Volume Properties of Ground-State Nuclear Matter*

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An approximation developed by one of the authors for treating macroscopic bound systems in the ground state is applied to a homogeneous system of nuclear matter using a simple separable potential. More extensive numerical results are given for the density, energy density, chemical potential, effective potential, momentum distribution, pressure, and density correlation function. In addition, we discuss an ambiguity in the calculation of the pressure arising directly from the approximation. This ambiguity and its effect on the nuclear parameters at the physically meaningful point of zero pressure is treated in detail. In particular, three different expressions for the pressure are derived which lead to zero-pressure values of -14.4, -17.0, and -17.5 MeV for the binding energy/particle and 0.87, 0.95, and 1.01 F for the interparticle spacing. A numerical examination of the density correlation function suggests, as expected, that correlations of more than two particles are rare at nuclear densities.

I. INTRODUCTION

IN a previous paper,[1] one of the authors developed an approximation for calculating the ground-state properties of nuclear matter based upon the general formalism of Martin and Schwinger.[2] In this paper we present more extensive numerical results for the volume properties of nuclear matter, and discuss certain ambiguities which appear in the calculation of the pressure. A later paper will be devoted to an extension of the approximation to the calculation of surface properties.

The basic approximation involves a treatment of two-particle correlations which is sufficient to remove hard-core divergences and which allows for the effect of these correlations on both the energy-momentum relation and the momentum distribution. The approximation leads to a two-particle scattering matrix which is independent of many-body effects, so that only one-particle equations need to be solved self-consistently. As a result, the necessary numerical computations are straightforward, particularly for a simple interparticle potential.

The development of the approximation leads to a set of equations which determine various parameters of nuclear matter as functions of the density. To determine these parameters for the physically meaningful case of an unrestrained system, one must impose the requirement that the pressure be zero. This requirement may be formulated in several ways which are completely equivalent in an exact theory. For

example, we know that at zero temperature and pressure the chemical potential equals the total energy per particle. An equivalent statement of the $T=0$, $P=0$ condition is that the slope of $(E/N)(\rho)$ be zero. The former condition was the first imposed in connection with the application of the Martin-Schwinger formalism to nuclear matter.[1] The second condition was used by Falk and Wilets[3] in their investigation of nuclear compressibility. Numerical results for the parameters of interest were different in the two cases. Such differences are to be expected unless the approximation is capable of giving correctly both the chemical potential and the slope of the $(E/N)(\rho)$ curve at saturation. The extent of these differences reflects the error involved in our approximation.

There are, of course, other elementary relations among the thermodynamics quantities which in principle can be used to find the density of the zero-pressure system. One can also use for the pressure an expression which we will derive from momentum transport considerations. All of these relations would give the same results in an exact calculation, but the answers may differ to some extent in any approximation. The best choice probably depends on just what quantity is to be calculated, and it is our feeling that there is no clear-cut basis at present for preferring one relation over another. We treat on an equal footing the choices used in references 1 and 3 together with the expression which we will derive from momentum transport considerations.

In the following sections we discuss the general derivation of the approximation, the calculation of the parameters for a homogeneous system as functions of the density, the calculation of the pressure and the parameters of the system at zero pressure, and the calculation and interpretation of the density correlation function.

[1] R. D. Puff, Ann. Phys. (N.Y.) **13**, 317 (1961).
[2] P. C. Martin and J. Schwinger, Phys. Rev. **115**, 1342 (1959).
[3] D. S. Falk and L. Wilets, Phys. Rev. **124**, 1887 (1961).

II. THE APPROXIMATION

Since a detailed derivation of the approximation which we use has been given in reference 1, we will limit ourselves to a brief recapitulation. We deviate from reference 1 by postponing the transformation into momentum space to as late a stage as possible (anticipating the extension of the approximation to inhomogeneous systems) and by avoiding the use of Fourier series expansions over a finite time interval, which have rather obscure physical significance.

A many-fermion system may be described by a sequence of n-particle Green's functions describing the propagation of n particles through a many-body background. These functions are defined as expectation values of time-ordered products of field operators,

$$G_n(1\cdots n; 1'\cdots n') \equiv (-i)^n \exp[i\mu \sum_i (t_i - t_i')]$$
$$\times \langle T(\psi(1)\cdots\psi(n)\psi^\dagger(n')\cdots\psi^\dagger(1'))\rangle,$$

where each numerical argument is used to indicate a set of space-time coordinates (as well as a discrete internal coordinate specifying spin and isospin, which we will usually neglect). T is the usual time-ordering symbol, and the exponential phase factor is inserted to simplify boundary conditions.

A set of coupled integrodifferential equations for the Green's functions may be obtained from the equation of motion for the field operators. For a nonlocal potential v these equations are

$$\left[i\frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2m} + \mu\right]G_n(1\cdots n; 1'\cdots n') + i\int \langle 1, n+1|v|1'', n+1''\rangle G_{n+1}(1''2\cdots n, n+1''; 1'\cdots n', n+1^+)$$

$$= \sum_{j=1}^n (-1)^{j-1}\delta(1-j')G_{n-1}(2\cdots n; 1'\cdots j-1', j+1'\cdots n'). \quad (1)$$

The integrations extend over space and time coordinates and include an implied sum over internal coordinates. The notation n^+ refers to $t_n^+ = t_n + 0$, and it indicates the correct ordering of field operators in the potential term.

The expectation values of various operators can be expressed in terms of the G's; in particular the number and Hamiltonian operators may both be written in terms of G_1. Furthermore, the nature of the expectation value used in defining the Green's functions acts as a boundary condition on Eqs. (1). We are actually interested in the expectation value

$$\langle X\rangle_{0N} = \langle N,E_0(N)|X|N,E_0(N)\rangle$$

for a state of N particles (in a fixed volume) with ground-state energy $E_0(N)$. However, it is convenient to treat this as the low-temperature limit of a grand canonical expectation value, $\langle X\rangle^{i\tau,\mu} = \text{Tr}[e^{-i\tau(H-\mu N)}X]/\text{Tr}[e^{-i\tau(H-\mu N)}]$, which is taken over a complete ensemble of states with temperature $1/i\tau$ and chemical potential μ. As $i\tau \to \infty$ the weighting exponential picks out the state or states for which $E - \mu N$ is a minimum. This occurs at $E = E_0(N)$, where N is such that

$$dE_0(N)/dN = \mu. \quad (2)$$

The advantage of the canonical expectation value lies in the cyclic properties of the trace, which lead to

antiperiodic boundary conditions on the Green's functions for real τ. Thus,[2]

$$G_n(t_1 = \tau) = -G_n(t_1 = 0), \quad (3)$$

if τ is real and positive and all time coordinates lie in the interval 0 to τ.

The use of these boundary conditions for real τ necessitates using analytic continuation to obtain the limit $i\tau \to \infty$. For a one-particle function such as G_1 this continuation may be carried out by expressing the Green's function in terms of a spectral function. Thus, one defines two continuous functions $G_>$ and $G_<$ by

$$G_1(1; 1') = G_>(1; 1'), \quad t_1 > t_1'$$
$$= G_<(1; 1'), \quad t_1 < t_1'. \quad (4)$$

Then, an expansion of $G_>$ and $G_<$ as traces shows that for arbitrary complex τ

$$G_>(\mathbf{r}, t+\tau; \mathbf{r}'t') = -G_<(\mathbf{r}t; \mathbf{r}'t'). \quad (5)$$

This relation establishes a proportionality between the Fourier time-transforms of $G_>$ and $G_<$. By expressing these transforms symmetrically in terms of a function A, called the spectral function, one may write G_1 in a form which insures the satisfaction of the boundary conditions,

$$G_1(1; 1') = \int \frac{d\omega}{2\pi i} e^{-i\omega(t_1 - t_1')} A(\mathbf{r}_1\mathbf{r}_1'\omega) \times \begin{cases} \dfrac{1}{1+e^{-i\omega\tau}}, & t > t' \\[2ex] \dfrac{-1}{1+e^{i\omega\tau}}, & t < t'. \end{cases} \quad (6)$$

The magnitude of the discontinuity in G_1 at $t_1=t_1'$, which is determined by the field operator anticommutation relations, establishes a normalization condition on the spectral function,

$$\int \frac{d\omega}{2\pi} A(\mathbf{r}_1\mathbf{r}_1'\omega) = \delta(\mathbf{r}_1 - \mathbf{r}_1'). \tag{7}$$

The number and energy averages may be written in terms of A as

$$\langle N \rangle = \int d\mathbf{r}\, \rho(\mathbf{r}); \quad \langle H \rangle = \int d\mathbf{r}\, \epsilon(\mathbf{r}), \tag{8}$$

where

$$\rho(\mathbf{r}) = \int \frac{d\omega}{2\pi} \frac{A(\mathbf{r}\mathbf{r}\omega)}{1+e^{i\omega\tau}};$$

and

$$\epsilon(\mathbf{r}) = \int \frac{d\omega}{2\pi} \lim_{\mathbf{r}'\to\mathbf{r}} \frac{1}{2}\left[\omega - \frac{\nabla^2}{2m} + \mu\right] \frac{A(\mathbf{r}\mathbf{r}'\omega)}{1+e^{i\omega\tau}} \tag{9}$$

are particle and energy densities which will be constant for a homogeneous system.

