + \frac{1}{\sigma^2} \int_{-1}^{1} \kappa(y,x) r(x) dx, \quad (B11a)$$

$$\kappa(y,x) = \sum_{i=0}^{\infty} \sigma^{-i} K^{(i+1)}(y,x). \quad (B11b)$$

The eigenvalues of (B2) are, thus, all $\geq 1/(\pi - \epsilon)$ and those of (B1) are $\leq 2\pi - 2\epsilon$. Property (a) of Sec. III, that is, the existence of a unique solution of Eq. (B1) for $\sigma = 2\pi$, is thus proved. From the uniform convergence of the series in (B9) with $j \geq 1$ follows the analyticity of the resolvent kernel with respect to $s$ and $t$.

(iii) Let $\lambda = z$ be complex in the expressions for the kernel $K$ and its iterates. Consider a closed circular domain $\bar{D}$ defined by $|z - \lambda_1| \leq \delta$, with real $\lambda_1$, $\delta$ and $0 < \delta < \lambda_1/\sqrt{2}$. Then, if $y$ is real, it is easy to see that $|K(y;z)| \leq 2(\lambda_1 + \delta)/(\lambda_1 - \delta)^2$ for any $z \in \bar{D}$. Let $[\pi - \epsilon_1(\delta)] = 2 \arctan[2/(\lambda_1 - \delta)]$, $\epsilon_1(\delta) > 0$. As $\delta \to 0$, the quantity $q(\delta) = [(\lambda_1 + \delta)/(\lambda_1 - \delta)] \cdot [\pi - \epsilon_1(\delta)]$ decreases monotonically to $(\pi - \epsilon) = 2 \arctan(2/\lambda_1)$, where $\epsilon_1(\delta) \leq \epsilon$. Thus, if $0 < \epsilon_2 < \epsilon$, $q(\delta) \leq \pi - \epsilon_2$ for sufficiently small $\delta$. Now assume that, for a certain $i \geq 1$ and real $y$, $x$, $\xi$,

$$|K^{(i)}(y,x;z)| \leq 2^i \frac{\lambda_1 + \delta}{(\lambda_1 - \delta)^2} (\pi - \epsilon_2)^{i-1}. \quad (B12)$$

Then,

$$|K^{(i+1)}(y,x;z)| = \left| \int_{-1}^{1} K^{(i)}(y,\xi;z) K(\xi,x;z) d\xi \right|$$

$$\leq 2^i \frac{\lambda_1 + \delta}{(\lambda_1 - \delta)^2} (\pi - \epsilon_2)^{i-1} \int_{-1}^{1} \frac{2(\lambda_1 + \delta) d\xi}{(\lambda_1 - \delta)^2 + (\xi - x)^2},$$

and

$$\int_{-1}^{1} \frac{2(\lambda_1 + \delta) d\xi}{(\lambda_1 - \delta)^2 + (\xi - x)^2} = \frac{2(\lambda_1 + \delta)}{(\lambda_1 - \delta)} \int_{-1}^{1} \frac{(\lambda_1 - \delta) d\xi}{(\lambda_1 - \delta)^2 + (\xi - x)^2}$$

$$\leq 2q(\delta) \leq 2(\pi - \epsilon_2);$$

thus, (B12) holds for $i+1$. As shown above, (B12) does hold for $i=1$, and by induction it follows that (B12) holds for all $i \geq 1$. As a consequence, the Neumann series (B11b) with $\lambda = z$ converges uniformly with respect to $z$ in $\bar{D}$. The kernel $K$ and its iterates are analytic functions of $z$ in $\bar{D}$ because $\bar{D}$ does not intersect the imaginary axis, the only place where $K$ and its iterates could have singularities. Thus, for $\sigma \geq 1/2\pi$, (B11b) represents a uniformly converging series of analytic functions and, as a consequence,[13] $\kappa(y,x;z)$ is analytic in $z$ in the neighborhood of any $\lambda_1 > 0$. In particular, $\kappa$ is analytic in $\lambda$ in the real sense. Finally, from (B11a) follows the analyticity of $g(y)$ as a function of $\lambda$. This completes the proof of statement (b) of Sec. III, that is $g(y)$ is analytic in $\lambda$ for $\sigma = 1/2\pi$.

[13] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, New York, 1952), p. 91.

