Nuclear Properties of $_{25}Mn^{55}$ (Semiatomic Model)

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The nuclear properties of $_{25}Mn^{55}$ and some other nuclei in the region of $(1f_{7/2})$ shell are investigated in detail by using a new model (called semiatomic model) which is essentially an extremely weak-coupling formulation of the unified model in contrast to the semimolecular formulation of Bohr and Mottelson. By restricting the variation range of the deformation variable γ to the interval $0 \leqslant \gamma < \pi/3$, a "zero-point" surface-particle interaction which does not exist in the usual weak-coupling theory is naturally introduced. The theoretical basis for this interaction and the problem of the range of γ are discussed. A theoretical relation for the magnetic moment ratio of two odd-A nuclei with conjugate configurations is tested in the region of the $(1f_{7/2})$ shell. The theoretical predictions are in good agreement with the experimental results. An explanation for the positive-negative asymmetry of the quadrupole moment distribution of odd-A nuclei is suggested. It seems that this asymmetry has been so far overlooked without explanation. It is shown that the $E2$ reduced transition probability in the semiatomic model is given by the following expression:

$$B(E2; J_i \rightarrow J_f) = \frac{2J_f+1}{2J_i+1} B^s(E2; J_i \rightarrow J_f),$$

where $B^s(E2; J_i \rightarrow J_f)$ is the value which one obtains in the usual shell theory. The calculated $E2$ reduced transition probability for $_{23}V^{51}$ in the transition $J_0 = 5/2 \rightarrow J = 7/2$ is in excellent agreement with the value from Coulomb excitation measurements.

I. INTRODUCTION

THE "many-particle" shell model of nuclei (the usual shell model including more general residual interaction of the extra-core nucleons) has been reasonably successful in recent years. The calculations with this model, which are, in general, very tedious, become much simpler for those nuclei in which the $(1f_{7/2})$ shell is being filled, because in such cases the single-particle level of the extra-core nucleons is so well separated from the neighboring single-particle levels available that the configuration interaction may be neglected. In the following discussion, we shall confine ourselves to these simple cases.

Furthermore, we should also be able to describe the motion of the core and its interaction with the motion of the extra-core nucleons. In view of the close spacing of the single-particle levels of the core, a complete description of its motion should be very difficult in the language of the shell model. Fortunately, this difficulty may be surmounted, at least in the low-energy region, by ascribing to the core certain collective degrees of freedom as originally suggested by Rainwater[1] and later fully developed by Bohr and Mottelson.[2,3] In the Bohr and Mottelson theory, the motion of the core is described as surface-phonon (or surfon) excitation of an incompressible fluid droplet and, to a good approximation, each surfon carries an angular momentum $2\hbar$. The interaction mentioned previously is, then, attributed to the deformation of the average central potential (for the extra-core nucleons). The shape of the average central potential may be associated with the mass distribution of the core.

The problem of a single nucleon moving in a strongly deformed potential without considering the residual interaction has been discussed in detail by Nilsson.[5] However, it has been shown that the effect of the residual interaction is always to reduce the deformation of the average central potential,[6] which is supposed to exist for completely independent particle motion as treated in Nilsson's paper. Therefore, the deformation of the average central potential also depends upon the intrinsic structure of the extra-core nucleons. For better approximation, we shall assume that the total deformation of the average central potential consists of two parts, namely, the intrinsic deformation and the residual deformation. The former is due to the intrinsic structure of the core and the latter is due to that of the extra-core nucleons.

In separating the nucleus into two parts, namely, the core and the extra-core nucleons, our problem becomes rather similar to the so-called "polaron problem" which deals with the motion of an electron in a polar crystal. The electron tends to polarize the lattice in its vicinity and the polarization energy can be described in terms of phonon excitation in the vicinity of the electron. In other words, the motion of the electron is coupled to the optical branch of the lattice vibration. From quite general arguments, Landau[7] has shown that the energy spectrum of a Fermi fluid can have a "Bose branch" in