The use of a spectral representation for G_1 is a special case of a general method for analytically continuing any function $F(t-t')$ which is discontinuous only at $t=t'$ and satisfies either periodic or antiperiodic boundary conditions within 0 to τ. These conditions imply that the function may be expressed in terms of a spectral function $\mathfrak{F}(\omega)$ by

$$F(t-t') = F_0\delta(t-t') + \int \frac{d\omega}{2\pi i} e^{-i\omega(t-t')}\mathfrak{F}(\omega) \times \begin{cases} \dfrac{1}{1\mp e^{-i\omega\tau}}, & t>t' \\[2mm] \dfrac{-1}{1\mp e^{i\omega\tau}}, & t<t' \end{cases} \tag{10}$$

where the upper and lower signs are appropriate for periodic and antiperiodic functions, respectively. We may then define a function of complex energy,

$$F(\omega) = F_0 + \int \frac{d\omega'}{2\pi} \frac{\mathfrak{F}(\omega')}{\omega-\omega'}, \tag{11}$$

such that the spectral function is just the discontinuity in $F(\omega)$ across the real axis,

$$\mathfrak{F}(\omega) = \lim_{\epsilon\to+0} (1/i)[F(\omega-i\epsilon) - F(\omega+i\epsilon)]. \tag{12}$$

The replacement $F(t-t') \to \mathfrak{F}(\omega) \to F(\omega)$ is a linear transformation which obeys general convolution and product rules. One may show that if

$$F(t-t') = \int_0^\tau dt''\, G(t-t'')H(t''-t'), \tag{13}$$

when $0<t$, $t'<\tau$, and F, G, and H are all periodic or all antiperiodic, then

$$F(\omega) = G(\omega)H(\omega). \tag{14}$$

On the other hand, if

$$F(t-t') = G(t-t')H(t-t'), \tag{15}$$

where neither G nor H contains delta functions, then

$$F(\omega) = \frac{1}{i} \int \frac{d\omega_1 d\omega_2}{(2\pi)^2} \frac{\mathcal{G}(\omega_1)\mathcal{H}(\omega_2)}{\omega-\omega_1-\omega_2} \{\tfrac{1}{2}[\coth(\tfrac{1}{2}i\omega_1\tau)]^{\epsilon_1} + \tfrac{1}{2}[\coth(\tfrac{1}{2}i\omega_2\tau)]^{\epsilon_2}\}$$

$$= \int \frac{d\omega'}{2\pi i}\{\tfrac{1}{2}[\coth(\tfrac{1}{2}i\omega'\tau)]^{\epsilon_1}\mathcal{G}(\omega')H(\omega-\omega') + \tfrac{1}{2}[\coth(\tfrac{1}{2}i\omega'\tau)]^{\epsilon_2}G(\omega-\omega')\mathcal{H}(\omega')\}, \tag{16}$$

where the script letters designate spectral functions and $\epsilon_1=+1$ if G is periodic or -1 if G is antiperiodic, while ϵ_2 depends similarly on H. Finally, if

$$F(t-t') = G(t'-t), \tag{17}$$

then

$$\mathfrak{F}(\omega) = -\mathcal{G}(-\omega), \tag{18}$$

and

$$F(\omega) = G(-\omega). \tag{19}$$

To produce a computable approximation, the infinite set of coupled equations (1) must be truncated by approximating some G_n by a combination of smaller G's. For this purpose the equations for G_1, G_2, and G_3 may be manipulated to obtain an equation for G_2 which is symmetric in variables 1 and 2,

$$\left[i\frac{\partial}{\partial t_1}+\frac{\nabla_1^2}{2m}+\mu\right]\left[i\frac{\partial}{\partial t_2}+\frac{\nabla_2^2}{2m}+\mu\right][G_2(12;1'2')-G_1(1;1')G_1(2;2')+G_1(1;2')G_1(2:1')]-i\int\langle12|v|\bar1\bar2\rangle G_2(\bar1\bar2;1'2')$$

$$=-\int\langle13|v|\bar1\bar3\rangle\langle24|v|\bar2\bar4\rangle\{G_4(\bar1\bar2\bar3\bar4;1'2'3^+4^+)-G_2(\bar1\bar3;1'3^+)G_2(\bar2\bar4;2'4^+)+G_2(\bar1\bar3;2'3^+)G_2(\bar2\bar4;1'4^+)\}. \quad (20)$$

Our approximation is obtained by neglecting the right-hand side of this equation, and a discussion of its interpretation can be found in reference 1. We may rewrite both the approximate expression for G_2 and the exact equation for G_1 in integral form by introducing the function G_1^0 which satisfies Eq. (1) without the interaction term, together with the boundary condition (5). Thus, we have

$$G_2(12;1'2')\approx G_1(1;1')G_1(2;2')-G_1(1;2')G_1(2;1')+i\int G_1^0(1;\bar1)G_1^0(2;\bar2)\langle\bar1\bar2|v|\bar1'\bar2'\rangle G_2(\bar1'\bar2';1'2'), \quad (21)$$

and

$$G_1(1;1')=G_1^0(1;1')-i\int G_1^0(1;\bar1)\langle\bar12|v|\bar1'2'\rangle G_2(\bar1'2';1'2^+), \quad (22)$$

where the time integrations as well as the free time variables are restricted to the interval 0 to τ.

We then define the function T by

$$\int\langle12|T|\bar1\bar2\rangle G_1(\bar1;1')G_1(\bar2;2')=\int\langle12|v|\bar1\bar2\rangle G_2(\bar1\bar2;1'2'), \quad (23)$$

and so obtain

$$\langle12|T|1'2'\rangle=\langle12|v|1'2'\rangle-\langle12|v|2'1'\rangle+i\int\langle12|v|\bar1\bar2\rangle G_1^0(\bar1;\bar1')G_1^0(\bar2;\bar2')\langle\bar1'\bar2'|T|1'2'\rangle. \quad (24)$$

T is symmetric in the unprimed and primed coordinates and also independent of G_1. The substitution of its definition into the exact differential equation for G_1 gives

$$\left[i\frac{\partial}{\partial t_1}+\frac{\nabla_1^2}{2m}+\mu\right]G_1(1;1')$$

$$-\int\Sigma(1;\bar1)G_1(\bar1;1')=\delta(1-1'), \quad (25)$$

where

$$\Sigma(1;\bar1)=-i\int\langle12|T|\bar1\bar2\rangle G_1(\bar2;2^+) \quad (26)$$

is the usual self-energy function (or effective potential), which is nonlocal in both space and time.

It is, of course, possible to investigate the Green's function equations by manipulating the primed indices. The exact expression for G_1 is

$$G_1(1:1')=G_1^0(1;1')$$

$$-i\int G_2(12'^-;\bar1\bar2)\langle\bar1\bar2|v|\bar1'2'\rangle G_1^0(\bar1';1'), \quad (27)$$

and the G_2 equation, within our approximation on the higher order functions, is

$$G_2(12;1'2')\approx G_1(1:1')G_1(2;2')-G_1(1;2')G_1(2;1')$$

$$+i\int G_2(12;\bar1\bar2)\langle\bar1\bar2|v|\bar1'\bar2'\rangle G_1^0(\bar1';1')G_1^0(\bar2';2'). \quad (28)$$

Now, we find $G_2v=G_1G_1T$ (instead of 23), and G_1 is again given in terms of T by Eqs. (25) and (26). However, although this procedure gives exactly the same G_1, it is clear that the G_2 given by (21) is not the same as the solution to (28), even though both forms have the proper symmetry upon interchange of indices 1 and 2 or 1' and 2'. Baym and Kadanoff have shown[4] that if G_1 satisfies both (22) and (27) with the same approximate G_2, and if that G_2 has the correct 1-2 and 1'-2' symmetry, then all of the basic conservation laws and simple sum rules which follow directly from the definition of G_1 are satisfied. The fact that our approximation will give the same G_1 from (22) and its "adjoint" equation (27) only by using two different expressions for G_2 shows that we have not satisfied the Baym-Kadanoff sufficiency conditions. Since the conservation laws are, of course, used in the derivation of many relations between thermodynamics quantities, this is a

[4] G. Baym and L. P. Kadanoff, Phys. Rev. **124**, 287 (1961); G. Baym, *ibid.* **127**, 1391 (1962).

possible source of the discrepancy between numerical results obtained from the same approximation when different exact thermodynamic relations are used for finding the point of zero pressure. It is certainly to be regarded as a flaw in the approximation, since the ambiguity in the proper pressure relation is conceptually disturbing even in cases where it is not numerically

important.[5] In any case, we investigate the importance of this difficulty for nuclear matter.

The fact the v is an instantaneous interaction allows the substitution

$$\langle 12|T|1'2'\rangle = \delta(t_1-t_2)\langle \mathbf{r}_1\mathbf{r}_2 t_1|T|\mathbf{r}_1'\mathbf{r}_2't_1'\rangle\delta(t_1'-t_2'), \quad (29)$$

where the common-time T satisfies

$$\langle \mathbf{r}_1\mathbf{r}_2 t|T|\mathbf{r}_1'\mathbf{r}_2't'\rangle = \{\langle \mathbf{r}_1\mathbf{r}_2|v|\mathbf{r}_1'\mathbf{r}_2'\rangle - \langle \mathbf{r}_1\mathbf{r}_2|v|\mathbf{r}_2'\mathbf{r}_1'\rangle\}\delta(t-t')$$

$$+ \int \langle \mathbf{r}_1\mathbf{r}_2|v|\mathbf{r}_1''\mathbf{r}_2''\rangle\Lambda^{00}(\mathbf{r}_1''\mathbf{r}_2''t;\mathbf{r}_1'''\mathbf{r}_2'''t'')\langle \mathbf{r}_1'''\mathbf{r}_2'''t''|T|\mathbf{r}_1'\mathbf{r}_2't'\rangle, \quad (30)$$

and

$$\Lambda^{00}(\mathbf{r}_1\mathbf{r}_2 t;\mathbf{r}_1'\mathbf{r}_2't') = iG_1^0(\mathbf{r}_1 t;\mathbf{r}_1't')G_1^0(\mathbf{r}_2 t;\mathbf{r}_2't'). \quad (31)$$