(iv) Using $1/(1+4a^2) \leq |R(s,t)| \leq 1$ and inequality (B7) one can prove by induction that

$$[1/(1+4a^2)](\pi/2)^{i-1} \leq |R^{(i)}(s,t)| \leq (\pi-\epsilon)^{i-1}.$$

Thus, there exist constants, $b_1$ and $b_2$, such that, for $\mu \leq 1/\pi$,

$$0 < b_1 = [1/(1+4a^2)]\frac{1}{1-\mu(\pi/2)} \leq |\rho(s,t)|$$

$$\leq \frac{1}{1-\mu(\pi-\epsilon)} = b_2 < \infty \quad \text{(B.13)}$$

If there exist constants $b_3$ and $b_4$ such that $b_3 \leq v(s) \leq b_4$, then one obtains from (B10)

$$b_3(1+2ab_1) \leq u(s) \leq b_4(1+2a\mu b_2), \quad \text{(B14)}$$

which proves statement (d) of Sec. III, that is if $r(y)$ is bounded above or below, then so is the solution $g(y)$. The proof of statement (c) of Sec. III [that is, if $r(y)$ is positive definite, then so is the solution $g(y)$] follows

immediately from (B14) as $b_3$ can be chosen positive in this case.

## II. Numerical Solution

The basic integral equation (B2) has been solved numerically by applying Simpson's rule to the integral on a grid $\{s_i = -a + (i-1)h; h = a/n; i = 1, \cdots, 2n+1\}$. This yields a system of $(2n+1)$ linear algebraic equations for the $(2n+1)$ discrete approximate values $u_i \approx u(s_i)$ which can be solved by a standard method. The quadratures involved in calculating $\gamma$ and $e$ also have been carried out by Simpson's rule. The functions $K(\gamma)$ and $e(\gamma)$ are obtained in parametric form, that is $(K(\lambda), \gamma(\lambda))$ and $(e(\lambda), \gamma(\lambda))$. To obtain $\mu$, the quantities $e$ and $\gamma$ are evaluated on a sufficiently fine grid of equidistant $\lambda$ values. Then,

$$\mu = 3e - \gamma\frac{de}{d\gamma} = 3e - \gamma\frac{de}{d\lambda}\Big/\frac{d\gamma}{d\lambda}$$

can be calculated by numerical differentiation.

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# Exact Analysis of an Interacting Bose Gas. II. The Excitation Spectrum

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We continue the analysis of the one-dimensional gas of Bose particles interacting via a repulsive delta function potential by considering the excitation spectrum. Among other things we show that: (i) the elementary excitations are most naturally thought of as a double spectrum, not a single one; (ii) the velocity of sound derived from the macroscopic compressibility is shown to agree with the velocity of sound derived from microscopic considerations, i.e., from the phonon spectrum. We also introduce a distinction between elementary excitations and quasiparticles, on the basis of which we give some heuristic reasons for expecting the double spectrum to be a general feature, even in three dimensions, and not an exception.

## I. INTRODUCTION

IN the preceding paper[1] we introduced a soluble model of a Bose gas interacting via a repulsive δ-function potential. We discussed the nature of the eigenfunctions and explicitly calculated the ground-state energy and other properties of the ground state in the limit of a large system.

In this paper we discuss the nature of the excitation spectrum for a large system of $N$ particles. The surprising result, as we stated but did not show in I, is that for all values of the potential strength, the most convenient and natural way to view the spectrum is to regard it as a *double* spectrum of elementary boson excitations. While Bogoliubov's perturbation theory[2,3]

gives one of the spectra quite accurately for a weak potential, the second spectrum is entirely unaccounted for (see Figs. 3 and 4). The second spectrum exists only for values of the momentum satisfying $|p| \leq \pi\rho$.

We may summarize the results of this paper as follows: (i) In Sec. II we discuss the nature of the energy spectrum of the problem and show that there are two elementary spectra. These are always well defined and are explicitly calculated. We show that there is no energy gap and that the two spectra have a common slope at $p=0$ which means that they propagate sound at the same velocity. The velocity of sound at absolute zero derived in this way from an atomic picture[4-6] is shown to be identical with the velocity of sound defined by the usual macroscopic considerations [cf. Eqs. (1.1) and (1.4)].

[1] E. Lieb and W. Liniger, Phys. Rev. **130**, 1605 (1963) (referred to here as I).

[2] See I, Sec. IV.

[3] See, for example, *The Many Body Problem*, edited by C. DeWitt (John Wiley & Sons, Inc., New York, 1958), pp. 347–355.

[4] R. P. Feynman, Phys. Rev. **91**, 1291 (1953).

[5] R. P. Feynman, Phys. Rev. **91**, 1301 (1953).

[6] R. P. Feynman, Phys. Rev. **94**, 262 (1954).