[1] J. Rainwater, Phys. Rev. **79**, 432 (1950).
[2] A. Bohr, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **26**, 14 (1952).
[3] A. Bohr and B. R. Mottelson, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **27**, 16 (1953).
[4] M. G. Mayer and J. H. D. Jensen, *Elementary Theory of Nuclear Shell Structure* (John Wiley & Sons, Inc., New York, 1955).
[5] S. G. Nilsson, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **29**, 16 (1955).
[6] A. Bohr and B. R. Mottelson, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **30**, 1 (1955).
[7] L. D. Landau, Zh. Eksperim. i Teor. Fiz. **32**, 59 (1957) [translation: Soviet Phys.—JETP **5**, 101 (1957)].

the form of phonons. Therefore, if we consider the core as a droplet of Fermi fluid, we may argue that the extra-core nucleons tend to deform the core, and the deformation energy can be described in terms of phonon excitation in the vicinity of the extra-core nucleons. Consequently, the motion of the extra-core nucleons is coupled to the "Bose branch" of the core and the residual deformation is simply the equilibrium deformation of the core, caused by the intrinsic structure of the extra-core nucleons.

The concept of the residual deformation, which is essentially the original idea of Rainwater,[1] may be further explained by using the arguments given by Mayer and Jensen.[4] Since the total angular momentum operator has the same structure in momentum space (plus spin space) as in ordinary space (plus spin space), the extra-core nucleons in a state of nonvanishing total angular momentum ($>\frac{1}{2}\hbar$) always exert pressure on the wall of the average central (in general not spherical) potential, which follows the mass distribution of the core. Therefore, whenever the core remains in the state of zero-point oscillation, the anisotropy of the mass distribution of the core always follows the anisotropy of the mass distribution of the extra-core nucleons. From these arguments, it becomes clear that the residual deformation is essentially a sort of self-energy correction (or effective mass correction) for the extra-core nucleons. However, the magnitude of this correction should depend upon the intrinsic structure of the core. In the theory of Bohr and Mottelson, this dependence is characterized by the mass parameter B and the elastic parameter C of the core.

The formulation given by Bohr and Mottelson is actually equivalent to the picture discussed above, except for the residual deformation of the core. In the so-called strong-coupling case, detail analysis of the residual deformation is rather complicated, because the intrinsic motion of the extra-core nucleons cannot be separated from the surfon excitation in the zeroth order. However, if the surfon excitation is much higher than the intrinsic excitation, the simple perturbation method can apply and the effect of the residual deformation is simply the relative shift of the well-defined intrinsic states of the extra-core nucleons.

In the present paper, we are particularly interested in the extremely limiting case in which the off-diagonal elements of the interaction between the core and the extra-core nucleons can be entirely neglected. In other words, the core can be considered as remaining in the state of zero-point oscillation. In the usual Bohr-Mottelson theory this zero-point state is taken as the vacuum state of the whole nuclear system. This is correct if the residual deformation is not included. However, since the residual deformation is a sort of self-energy correction for the extra-core nucleons, the vacuum state of the whole nuclear system is not just the zero-point oscillation of the core. Therefore, the intrinsic motion of the extra-core nucleons should deviate from the shell-model description even when no surfon excitation occurs.

Now, taking into account the most important component of the residual deformation, namely, the spherical harmonic of second order $Y_{2\mu}$, we can easily show by a simple perturbation calculation that the energy shift of a state with total angular momentum J (assuming that there is no surfon excitation) is given by[4]

$$\Delta E_J = -\epsilon V_0 \frac{R^3}{3} |\varphi(R)|^2 Q_J^s / \langle r^2 \rangle, \tag{1}$$

where Q_J^s is the shell-model quadrupole moment in the state J and the other symbols are the same as in reference 4.

It should be noted that the derivation of Eq. (1) is originally based on a static model; but the identical result can be obtained also in a dynamical theory, if one recalls that, in the body-fixed coordinate system of the core with no surfon excitation, the quantum projection of $\hat{J}$ on the body-fixed 3-axis must be J (or $-J$) and, therefore, only the diagonal matrix element of Y_{20} can survive in the perturbation calculation. It is important to observe that the existence of the body-fixed coordinate system, which is the most natural system to define the symmetry of the average central potential for the extra-core nucleons, is guaranteed by the existence of the residual deformation.

It is emphasized that Eq. (1) is not a direct consequence of the usual Bohr-Mottelson theory. It will be shown in Sec. II, however, that this minor shortcoming can be easily removed by restricting the variation range of the deformation variable γ to the interval $0 \leq \gamma < \pi/3$. The theoretical basis for this restriction will be discussed later.