Since the functions in this equation depend only upon a single time difference, we may perform the transformation to functions of complex energy which is described by Eqs. (10) through (12). We note that Λ^{00}, and therefore T, is periodic over 0 to τ since it is the product of two antiperiodic functions. The transformed equation is

$$\langle \mathbf{r}_1\mathbf{r}_2|T(\omega)|\mathbf{r}_1'\mathbf{r}_2'\rangle = \langle \mathbf{r}_1\mathbf{r}_2|v|\mathbf{r}_1'\mathbf{r}_2'\rangle - \langle \mathbf{r}_1\mathbf{r}_2|v|\mathbf{r}_2'\mathbf{r}_1'\rangle + \int \langle \mathbf{r}_1\mathbf{r}_2|v|\mathbf{r}_1''\mathbf{r}_2''\rangle\Lambda^{00}(\mathbf{r}_1''\mathbf{r}_2'';\mathbf{r}_1'''\mathbf{r}_2'''\omega)\langle \mathbf{r}_1'''\mathbf{r}_2'''|T(\omega)|\mathbf{r}_1'\mathbf{r}_2'\rangle, \quad (32)$$

where

$$\Lambda^{00}(\mathbf{r}_1\mathbf{r}_2;\mathbf{r}_1'\mathbf{r}_2'\omega) = \int \frac{d\omega_1 d\omega_2}{(2\pi)^2}\frac{A^0(\mathbf{r}_1\mathbf{r}_1'\omega_1)A^0(\mathbf{r}_2\mathbf{r}_2'\omega_2)}{\omega-\omega_1-\omega_2}[\tfrac{1}{2}\tanh(\tfrac{1}{2}i\omega_1\tau)+\tfrac{1}{2}\tanh(\tfrac{1}{2}i\omega_2\tau)], \quad (33)$$

and A^0 is the spectral function for noninteracting particles.

This result may be further simplified by the introduction of center of mass and relative coordinates and a transformation into momentum space,

$$\langle \mathbf{r}_1\mathbf{r}_2|T(\omega)|\mathbf{r}_1'\mathbf{r}_2'\rangle = \int \frac{d\mathbf{K}}{(2\pi)^3}\exp\left[i\mathbf{K}\cdot\left(\frac{\mathbf{r}_1+\mathbf{r}_2}{2}-\frac{\mathbf{r}_1'+\mathbf{r}_2'}{2}\right)\right]\langle \mathbf{r}_1-\mathbf{r}_2|T_\mathbf{K}(\omega)|\mathbf{r}_1'-\mathbf{r}_2'\rangle$$

$$= \int \frac{d\mathbf{K}}{(2\pi)^3}\int \frac{d\mathbf{k}d\mathbf{k}'}{(2\pi)^3}\exp\left[i\mathbf{K}\cdot\left(\frac{\mathbf{r}_1+\mathbf{r}_2}{2}-\frac{\mathbf{r}_1'+\mathbf{r}_2'}{2}\right)+i\mathbf{k}\cdot(\mathbf{r}_1-\mathbf{r}_2)-i\mathbf{k}'\cdot(\mathbf{r}_1'-\mathbf{r}_2')\right]\langle \mathbf{k}|T_\mathbf{K}(\omega)|\mathbf{k}'\rangle. \quad (34)$$

This reduces the T equation to

$$\langle \mathbf{k}|T_\mathbf{K}(\omega)|\mathbf{k}'\rangle = \langle \mathbf{k}|v|\mathbf{k}'\rangle - \langle \mathbf{k}|v|-\mathbf{k}'\rangle + \int d\mathbf{k}'' \langle \mathbf{k}|v|\mathbf{k}''\rangle\Lambda_\mathbf{K}^{00}(\mathbf{k}''\omega)\langle \mathbf{k}''|T_\mathbf{K}(\omega)|\mathbf{k}'\rangle, \quad (35)$$

where the use of the momentum-space spectral function for free particles $A^0(\mathbf{k}\omega) = 2\pi\delta(\omega+\mu-\mathbf{k}^2/2m)$ gives

$$\Lambda_\mathbf{K}^{00}(\mathbf{k}\omega) = \left[\omega+2\mu-\frac{\mathbf{K}^2}{4m}-\frac{\mathbf{k}^2}{m}\right]^{-1}\left\{\frac{1}{2}\tanh\left[\frac{i\tau}{2}\left(\frac{1}{2m}(\tfrac{1}{2}\mathbf{K}+\mathbf{k})^2-\mu\right)\right]+\frac{1}{2}\tanh\left[\frac{i\tau}{2}\left(\frac{1}{2m}(\tfrac{1}{2}\mathbf{K}-\mathbf{k})^2-\mu\right)\right]\right\}. \quad (36)$$

In the zero-temperature limit, providing μ is negative (as will be the case for nuclear matter) the statistical factor in brackets will become unity. Under these conditions T will satisfy

$$\langle \mathbf{k}|T_\mathbf{K}(\omega)|\mathbf{k}'\rangle = \langle \mathbf{k}|v|\mathbf{k}'\rangle - \langle \mathbf{k}|v|-\mathbf{k}'\rangle$$

$$+ \int d\mathbf{k}'' \frac{\langle \mathbf{k}|v|\mathbf{k}''\rangle\langle \mathbf{k}''|T_\mathbf{K}(\omega)|\mathbf{k}'\rangle}{\omega+2\mu-\mathbf{K}^2/4m-\mathbf{k}''^2/m}. \quad (37)$$

This is simply the equation for the energy-dependent scattering matrix of two nucleons in a center-of-mass system, apart from the antisymmetrization of the

potential and the addition of $2\mu-\mathbf{K}^2/4m$ to the energy. This result reveals the essential simplicity of the approximation; that the T matrix is isolated from any consideration of many-body effects. It also yields an immediate knowledge of the singularities of T, which will have a branch line along the real ω axis for $\omega > \mathbf{K}^2/4m-2\mu$, and a simple pole at $\omega=\omega_d+\mathbf{K}^2/4m-2\mu$, where ω_d is the binding energy of the deuteron.

[5] Similar objections can, of course, be made against both the Hartree and random-phase approximations, but where these approximations are useful the neglected symmetry (exchange) is not numerically important.

The remainder of the approximation is the relation between G_1 and the self-energy Σ expressed by Eqs. (25) and (26). By applying the usual energy transformation to (26) one may obtain

$$\Sigma(\omega) = \int \frac{d\omega'}{2\pi} \left[T(\omega+\omega')\frac{A(\omega')}{1+e^{i\omega'\tau}} \right.$$

$$\left. + \frac{\mathcal{T}(\omega')}{1-e^{i\omega'\tau}}G(\omega'-\omega) \right], \quad (38)$$

where $\mathcal{T}$ is the spectral function corresponding to T. However, for ground-state nuclear matter this result is subject to an important simplification. On the one hand, $\mathcal{T}$ will vanish at all real ω' for which T is analytic, i.e., for $\omega' < \omega_d - 2\mu$, while on the other hand the factor $(1-e^{i\omega'\tau})^{-1}$ will vanish for $\omega' > 0$ as $i\tau \to \infty$. Thus, the second term will be zero if $2\mu < \omega_d$, which we find to be true for nuclear matter at reasonable densities. When the appropriate spatial variables are inserted into the remaining term and the low-temperature limit is taken, the equation becomes

$$\Sigma(\mathbf{r}_1\mathbf{r}_1'\omega) = \int_{-\infty}^{0} \frac{d\omega'}{2\pi} \int d\mathbf{r}_2 d\mathbf{r}_2' \int \frac{d\mathbf{K}}{(2\pi)^3}$$

$$\times \exp\left[i\mathbf{K} \cdot \left(\frac{\mathbf{r}_1+\mathbf{r}_2}{2} - \frac{\mathbf{r}_1'+\mathbf{r}_2'}{2} \right) \right]$$

$$\times \langle \mathbf{r}_1 - \mathbf{r}_2 | T_{\mathbf{K}}(\omega+\omega') | \mathbf{r}_1' - \mathbf{r}_2' \rangle A(\mathbf{r}_2'\mathbf{r}_2\omega'). \quad (39)$$

It should be noted that this equation implies that $\Sigma(\omega)$ is continuous across the real axis in the region $\omega < \omega_d - 2\mu$, which includes the region $\omega < 0$.

Finally, Eq. (25) may be transformed into

$$\left[\omega + \frac{\nabla^2}{2m} + \mu \right] G(\mathbf{r}\mathbf{r}'\omega)$$

$$- \int d\mathbf{r}'' \, \Sigma(\mathbf{r}\mathbf{r}''\omega)G(\mathbf{r}''\mathbf{r}'\omega) = \delta(\mathbf{r}-\mathbf{r}'), \quad (40)$$

which establishes $G(\omega)$ as the energy-dependent Green's function for the self-energy Σ. The spectral function A is then obtained as the discontinuity in G across the real ω axis.