Finally, some remarks about the role played by the adiabatic principle in the ensuing discussion are in order. In the strong-coupling case the surfon excitation has always been assumed much lower than the intrinsic excitation of the extra-core nucleons. Then, applying the adiabatic principle, we can average the collective dynamic variables, which describe the surfon excitation, over an intrinsic state; and these averaged dynamic variables can be treated as parameters. This situation is similar to that of the molecular spectra and the strong-coupling version of Bohr and Mottelson's theory may be called the semimolecular model. In the molecular case, there is strong basis for assuming that the motion of nuclei in a molecule is very slow compared with the electronic motion. However, for a nucleus, there is no apparent reason why the opposite case cannot occur. The physical basis for the following discussion is the assumption that this opposite case does exist. To distinguish it from the previous theories, our formulation will be called the semiatomic model (not identical to the shell model).

To recapitulate, we may say that, when the surfon excitation is much lower than the intrinsic excitation, the nuclear system can be approximated by the semi-molecular model; and in the opposite case, it will be more closer to the semiatomic model.

A more rigorous formulation, which includes Eq. (1) as its special case, will be presented in Sec. II. Then the results of this formulation will be applied to the calculation of the nuclear properties of $_{25}Mn^{55}$ and some other nuclei in the region of $(1f_{7/2})$ shell.

II. FORMULATION OF THE PROBLEM

For simplicity, let us consider a nucleus with ν extra-core nucleons of the same type in the shell with total angular momentum j. Following the prescription given in the introduction, the total Hamiltonian of this nucleus may be written as follows[2,3,8]

$$\hat{H}=\hat{H}_p+\hat{H}_c, \tag{2}$$

$$\hat{H}_p=\sum_{i=1}^{\nu}\frac{\mathbf{p}_i^2}{2m}+\sum_{i=1}^{\nu}V_1(r_i,\alpha_{l\mu})+\tfrac{1}{2}\sum_{i=1}^{\nu}\{V_2(r_1,\alpha_{l\mu})\mathbf{l}_i\cdot\mathbf{s}_i+\mathbf{l}_i\cdot\mathbf{s}_iV_2(r_i,\alpha_{l\mu})\}+\sum_{i<j}^{\nu}V_3(|\mathbf{r}_i-\mathbf{r}_j|), \tag{3}$$

$$\hat{H}_c=\tfrac{1}{2}\sum_{l\mu}B_l|\dot{\alpha}_{l\mu}|^2+\tfrac{1}{2}\sum_{l\mu}C_l|\alpha_{l\mu}|^2, \tag{4}$$

where $\hat{H}_p$ is the Hamiltonian of the extra-core nucleons and $\hat{H}_c$ is that of the core. The Hamiltonian $\hat{H}_c$ is given in reference 2 with the Racah tensors $\alpha_{l\mu}$ playing the role of the normal coordinates of the surface oscillation of the core.

In Eq. (4), the total number of terms has not been specified. In order to avoid the difficulty of redundancy of coordinates, we shall assume that the total number of terms in Eq. (4) should not exceed the total number of degrees of freedom of the core ($3A'$ and A' is the number of nucleons in the core). However, practically, only the terms with $l=2$ should be considered, so actually there is a deficiency of coordinates. In view of the close packing of the core, this deficiency may be explained by assuming that most of the intrinsic degrees of freedom of the core have been "frozen up," so to speak, by the exclusion principle, except for those near the surface of the core. (Of course, these arguments can apply to the low-energy region only.) It is noted that a term which accounts for the binding energy of the core should be added to $\hat{H}_c$; however, in the low-energy region this term is of no relevance in the dynamical sense.

[8] J. P. Davison and E. Feenberg, Phys. Rev. **89**, 856 (1953); C. Levinson and K. W. Ford, *ibid.* **99**, 792 (1955).

The Hamiltonian $\hat{H}_p$ is the usual shell-model Hamiltonian except for the assumption that both the single-particle central potential $V_1(r_i,\alpha_{l\mu})$ and the spin-orbit interaction potential $V_2(r_i,\alpha_{l\mu})$ are nonspherical. It should be pointed out that these two potentials should in general also depend on the time derivative of $\alpha_{l\mu}$. However, for two limiting cases, this dependence may be neglected. One is the case in which the surfon excitation is very low and consequently $\dot{\alpha}_{l\mu}$ is very small; and the other is that in which the surfon excitation is so high that, after time average, the intrinsic deformation drops out and only the residual deformation remains. In the present paper, the latter case will be our particular concern.