Throughout our derivation we have neglected the effects of spin and isospin, which produce a few modifications of the final formulas; we will state these without proof. We assume that the interaction between particles conserves the total spin and isospin and is symmetric in these variables, i.e., that there is no spin-orbit coupling or electromagnetic effects. In this case the T matrix becomes a combination of the two-particle scattering matrices S_{ss}, S_{ts}, S_{st}, and S_{tt} for singlet (spin)-singlet (isospin), triplet-singlet, singlet-triplet, and triplet-triplet scattering, respectively. If each S

satisfies

$$\langle \mathbf{k} | S(\omega) | \mathbf{k}' \rangle = \langle \mathbf{k} | v | \mathbf{k}' \rangle$$

$$+ \int d\mathbf{k}'' \frac{\langle \mathbf{k} | v | \mathbf{k}'' \rangle \langle \mathbf{k}'' | S(\omega) | \mathbf{k}' \rangle}{\omega - \mathbf{k}''^2/m} \quad (41)$$

for the appropriate matrix element of the interaction, then

$$\langle \mathbf{r} | T_{\mathbf{K}}(\omega) | \mathbf{r}' \rangle = \langle \mathbf{r} | \left[\tfrac{1}{4}S_{ss}(\omega') + \tfrac{3}{4}S_{ts}(\omega') \right.$$

$$+ \tfrac{3}{4}S_{st}(\omega') + (9/4)S_{tt}(\omega') \right] | \mathbf{r}' \rangle$$

$$+ \langle \mathbf{r} | \left[-\tfrac{1}{4}S_{ss}(\omega') + \tfrac{3}{4}S_{ts}(\omega') + \tfrac{3}{4}S_{st}(\omega') \right.$$

$$\left. - (9/4)S_{tt}(\omega') \right] | -\mathbf{r}' \rangle |_{\omega'=\omega+2\mu-\mathbf{K}^2/4m}. \quad (42)$$

A second consequence of the internal variables is that Eqs. (9) for the particle and energy densities must be multiplied by a degeneracy factor of 4. The numerical results which we give for these quantities do not include this factor.

III. RESULTS FOR HOMOGENEOUS NUCLEAR MATTER

For a homogeneous system, single-particle functions such as $A(\mathbf{rr}'\omega)$ depend only upon the difference of their spatial variables and may be expressed by Fourier transforms in one momentum variable, such as

$$A(\mathbf{rr}'\omega) = \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} A(\mathbf{k}\omega).$$

Equation (39) and (40) can then be written in momentum space and the function $A(\mathbf{k}\omega)$ is given by the discontinuity in $G(\mathbf{k}\omega)$ across the real ω axis. The expressions for Σ and for the particle and energy densities require a knowledge of A for negative frequencies only. In this region, however, the properties of T imply that $\Sigma(\omega)$ is continuous, so that A will vanish except in the neighborhood of $\omega=\omega_0(\mathbf{k})$, where $\omega_0(\mathbf{k})$ is the solution to

$$\omega_0(\mathbf{k}) = \mathbf{k}^2/2m - \mu + \Sigma(\mathbf{k},\omega_0(\mathbf{k})). \quad (43)$$

Using the Taylor series expansion of $\Sigma(\omega)$ about this point, we find

$$A(\mathbf{k}\omega) = 2\pi\rho(\mathbf{k})\delta[\omega-\omega_0(\mathbf{k})], \quad (44)$$

where

$$\rho(\mathbf{k}) = \{1 - \partial\Sigma(\mathbf{k},\omega_0(\mathbf{k}))/\partial\omega\}^{-1}. \quad (45)$$

This procedure is carried out in detail and the uniqueness of the solution is discussed in reference 1. It is also shown that $\partial\Sigma/\partial\omega$ is negative, so that $0 < \rho(\mathbf{k}) < 1$.

Since the spectral function $A(\mathbf{k}\omega)$ describes the energy spectrum of a single-particle excitation of momentum $\mathbf{k}$, we see that the negative region of this spectrum either consists of a single delta function of strength $\rho(\mathbf{k})$ or else vanishes completely, depending on whether k is smaller or larger than a cutoff mo-

mentum k_f determined by $\omega_0(k_f)=0$. Of course, since $\rho(\mathbf{k})<1$, the normalization condition (7) implies that $A(\mathbf{k}\omega)$ is nonzero in the positive energy region. However, this portion of the spectrum, which describes hole excitations, does not contribute to Σ or to the particle or energy densities.

For a homogeneous system, the momentum distribution is given by the integral of A over negative ω, which gives $\rho(\mathbf{k})$ for $k<k_f$ and 0 for $k>k_f$. As in conventional perturbation approximations there is a sharp cutoff momentum k_f, but the distribution below the cutoff is now decreased by the factor $\rho(\mathbf{k})$. This implies a corresponding increase of the cutoff momentum beyond the conventional Fermi momentum.

It is convenient to define an effective potential which is a function of momentum alone by $V(\mathbf{k})=\Sigma(\mathbf{k},\omega_0(\mathbf{k}))$. Then $V(\mathbf{k})$ and $\rho(\mathbf{k})$ are solutions of the simultaneous equations

$$V(\mathbf{k}_1)=\int_{|\mathbf{k}_2|<k_f} d\mathbf{k}_2\,\rho(\mathbf{k}_2)\left\langle\frac{\mathbf{k}_1-\mathbf{k}_2}{2}\left|T_{\mathbf{k}_1+\mathbf{k}_2}\left(V(\mathbf{k}_1)+V(\mathbf{k}_2)+\frac{\mathbf{k}_1^2+\mathbf{k}_2^2}{2m}-2\mu\right)\right|\frac{\mathbf{k}_1-\mathbf{k}_2}{2}\right\rangle, \tag{46}$$

and

$$\rho(\mathbf{k}_1)=\left\{1-\int_{|\mathbf{k}_2|<k_f}d\mathbf{k}_2\,\rho(\mathbf{k}_2)\left\langle\frac{\mathbf{k}_1-\mathbf{k}_2}{2}\left|T_{\mathbf{k}_1+\mathbf{k}_2}'\left(V(\mathbf{k}_1)+V(\mathbf{k}_2)+\frac{\mathbf{k}_1^2+\mathbf{k}_2^2}{2m}-2\mu\right)\right|\frac{\mathbf{k}_1-\mathbf{k}_2}{2}\right\rangle\right\}^{-1}, \tag{47}$$

where T' is the energy derivative of T, and k_f is related to μ by

$$\mu=k_f^2/2m+V(k_f). \tag{48}$$

From Eqs. (9) (in the zero-temperature limit) we may obtain the particle and energy densities as

$$\rho=\int_{|\mathbf{k}|<k_f}\frac{d\mathbf{k}}{(2\pi)^3}\rho(\mathbf{k}), \tag{49}$$

and

$$\epsilon=\int_{|\mathbf{k}|<k_f}\frac{d\mathbf{k}}{(2\pi)^3}\rho(\mathbf{k})\left[\frac{\mathbf{k}^2}{2m}+\tfrac{1}{2}V(\mathbf{k})\right]. \tag{50}$$

The numerical solution of these equations was performed on an electronic computer. The solutions of Eqs. (46) and (47) were obtained for a range of k_f, and these results were used to evaluate Eqs. (48) through (50). The integrations were approximated by Gaussian quadrature and the resulting nonlinear matrix equations were solved by a multidimensional Newton-Raphson technique.

As in reference 1, an interparticle potential was used which is the sum of three separable terms; an S-state hard shell, and two Yamaguchi potentials acting in single and triplet spin states,

$$v=v_c+v_{Ys}P_s^\sigma+v_{Yt}P_t^\sigma, \tag{51}$$

where the P's are spin-state projection operators and

$$v_c(\mathbf{rr}')=\lim_{\lambda_c\to\infty}\frac{\pi\lambda_c}{2m}\frac{1}{rr'}\delta(r-r_c)\delta(r'-r_c),$$

$$v_{Y(s,t)}(\mathbf{rr}')=-\frac{\pi\lambda_{s,t}}{2m}\frac{1}{rr'}\exp[-\alpha_{s,t}(r+r')]. \tag{52}$$

Here α and λ are the range and depth parameters for the attractive potential in the appropriate spin state and r_c is the hard shell radius. For a potential of this sort the scattering matrix may be obtained analytically.[1] The values used for the parameters are those given in reference 1 and were obtained by fitting the singlet phase shift at 310 MeV, as well as the usual low-energy scattering data and the binding energy of the deuteron. We have

$$\alpha_s=2.004\text{ F}^{-1}, \quad \lambda_s=3.64037\text{ F}^{-3},$$

$$\alpha_t=2.453\text{ F}^{-1}, \quad \lambda_t=8.6949\text{ F}^{-3}, \quad r_c=0.45\text{ F}, \tag{53}$$

while the average nucleon mass was taken as $(2m)^{-1}=20.7347$ MeV-F^2.

The results of the calculations are shown in Figs. 1 to 4. Figures 1 and 2 show the values of ϵ, ρ, $\mu=k_f^2/2m+V(k_f)$, $E/N=\epsilon/\rho$, and $dE/dN=d\epsilon/d\rho$ (computed from the values of ϵ and ρ) as functions of k_f. Figures 3 and 4 show $V(k)$ and $\rho(k)$ for selected values of k_f. The results for E/N appear to agree with those given by Falk and Wilets[3] for a more limited range of k_f.

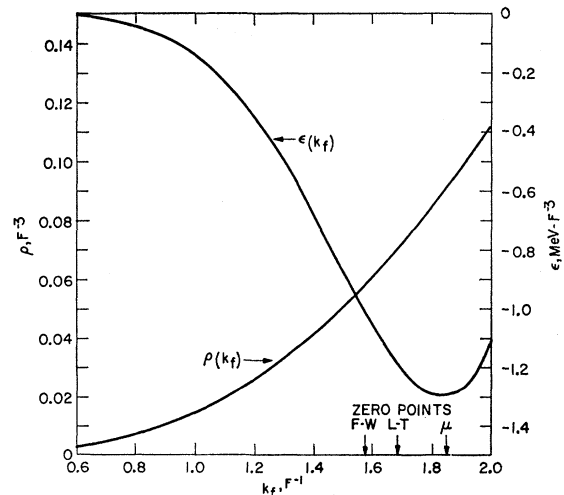As shown in Fig. 2, the numerical results for μ, as given in our approximation by Eq. (48), violate the

FIG. 1. Particle density ρ and energy density ϵ as functions of the cutoff momentum k_f. Both ρ and ϵ must be multiplied by a degeneracy factor of 4.
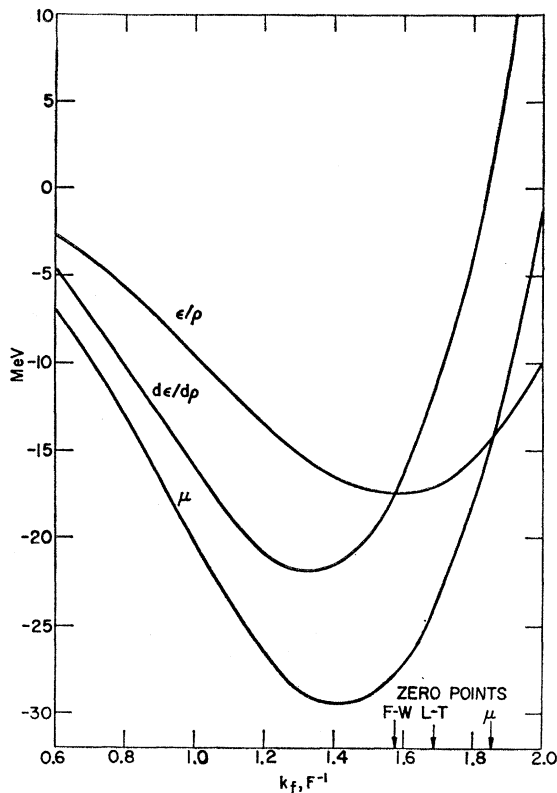
FIG. 2. Chemical potential $\mu = k_f^2/2m + V(k_f)$, binding energy per particle ϵ/ρ, and the derivative of energy with respect to particle number at constant volume $d\epsilon/d\rho$ as functions of the cutoff momentum k_f.

exact thermodynamic relation $\mu = dE/dN$. It is the extent of this failure and its effect on the calculation of nuclear parameters which we want to investigate.

IV. COMPUTATION OF THE PRESSURE

The results of our calculation specify the properties of nuclear matter as functions of the cutoff momentum, or equivalently of the density. We must now determine the point on these curves which corresponds to an actual nucleus. As we stated before, the necessary physical condition is that the nuclear material is un-compressed, i.e., that the pressure of the system is zero.

One equation for the pressure may be obtained from its definition as the negative rate of change of the energy of the system with respect to its volume, with the number of particles held constant. If surface effects are neglected, the energy density $\epsilon = E_0/V$ will depend on the volume only through the particle density $\rho = N/V$, so that the pressure will be

$$P = -\left(\frac{\partial E_0}{\partial V}\right)_N = -\left(\frac{\partial}{\partial V}[V\epsilon(N/V)]\right)_N = \rho\frac{d\epsilon}{d\rho} - \epsilon, \quad (54)$$

or, by using $\mu = d\epsilon/d\rho$,

$$P = \rho\mu - \epsilon. \quad (55)$$

(Both these expressions should be multiplied by a degeneracy factor of 4, which will be omitted from our equations and numerical results.)

Unfortunately, since our calculated μ violates the relation $\mu = d\epsilon/d\rho$, Eqs. (54) and (55) lead to different values of the pressure and thus to different densities for uncompressed matter. We refer to (55), which was used in reference 1, as the μ pressure and to (54), which was used by Falk and Wilets,[3] as the F-W pressure. The corresponding values of k_f at zero pressure will be called the μ and F-W zero points.

In choosing between expressions for a quantity which differ only because of the approximate nature of a calculation, it is impossible to single out one relation as correct in any absolute sense. We might expect both μ and $d\epsilon/d\rho$ to be less accurately given by any approximation than ϵ or ρ. Indeed, we can see no reliable basis for preferring either (54) or (55) and suggest that the discrepancy between results at the two zero points should be viewed as a reflection of the inaccuracy which is inherent in the approximation. Nevertheless, we wish to derive and use a third expression for the pressure which, although not inherently better in any sense, is at least a plausible alternative, and which will be required for consistency in the extension of the approximation to inhomogeneous matter.

This third expression is derived from the definition of pressure in terms of the local transport of momentum. The local momentum density $\mathcal{G}(rt)$ is given in terms of
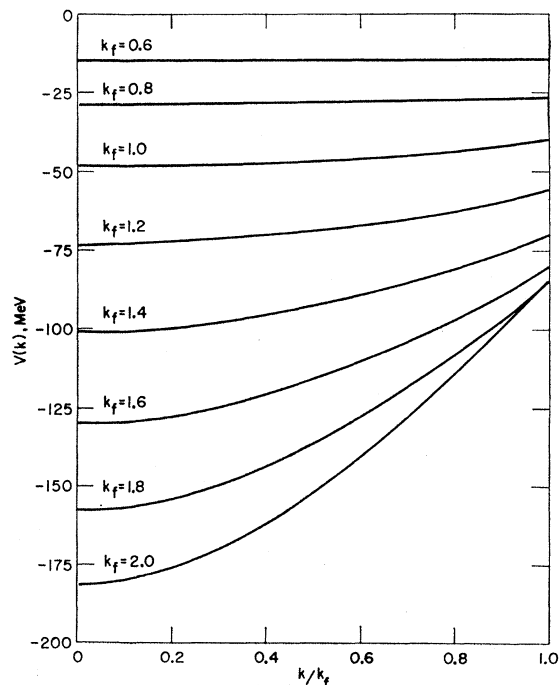

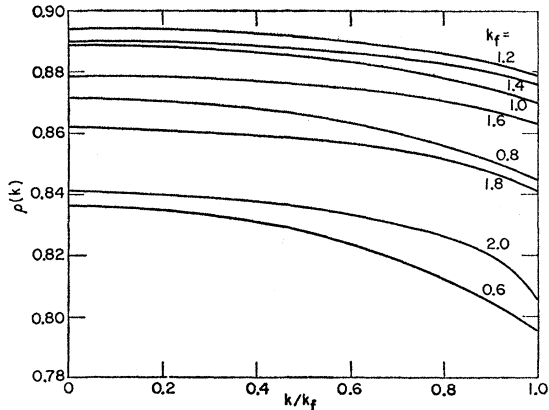FIG. 3. Effective potential $V(k)$ as a function of k/k_f for discrete values of k_f.

FIG. 4. Momentum distribution $\rho(k)$ as a function of k/k_f for discrete values of k_f.

G_1 by

$$\mathcal{G}(\mathbf{r}t) = \frac{1}{2i}\lim_{\mathbf{r}'\to\mathbf{r}} (\nabla-\nabla')\langle\psi^\dagger(\mathbf{r}'t)\psi(\mathbf{r}t)\rangle$$

$$= -\tfrac{1}{2}\lim_{\mathbf{r}'\to\mathbf{r}} (\nabla-\nabla')G_1(\mathbf{r}t;\mathbf{r}'t^+), \qquad (56)$$

which has the time derivative

$$\frac{\partial}{\partial t}\mathcal{G}(\mathbf{r}t) = \frac{i}{2}\lim_{\mathbf{r}'\to\mathbf{r},t'\to t+} (\nabla-\nabla')\left[i\frac{\partial}{\partial t}+i\frac{\partial}{\partial t'}\right]G_1(\mathbf{r}t;\mathbf{r}'t'). \quad (57)$$

(Of course, this derivative must vanish for a stationary system; this will lead to the requirement that the pressure be constant over the system. A rigorous derivation would require an explicit extension of the Green's function formalism to a stationary inhomogeneous system, but we will ignore this complication at present.)

To obtain the time derivative of G_1 we would normally use Eq. (22) along with the corresponding adjoint equation (27). However, as we discussed previously, (22) and (27) do not lead to the same G_1 when we use our approximate G_2 in both equations. Therefore, we

must instead use the adjoint of Eq. (25),

$$\left[-i\frac{\partial}{\partial t_1'}+\frac{\nabla_1'^2}{2m}+\mu\right]G_1(1;1')$$

$$-\int G_1(1;\bar{1})\Sigma(\bar{1};1')=\delta(1-1'), \quad (58)$$

which is satisfied because of the symmetry of T in unprimed and primed coordinates.

The use of (25) and (58) in (57) allows us to express the time derivative of the momentum density as a sum of kinetic and potential contributions. These contributions must then be rewritten as divergences of a stress tensor in order to obtain the pressure. Thus,

$$\frac{\partial}{\partial t}\mathcal{G}(\mathbf{r}t) = -\nabla\cdot\mathbf{T}^{(0)}(\mathbf{r})-\nabla\cdot\mathbf{T}^{(1)}(\mathbf{r}). \qquad (59)$$