For simplicity, the residual interaction [last term of Eq. (3)] has been assumed as of two-body type. It is important to observe that the residual interaction so assumed does not depend upon the deformation of the central potential.

The origin of the coordinate system to which all operators in Eqs. (3) and (4) are referred coincides with the center of mass of the core. The three axes of this coordinate system are fixed in space; and this coordinate system will be called the space-fixed system.

The shape of the central potential $V_1(r_i,\alpha_{l\mu})$ is assumed to follow the mass distribution of the core, which to a certain extent is also affected by the mass distribution of the extra-core nucleons (because of the residual deformation). We shall assume that the total deformation of the core is given by the equation[2,3]

$$R'=R_0'[1+\sum_{l\mu}\alpha_{l\mu}^*Y_{l\mu}(\theta,\varphi)]=R_0'[1+\sum_{l\mu}(-1)^{\mu}\alpha_{l\mu}Y_{l-\mu}(\theta,\varphi)], \tag{5}$$

where R' is the radius of the core and R_0' is the average value of R' ($R_0'=1.5\times10^{-13}A'^{1/3}$ cm). Then, treating the core as a Fermi fluid, the average central potential may be written as follows[9]

$$V_1(r_i,\alpha_{l\mu})=-V_0f(E_Fr_i/R')=-V_0\Big/\left\{1+\exp\left[\frac{E_F}{\tau R'}(r_i-R')\right]\right\}, \tag{6}$$

where V_0 is the depth of the central potential and $f(E_Fr_i/R')$ is the Fermi distribution function. In writing the central potential in this form, the parameters E_F and τ (in energy) may be loosely interpreted as the Fermi energy and the nuclear temperature, respectively.

From Eqs. (5) and (6) we can easily show that

$$V_1(r_i,\alpha_{l\mu})=-V_0f(E_Fr_i/R_0')-V_0R_0'\delta(r_i-R_0')\sum_{l\mu}\alpha_{l\mu}^*Y_{l\mu}(\theta,\varphi), \tag{7}$$

[9] A. A. Rose, H. Mark, and R. D. Lawson, Phys. Rev. **102**, 1613 (1956).

where higher orders of $\alpha_{l\mu}$ have been neglected. The first term of Eq. (7) is isotropic and has the proper form as required by the existence of the magic numbers, and the second term is essentially the surface-particle interaction operator proposed by Rainwater *et al.*[1,10]

Following Maris' treatment[11] of the spin-orbit coupling in a nonspherical central potential, we may write

$$\tfrac{1}{2}\{V_2\mathbf{l}_i\cdot\mathbf{s}_i+\mathbf{l}_i\cdot\mathbf{s}_iV_2\}=\frac{\lambda\hbar}{4m^2c^2}\{\nabla V_1\times\mathbf{p}_i-\mathbf{p}_i\times\nabla V_1\}\cdot\mathbf{s}_i, \quad (8)$$

where λ is a parameter. If the central potential V_1 is spherical, the right-hand side of Eq. (8) reduces to the familiar Thomas form. However, using Eqs. (6) and (8), it can easily be shown that the Thomas form is still valid even to the first order of $\alpha_{l\mu}$. Therefore, the spin-orbit interaction is not sensitive to the deformation of the central potential. Neglecting the higher orders of $\alpha_{l\mu}$, we then have

$$\tfrac{1}{2}\{V_2\mathbf{l}_i\cdot\mathbf{s}_i+\mathbf{l}_i\cdot\mathbf{s}_iV_2\}=-\frac{\lambda\hbar^2V_0}{2m^2c^2r_i}\delta(r_i-R_0')\mathbf{l}_i\cdot\mathbf{s}_i. \quad (9)$$

The simple form of Eq. (9) may be useful in estimating the strength of the spin-orbit interaction. For example, by using the oscillator eigenfunction,[4] it is found that the strength of the spin-orbit interaction is proportional to

$$A^{-2/3}(l+\tfrac{3}{2})^{l+3/2}[2^{l+2}/(2l+1)!!]\exp\{-(l+\tfrac{3}{2})\}$$

for $n=1$, where n is the principal quantum number, l is the orbital quantum number, and $(2l+1)!!\equiv1\times3\times5\times\cdots\times(2l+1)$. By using this expression and the shell-model level scheme, we can estimate the relative strength of the spin-orbit coupling in different regions of the nuclear chart. For $l=5$ (in the region of $_{83}Bi^{209}$) and $l=1$ (in the region of $_{17}N^{15}$), the ratio is approximately ten. This may explain the fact that j-j coupling is more appropriate in the region of medium-weight and heavy nuclei.