The kinetic portion of the stress tensor is

$$\mathbf{T}^{(0)}(\mathbf{r}) = -\frac{i}{2}\frac{1}{2m}\lim_{\mathbf{r}'\to\mathbf{r}} (\nabla-\nabla')(\nabla-\nabla')G_1(\mathbf{r}t;\mathbf{r}'t^+)$$

$$= -\frac{1}{2}\frac{1}{2m}\lim_{\mathbf{r}'\to\mathbf{r}} (\nabla-\nabla')(\nabla-\nabla')$$

$$\times\int_{-\infty}^{0}\frac{d\omega}{2\pi}A(\mathbf{r}\mathbf{r}'\omega), \quad (60)$$

or for a homogeneous system

$$\mathbf{T}^{(0)} = 2\int_{-\infty}^{0}\frac{d\omega}{2\pi}\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{\mathbf{k}\mathbf{k}}{2m}A(\mathbf{k}\omega)$$

$$= \frac{2}{3}\delta\int_{|\mathbf{k}|<k_f}\frac{d\mathbf{k}}{(2\pi)^3}\frac{|\mathbf{k}|^2}{2m}\rho(\mathbf{k}), \qquad (61)$$

where δ is the unit tensor. However, the potential contribution of $\partial\mathcal{G}/\partial t$ cannot be generally expressed as a divergence. Nevertheless, for a short-range potential, it is possible to express this contribution as a divergence in a region of local uniformity. This may be done by integrating the potential contribution,

$$\nabla\cdot\mathbf{T}^{(1)}(\mathbf{r}_1) = -\frac{i}{2}\lim_{\mathbf{r}_1'\to\mathbf{r}_1,t_1'\to t_1+} (\nabla_1-\nabla_1')\int [\Sigma(1;\bar{1})G_1(\bar{1};1')-G_1(1;\bar{1})\Sigma(\bar{1};1')]$$

$$= \tfrac{1}{2}\lim_{\mathbf{r}_1'\to\mathbf{r}_1} (\nabla_1-\nabla_1')\int_{-\infty}^{0}\frac{d\omega d\omega'}{(2\pi)^2}\int d\mathbf{r}_1''d\mathbf{r}_2 d\mathbf{r}_2'\{\langle\mathbf{r}_1\mathbf{r}_2|T(\omega+\omega')|\mathbf{r}_1''\mathbf{r}_2'\rangle A(\mathbf{r}_2'\mathbf{r}_2\omega')A(\mathbf{r}_1''\mathbf{r}_1'\omega)$$

$$-A(\mathbf{r}_1\mathbf{r}_1''\omega)\langle\mathbf{r}_1''\mathbf{r}_2|T(\omega+\omega')|\mathbf{r}_1'\mathbf{r}_2'\rangle A(\mathbf{r}_2'\mathbf{r}_2\omega')\} \quad (62)$$

over a macroscopic region V.

We first note that since T is independent of G_1, it is independent of its average spatial coordinate, even in a system which is inhomogeneous. Using this fact and the symmetry of T and A in unprimed and primed coordinates, we obtain

$$\nabla\cdot\mathbf{T}^{(1)}(\mathbf{r}_1) = \int d\mathbf{r}_1''d\mathbf{r}_2 d\mathbf{r}_2'\langle\mathbf{r}_1\mathbf{r}_2|T|\mathbf{r}_1''\mathbf{r}_2'\rangle[-\nabla_1+\nabla_1''+\nabla_2+\nabla_2']A(\mathbf{r}_2'\mathbf{r}_2)A(\mathbf{r}_1''\mathbf{r}_1) \qquad (63)$$

(where the energy variables and integrations have been omitted). If we introduce the change of variables

$$\mathbf{r}_1=\mathbf{R}, \qquad \mathbf{r}_1''=\mathbf{R}-\mathbf{r},$$
$$\mathbf{r}_2=\mathbf{R}'-\mathbf{r}', \quad \mathbf{r}_2'=\mathbf{R}', \tag{64}$$

then the integral of the potential contribution over V may be written as

$$\int_V d\mathbf{R}\nabla\cdot\mathbf{T}^{(1)}(\mathbf{R})=\int d\mathbf{R}'d\mathbf{r}d\mathbf{r}'\int_V d\mathbf{R}\,\langle\mathbf{R},\,\mathbf{R}'-\mathbf{r}'|T|\mathbf{R}-\mathbf{r},\,\mathbf{R}'\rangle[-\nabla_{\mathbf{R}}+\nabla_{\mathbf{R}'}-2\nabla_{\mathbf{R}}]A(\mathbf{R}',\,\mathbf{R}'-\mathbf{r}')A(\mathbf{R}-\mathbf{r},\,\mathbf{R}). \tag{65}$$

Then by using a second change of variables

$$\mathbf{R}-\mathbf{r}=\mathbf{R}_{\mathrm{new}}, \quad \mathbf{R}'-\mathbf{r}'=\mathbf{R}'_{\mathrm{new}},$$
$$-\mathbf{r}=\mathbf{r}_{\mathrm{new}}, \qquad -\mathbf{r}'=\mathbf{r}'_{\mathrm{new}}, \tag{66}$$

and again employing the symmetry of T and A we obtain

$$\int_V d\mathbf{R}\nabla\cdot\mathbf{T}^{(1)}(\mathbf{R})=\int d\mathbf{R}'d\mathbf{r}d\mathbf{r}'\int_{V'} d\mathbf{R}\,\langle\mathbf{R},\,\mathbf{R}'-\mathbf{r}'|T|\mathbf{R}-\mathbf{r},\,\mathbf{R}'\rangle[\nabla_{\mathbf{R}}+\nabla_{\mathbf{R}'}+2\nabla_{\mathbf{r}}]A(\mathbf{R}',\,\mathbf{R}'-\mathbf{r}')A(\mathbf{R}-\mathbf{r},\,\mathbf{R}), \tag{67}$$

where V' is the region of $\mathbf{R}$ such that $\mathbf{R}-\mathbf{r}$ is in V. By averaging (65) and (67) we may write the integral as two terms,

$$\int_V d\mathbf{R}\,\nabla\cdot\mathbf{T}^{(1)}(\mathbf{R})=I+I', \tag{68}$$

where

$$I=\int d\mathbf{R}'d\mathbf{r}d\mathbf{r}'\int_V d\mathbf{R}\,\langle\mathbf{R},\,\mathbf{R}'-\mathbf{r}'|T|\mathbf{R}-\mathbf{r},\,\mathbf{R}'\rangle\nabla_{\mathbf{R}'}A(\mathbf{R}',\,\mathbf{R}'-\mathbf{r}')A(\mathbf{R}-\mathbf{r},\,\mathbf{R}), \tag{69}$$

and

$$I'=\frac{1}{2}\int d\mathbf{R}'d\mathbf{r}d\mathbf{r}'\int_{V'-V} d\mathbf{R}\,\langle\mathbf{R},\,\mathbf{R}'-\mathbf{r}'|T|\mathbf{R}-\mathbf{r},\,\mathbf{R}'\rangle[\nabla_{\mathbf{R}}+\nabla_{\mathbf{R}'}+2\nabla_{\mathbf{r}}]A(\mathbf{R}',\,\mathbf{R}'-\mathbf{r}')A(\mathbf{R}-\mathbf{r},\,\mathbf{R}).$$

Using partial integration and the fact that $\nabla_{\mathbf{R}}T=-\nabla_{\mathbf{R}'}T$, we may then write I as

$$I=\int d\mathbf{r}d\mathbf{r}'\int d\mathbf{R}'\int_V d\mathbf{R}\,A(\mathbf{R}',\,\mathbf{R}'-\mathbf{r}')A(\mathbf{R}-\mathbf{r},\,\mathbf{R})\tfrac{1}{2}[\nabla_{\mathbf{R}}-\nabla_{\mathbf{R}'}]\langle\mathbf{R},\,\mathbf{R}'-\mathbf{r}'|T|\mathbf{R}-\mathbf{r},\,\mathbf{R}'\rangle. \tag{70}$$

In this form the integrand is antisymmetric under the interchange $\mathbf{R}\leftrightarrow\mathbf{R}'$, $\mathbf{r}\leftrightarrow\mathbf{r}'$, so that the portion of the integral which arises from $\mathbf{R}'$ in V vanishes, and we may limit $\mathbf{R}'$ to the region outside V. Then, since the integrand vanishes when $\mathbf{R}-\mathbf{R}'$ is macroscopically large, we find that $\mathbf{R}$ and $\mathbf{R}'$ are limited to a region near the surface of V. For the portion of this region where the system has local uniformity we may write the integration as

$$\int_{\mathrm{non}-V} d\mathbf{R}'\int_V d\mathbf{R}=-\int dS\mathbf{n}\cdot\int(\mathbf{R}-\mathbf{R}')d(\mathbf{R}-\mathbf{R}')\times\begin{cases}1, & \mathbf{n}\cdot(\mathbf{R}-\mathbf{R}')<0 \\ 0, & \mathbf{n}\cdot(\mathbf{R}-\mathbf{R}')>0,\end{cases} \tag{71}$$

where S is the surface of V and $\mathbf{n}$ is a unit vector normal to the surface in the outward direction. Finally, using the antisymmetry of the integrand we have

$$I=\int dS\mathbf{n}\cdot\left\{-\frac{1}{2}\int d(\mathbf{R}-\mathbf{R}')d\mathbf{r}d\mathbf{r}'(\mathbf{R}-\mathbf{R}')A(\mathbf{R}',\,\mathbf{R}'-\mathbf{r}')A(\mathbf{R}-\mathbf{r},\,\mathbf{R})\tfrac{1}{2}[\nabla_{\mathbf{R}}-\nabla_{\mathbf{R}'}]\langle\mathbf{R},\,\mathbf{R}'-\mathbf{r}'|T|\mathbf{R}-\mathbf{r},\,\mathbf{R}'\rangle\right\}, \tag{72}$$

where the quantity in brackets is identified as the I contribution to $\mathbf{T}^{(1)}(\mathbf{R})$.

Next consider I'. In this case, the vanishing of the integrand for macroscopically large $\mathbf{r}$ limits the integration over $V'-V$ to a region near the surface of V, and for a locally uniform portion of this region, we have

$$\int_{V'-V} d\mathbf{R}=\int dS\,\mathbf{n}\cdot\mathbf{r}, \tag{73}$$

so that

$$I' = \int dS \, \mathbf{n} \cdot \left\{ \frac{1}{2} \int d\mathbf{R}' d\mathbf{r} d\mathbf{r}' \, \langle \mathbf{R}, \mathbf{R}'-\mathbf{r}' | T | \mathbf{R}-\mathbf{r}, \mathbf{R}' \rangle [\nabla_{\mathbf{R}} + \nabla_{\mathbf{R}'} + 2\nabla_{\mathbf{r}}] A(\mathbf{R}', \mathbf{R}'-\mathbf{r}') A(\mathbf{R}-\mathbf{r}, \mathbf{R}) \right\}, \quad (74)$$

where the quantity in brackets is the I' contribution to $\mathbf{T}^{(1)}(\mathbf{R})$.