The angular variables θ and φ in the second term of Eq. (7) is interpreted as the angular dependence of the deformation of the central potential in the space-fixed coordinate system, so if the extra-core nucleons can be treated as completely independent, we may need an index i for these angular variables to indicate that the operator $Y_{l\mu}(\theta_i,\varphi_i)$ is a single-particle operator acting on the ith extra-core nucleon only. However, if the correlation of the extra-core nucleons is not negligible, ambiguity would arise. In view of the pairing effect, it seems reasonable to assume that for even values of ν the correlation may be so strong that the extra-core nucleons interact coherently with the core and, therefore, the net effect of this interaction may be considered as due to a single particle with a mass of all the extra-core nucleons.[3] Then, the operator $Y_{l\mu}(\theta,\varphi)$ will be interpreted as acting on the center of mass of the extra-core nucleons. As for odd values of ν, we shall assume that the extra-core nucleons interact independently with the core.

The total Hamiltonian can be much simplified by transforming $\hat{H}_c$ and the surface-particle interaction to a coordinate system attached to the core (body-fixed coordinate system). This transformation has already been carried out by Bohr.[2] It is noted, however, that the magnitude of the position vector $\mathbf{r}_i$ is not affected by this transformation, because only a rotation of the space-fixed coordinate system about the center of mass of the core is involved.

Now, following the notations of Bohr, the total Hamiltonian may be written as follows:

$$\hat{H}=(\hat{H}_p{}^0+\hat{H}_p{}')+(\hat{H}_{\rm vib}+\hat{H}_{\rm rot})+\hat{H}_{\rm int}, \quad (10)$$

$$\hat{H}_p{}^0+\hat{H}_p{}'=\sum_{i=1}^{\nu}\left\{\frac{\mathbf{p}_i^2}{2m}-V_0f(E_Fr_i/R_0')-\frac{\lambda\hbar^2V_0}{2m^2c^2r_i}\delta(r_i-R_0')\mathbf{l}_i\cdot\mathbf{s}_i\right\}+\sum_{i<j}^{\nu}V_3(|\mathbf{r}_i-\mathbf{r}_j|), \quad (11)$$

$$\hat{H}_{\rm vib}+\hat{H}_{\rm rot}=\left\{-\frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\left(\beta^4\frac{\partial}{\partial\beta}\right)+\frac{1}{\beta^2\sin3\gamma}\frac{\partial}{\partial\gamma}\left(\sin3\gamma\frac{\partial}{\partial\gamma}\right)\right]+\tfrac{1}{2}C\beta^2\right\}+\sum_{\lambda=1}^{3}\hbar^2\hat{Q}^2/8B\beta^2\sin^2\left(\gamma-\frac{2\lambda\pi}{3}\right). \quad (12)$$

In writing Eq. (12), only the terms with $l=2$ in Eq. (4) have been considered.

It is noted that all operators in Eq. (11) are referred to space-fixed coordinate system. However, each term of this equation is invariant under space rotation, so all operators can also be considered as referred to the body-fixed coordinate system.

As mentioned previously, the interaction Hamiltonian $\hat{H}_{\rm int}$ in Eq. (10) may be written in two different forms, depending upon the intrinsic structure of the extra-core nucleons. We shall assume that, for even values of ν,[3,12,13]

$$\hat{H}_{\rm int}=-\frac{1}{4}\left(\frac{5}{4\pi}\right)^{1/2}\frac{1}{J(J+1)}\times K\beta[\cos\gamma(3\hat{J}_3{}'^2-\hat{J}^2)+3\sin\gamma(\hat{J}_1{}'^2-\hat{J}_2{}'^2)], \quad (J\neq0), \quad (13a)$$

$$=0, \quad (J=0),$$

[10] E. Feenberg and K. C. Hammack, Phys. Rev. **81**, 285 (1951).
[11] Th. A. J. Maris, Nucl. Phys. **3**, 213 (1957).
[12] A. S. Davydov and G. F. Fillippov, Zh. Eksperim. i Teor. Fiz. **36**, 1497 (1959) [translation: Soviet Phys.—JETP **9**, 1061 (1959)].
[13] K. W. Ford, Phys. Rev. **90**, 29 (1953).