For a homogeneous system (or within a locally uniform region) A will depend only upon the difference of its spatial coordinates, so that the gradients $\nabla_{\mathbf{R}}$ and $\nabla_{\mathbf{R}'}$ in the I' contribution will vanish. By using partial integration on the I' contribution, we may write the combined contributions as

$$\mathbf{T}^{(1)} = -\frac{1}{2} \int_{-\infty}^{0} \frac{d\omega d\omega'}{(2\pi)^2} \int d(\mathbf{R}-\mathbf{R}') d\mathbf{r} d\mathbf{r}' \, A(-\mathbf{r}\omega) A(\mathbf{r}'\omega')$$

$$\times [\tfrac{1}{2}(\mathbf{R}-\mathbf{R}')(\nabla_{\mathbf{R}}-\nabla_{\mathbf{R}'}) + 2\nabla_{\mathbf{r}}\mathbf{r}] \langle \mathbf{R}, \mathbf{R}'-\mathbf{r}' | T(\omega+\omega') | \mathbf{R}-\mathbf{r}, \mathbf{R}' \rangle, \quad (75)$$

which becomes in momentum space

$$\mathbf{T}^{(1)} = \frac{1}{2} \int_{-\infty}^{0} \frac{d\omega d\omega'}{(2\pi)^2} \int \frac{d\mathbf{k} d\mathbf{k}'}{(2\pi)^3} A(\mathbf{k}\omega) A(\mathbf{k}'\omega') [\delta + 2\mathbf{k}\nabla_{\mathbf{k}}] \left\langle \frac{\mathbf{k}-\mathbf{k}'}{2} \middle| T_{\mathbf{k}+\mathbf{k}'}(\omega+\omega') \middle| \frac{\mathbf{k}-\mathbf{k}'}{2} \right\rangle. \quad (76)$$

The substitution of the expressions for Σ and A in our approximation reduces this to

$$\mathbf{T}^{(1)} = \frac{1}{2}\delta \int_{|\mathbf{k}|<k_f} \frac{d\mathbf{k}}{(2\pi)^3} \rho(\mathbf{k}) \left[1 + \frac{2}{3} k \frac{\partial}{\partial k} \right] \Sigma(\mathbf{k}, \omega_0(\mathbf{k})). \quad (77)$$

The partial derivative of $\Sigma(\mathbf{k}, \omega_0)$ may be related to the total derivative of $V(\mathbf{k})$ by

$$\frac{\partial}{\partial k} \Sigma(\mathbf{k}, \omega_0(\mathbf{k})) = \frac{d}{dk} \Sigma(\mathbf{k}, \omega_0(\mathbf{k})) - \frac{\partial}{\partial \omega} \Sigma(\mathbf{k}, \omega_0(\mathbf{k})) \frac{d}{dk} \omega_0(\mathbf{k})$$

$$= \frac{d}{dk} V(\mathbf{k}) - [1 - \rho^{-1}(\mathbf{k})] \left[\frac{k}{m} + \frac{d}{dk} V(\mathbf{k}) \right]. \quad (78)$$

Substituting this result into (77) and integrating by parts gives

$$\mathbf{T}^{(1)} = \delta \left\{ \frac{k_f^3}{6\pi^2} V(k_f) + \int_{|\mathbf{k}|<k_f} \frac{d\mathbf{k}}{(2\pi)^3} \left[\frac{2}{3}[1-\rho(\mathbf{k})] \frac{\mathbf{k}^2}{2m} + [\tfrac{1}{2}\rho(\mathbf{k})-1] V(\mathbf{k}) \right] \right\}. \quad (79)$$

Finally, the pressure may be identified as the average diagonal element of the full stress tensor $\mathbf{T}^{(0)} + \mathbf{\Gamma}^{(1)}$,

$$P = \frac{k_f^5}{30\pi^2 m} + \frac{k_f^3}{6\pi^2} V(k_f) + \int_{|\mathbf{k}|<k_f} \frac{d\mathbf{k}}{(2\pi)^3} [\tfrac{1}{2}\rho(\mathbf{k})-1] V(\mathbf{k}). \quad (80)$$

We refer to this expression as the L-T (local transport) pressure. As usual, it must be multiplied by a degeneracy factor of 4 which will not be included in our equations or numerical results.

The numerical results presented in the preceding section were used to evaluate all three expressions for the pressure as functions of k_f. The results, given in Fig. 5, show distinct but similarly behaved functions, with $P_\mu < P_{\text{L-T}} < P_{\text{F-W}}$. Each pressure relation leads to a distinct zero point, and the basic nuclear parameters at each zero point are summarized in Table I. The functions $V(k)$ and $\rho(k)$ for the μ and local-transport zero points are given in Fig. 6, while analogous results

for the Falk-Wilets zero point may be found in reference 3.

The values for the energy per particle ϵ/ρ and the interparticle spacing constant $r_0 = [3/(16\pi\rho)]^{1/3}$ lie in the ranges -14.4 to -17.5 MeV and 0.871 to 1.01 F, respectively. These may be compared with values of $\epsilon/\rho = -15.75$ MeV, obtained from fitting nuclear masses,[6] and $r_0 = 1.1$ to 1.2 F, obtained from high-energy electron scattering data.[7,8] However, because of the

[6] A. E. S. Green, Phys. Rev. **95**, 1006 (1954).
[7] K. A. Brueckner and J. L. Gammel, Phys. Rev. **109**, 1023 (1958).
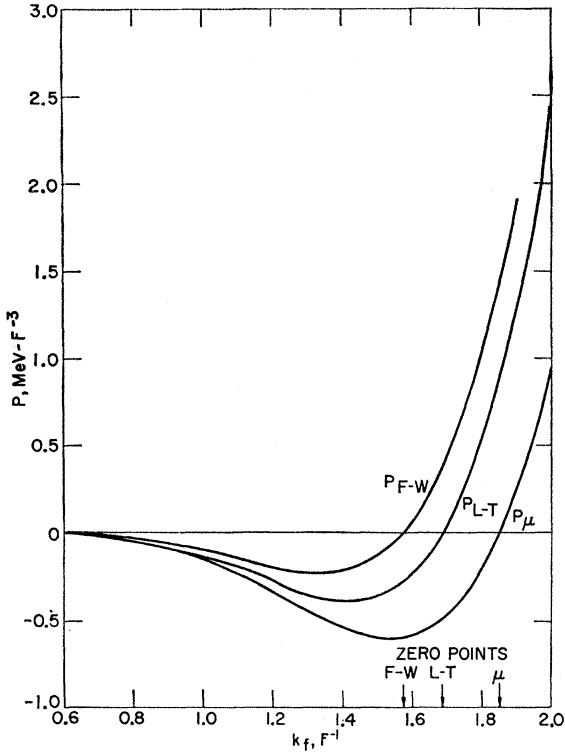[8] B. Hahn, D. G. Ravenhall, and R. Hofstadter, Phys. Rev. **101**, 1131 (1956).

FIG. 5. Pressure P as a function of the cutoff momentum k_f, as calculated from Eqs. (54) (F-W), (55) (μ), and (80) (L-T). These functions must be multiplied by a degeneracy factor of 4.

FIG. 6. Effective potential $V(k)$ and momentum distribution $\rho(k)$ as functions of k/k_f for the μ and local transport (L-T) zero points.

unrealistically simple potential which was used in our calculation, it is difficult to infer much from these results about the inherent accuracy of the approximation or the best choice among the pressure relations. It should be noted that at all three zero points the momentum distribution $\rho(k)$ shows a significant decrease from unity, although not as great as was expected in earlier work with the approximation.
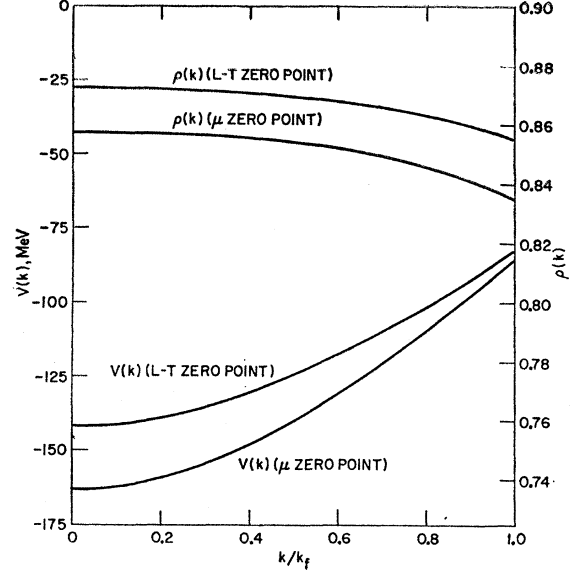
V. THE DENSITY CORRELATION FUNCTION

Since the basic task of a many-particle theory of nuclear matter is to treat particle correlations, it is desirable to investigate some measure of the correlation effects which occur in our approximation. The simplest description of these effects is given by the density correlation function $f(r_1\zeta_1, r_2\zeta_2)$, which measures the probability of simultaneously finding a particle at r_1 with spin and isospin ζ_1, and a second particle at r_2, with spin and isospin ζ_2. This function may be expressed in terms of G_2 as

$$f(r_1\zeta_1, r_2\zeta_2) = -G_2(r_1\zeta_1 t r_2\zeta_2 t ; r_1\zeta_1 t^+ r_2\zeta_2 t^+). \qquad (81)$$

From Eqs. (21) and (23) we have

$$f(r_1 r_2) = -G_1(r_1 t ; r_1 t^+)G_1(r_2 t ; r_2 t^+) + G_1(r_1 t ; r_2 t^+)G_1(r_2 t ; r_1 t^+)$$

$$+ i \int \Lambda^{00}(r_1 r_2 t ; r_1' r_2' t')\langle r_1' r_2' t' | T | r_1'' r_2'' t'' \rangle \Lambda (r_1'' r_2'' t'' ; r_1 r_2 t^+), \qquad (82)$$

where Λ^{00} is given by (31) and Λ by the corresponding equation with interacting G's. By introducing the usual energy transformations, we may obtain

$$f(r_1 - r_2) = \rho^2 - \left\{ \int_{-\infty}^{0} \frac{d\omega}{2\pi} A(r_1 r_2 \omega) \right\}^2 - \int_{-\infty}^{0} \frac{d\omega}{2\pi} \lim_{\epsilon \to +0} \frac{1}{i} [C(r_1 r_2, \omega - i\epsilon) - C(r_1 r_2, \omega + i\epsilon)], \qquad (83)$$

where

$$C(\omega) = \int \Lambda^{00}(\omega) T(\omega) \Lambda(\omega). \qquad (84)$$

However, since in the low-temperature limit Λ^{00} and T are continuous for $\omega < 0$, the effect of taking the discon-

tinuity of C across the real axis is simply to replace Λ by the corresponding spectral function, which is (at zero temperature)

$$\Lambda_{\text{spectral}}(\mathbf{r}_1\mathbf{r}_2\mathbf{r}_1'\mathbf{r}_2'\omega) = \int_{-\omega/2}^{\omega/2} \frac{d\omega'}{2\pi} A\left(\mathbf{r}_1\mathbf{r}_1'\frac{\omega}{2}+\omega'\right)A\left(\mathbf{r}_2\mathbf{r}_2'\frac{\omega}{2}-\omega'\right). \tag{85}$$

By transforming to momentum space and using (36), we, thus, obtain

$$f(\mathbf{r}) = \rho^2 - \left\{ \int_{-\infty}^0 \frac{d\omega}{2\pi} \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{r}} A(\mathbf{k}\omega) \right\}^2 + \int \frac{d\mathbf{k}d\mathbf{k}_1 d\mathbf{k}_2}{(2\pi)^6} e^{i\mathbf{k}\cdot\mathbf{r}}$$

$$\times \int_{-\infty}^0 \frac{d\omega_1 d\omega_2}{(2\pi)^2}\left[\omega_1+\omega_2+2\mu-\frac{(\mathbf{k}_1+\mathbf{k}_2)^2}{4m}-\frac{1}{m}\left(\mathbf{k}+\frac{(\mathbf{k}_1-\mathbf{k}_2)}{2}\right)^2\right]^{-1}$$

$$\times\left\langle \mathbf{k}+\frac{\mathbf{k}_1-\mathbf{k}_2}{2}\left|T_{\mathbf{k}_1+\mathbf{k}_2}(\omega_1+\omega_2)\right|\frac{\mathbf{k}_1-\mathbf{k}_2}{2}\right\rangle A(\mathbf{k}_1\omega_1)A(\mathbf{k}_2\omega_2), \tag{86}$$

and the substitution of (44) gives

$$f(\mathbf{r}) = \rho^2 - \left\{ \int_{|\mathbf{k}|<k_f} \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{r}}\rho(\mathbf{k}) \right\}^2 + \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{r}}\int_{|\mathbf{k}_1|,|\mathbf{k}_2|<k_f} \frac{d\mathbf{k}_1 d\mathbf{k}_2}{(2\pi)^3}\left[V(\mathbf{k}_1)+V(\mathbf{k}_2)-\frac{1}{m}\mathbf{k}\cdot(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2)\right]^{-1}$$

$$\times\left\langle \mathbf{k}+\frac{\mathbf{k}_1-\mathbf{k}_2}{2}\left|T_{\mathbf{k}_1+\mathbf{k}_2}\left(V(\mathbf{k}_1)+V(\mathbf{k}_2)+\frac{\mathbf{k}_1^2+\mathbf{k}_2^2}{2m}-2\mu\right)\right|\frac{\mathbf{k}_1-\mathbf{k}_2}{2}\right\rangle\rho(\mathbf{k}_1)\rho(\mathbf{k}_2). \tag{87}$$

The consideration of internal variables complicates this expression considerably for a general interaction, but for our particular choice of a potential, which is independent of isospin and acts only in the S state, the situation is fairly simple. There are four cases, depending on whether the correlated particles have the same or opposite spin and isospin. The second term in (87), which describes exclusion effects, vanishes for unlike particles, while the third term vanishes for identical particles, which are excluded from the S state and do not interact. For unlike particles, the combinations of the singlet and triplet scattering matrices which form T are

Spin	Isospin	
opposite	same	$T = S_s$
same	opposite	$T = S_t$
opposite	opposite	$T = (S_s+S_t)/2,$

where S_s and S_t are the appropriate scattering matrices.

The four correlation functions, as calculated for the local-transport zero point, are shown in Fig. 7, along with the function

$$N(\mathbf{r}) = \frac{1}{\rho}\int_{|\mathbf{r}'|<r} d\mathbf{r}' \sum_{\zeta} f(\mathbf{r}'\zeta), \tag{88}$$

TABLE I. Parameters of nuclear matter at zero pressure.

	Pressure expression set to zero:			
	μ (55)	L-T (80)	F-W (54)[a]	
Cutoff momentum k_f	1.8491	1.6865	1.575	F^{-1}
Chemical potential μ	-14.437	-24.028	-27.7	MeV
Density ρ[b]	0.09028	0.06991	0.0579	F^{-3}
Interparticle spacing r_0	0.871	0.949	1.01	F
Energy density ϵ[b]	-1.3034	-1.1903	-1.01	MeV-F^{-3}
Energy per particle ϵ/ρ	-14.437	-17.026	-17.5	MeV
μ pressure P_μ[b]	0.0	-0.4895	-0.59	MeV-F^{-3}
L-T pressure $P_{\text{L-T}}$[b]	0.9073	0.0	-0.27 approx	MeV-F^{-3}
F-W pressure $P_{\text{F-W}}$[b]	1.4 approx	0.38 approx	0.0	MeV-F^{-3}

[a] Quoted or calculated from results given in reference 3.
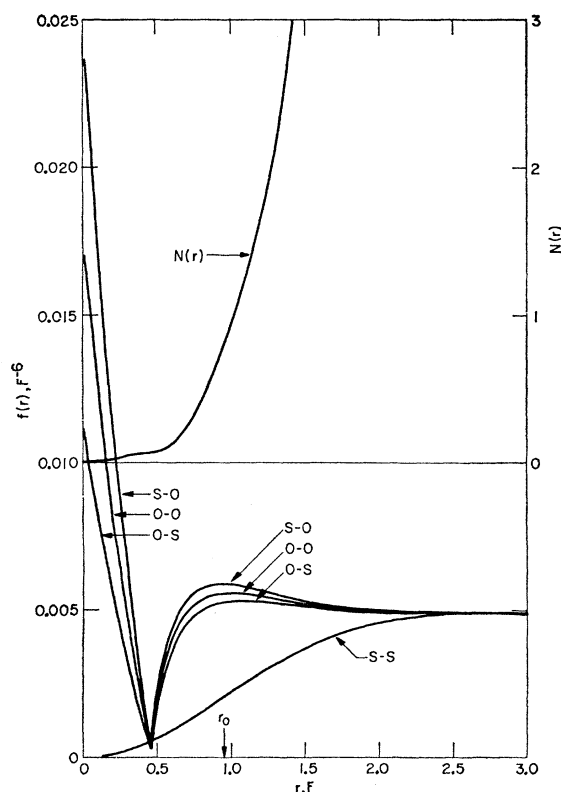[b] These quantities must be multiplied by a degeneracy factor of 4.

FIG. 7. Density correlation function $F(r)$ and its integral $N(r)$ as functions of the distance r. The density correlation function is given for particles with the same spin and isospin $(S\text{-}S)$, opposite spin and same isospin $(O\text{-}S)$, same spin and opposite isospin $(S\text{-}O)$, and opposite spin and isospin $(O\text{-}O)$. The function $N(r)$ [cf., Eq. (88)] is the expected number of additional particles within a radius r of a given particle. Note that $f(r)$ and $N(r)$ have ordinates with different origins.

which gives the expected number of additional particles to be found within the distance $\mathbf{r}$ of a given particle. It can be seen that there is only a 24% probability of finding an additional particle within 0.7 F, which is the approximate distance at which correlation effects other

than the exclusion principle begin to play a major role. In view of this result, it can be expected that correlations of more than two particles should be sufficiently rare as to have only slight effects on nuclear properties.

A rather disturbing result is the large value of the unlike-particle correlation functions within the core radius of 0.45 F. This is a consequence of our use of a hard shell to approximate a hard core, and represents the unphysical situation of particles bound inside the shell. However, the large size of the correlation functions is offset by the small size of the shell, so that $N(\mathbf{r})$ is only 6.3% at the shell radius.

In summary, by applying a rather straightforward approximation to a vary simple particle interaction, we have obtained qualitatively reasonable results for nuclear matter (using various pressure relations) of -14.4 to -17.5 MeV for the binding energy per particle and 0.87 to 1.01 F for the interparticle spacing or radius constant. The resulting momentum distribution is significantly smaller than a filled Fermi sphere and leads to a cutoff momentum which is 5.7% to 4.4% higher than the conventional Fermi momentum. The resulting density correlation function indicates that correlations of more than two particles should not exert any substantial effect at nuclear densities.

The major defect of the approximation is its violation of the exact relations between the calculated chemical potential and the total derivative with respect to particle number. This is presumably a result of the asymmetric form of our approximate G_2; however, it is this particular form which is needed to obtain a T matrix independent of many-particle effects and therefore calculable analytically.

ACKNOWLEDGMENTS