where $\hat{J}$ is the total angular momentum operator of the extra-core nucleons and $\hat{J}_\lambda'$ its projection on the body-fixed λ axis of the core; and for odd values of ν,

$$\hat{H}_{\text{int}}=-K\beta\cos\gamma\sum_{i=1}^{\nu}Y_{20}(\Omega_i')$$
$$-\frac{1}{\sqrt{2}}K\beta\sin\gamma\sum_{i=1}^{\nu}[Y_{22}(\Omega_i')+Y_{2-2}(\Omega_i')], \quad (13b)$$

where Ω_i' can be equivalently considered as the angular variables of the ith extra-core nucleon in the body-fixed coordinate system. The parameter K in Eqs. (13a) and (13b) is defined by the following equation

$$K=\int_0^\infty \varphi_{nlj}^*(r)[V_0 r\delta(r-R_0')]\varphi_{nlj}(r)r^2dr$$
$$=V_0R_0'^3|\varphi_{nlj}(R_0')|^2, \quad (14)$$

where $\varphi_{nlj}(r)$ is the radial eigenfunction of a single extra-core nucleon.

Now let us consider the solution of Eq. (10). Obviously $\hat{H}_p'$ and $\hat{H}_{\text{int}}$ should be very small in comparison with $\hat{H}_p^0$. What is not obvious here is the relative magnitudes of $\hat{H}_p'$ and $\hat{H}_{\text{int}}$ with respect to $\hat{H}_{\text{vib}}$ and $\hat{H}_{\text{rot}}$. Since no theoretical determination can be made on these relative magnitudes, the only thing we can do is to investigate all possible cases empirically. In the present paper, we shall consider the simplest case in which both $\hat{H}_p'$ and $\hat{H}_{\text{int}}$ are much smaller than $\hat{H}_{\text{vib}}$ and $\hat{H}_{\text{rot}}$. Under this specification, we then have a simple perturbation problem with the zero-order Hamiltonian given by

$$\hat{H}^0=\hat{H}_p^0+\hat{H}_{\text{vib}}+\hat{H}_{\text{rot}}. \quad (15)$$

Let $\chi_{J\mu}(\mathbf{x})$ be the orthonormal eigenfunction of $\hat{H}_p^0$, with definite angular momentum J and projection μ on the space-fixed z axis ($\mathbf{x}$ represents the configuration in the space-fixed coordinate system); and following the procedure of Bohr,[2] the orthonormal eigenfunction of $\hat{H}_{\text{vib}}+\hat{H}_{\text{rot}}$ will be expanded in terms of the rotation matrices $D_{\mu k}^Q(\xi_i)$,

$$\Phi_{Q\mu}^\tau(\beta,\gamma,\xi_i)=\sum_k \varphi_{Qk}^\tau(\beta,\gamma)D_{\mu k}^Q(\xi_i), \quad (16)$$

where Q is the total angular momentum of the core and μ its projection on space-fixed z-axis, and ξ_i ($i=1, 2, 3$) are the Euler angles specifying the body-fixed coordinates. Then from Eqs. (15) and (16) the zeroth-order wave function for a state with total angular momentum I and projection M can be written

$$\psi_{IM}(\tau,Q,J)=\sum_{k\mu}\langle QJ\mu\, M-\mu|IM\rangle\varphi_{Qk}^\tau(\beta,\gamma)$$
$$\times D_{\mu k}^Q(\xi_i)\chi_{J\,M-\mu}(\mathbf{x}), \quad (17)$$

where the bracket is the Clebsch-Gordan coefficient.

For the time being, the detailed structure of the eigenfunction $\chi_{J\mu}(\mathbf{x})$ is not necessary. The only property which concerns us here is that the $2J+1$ eigenfunctions $\chi_{J\mu}(x)$ with definite J and different projections (different μ) form a $(2J+1)$-dimensional representation of the rotation group. With this property, $\chi_{J\,M-\mu}(\mathbf{x})$ in Eq. (17) can be transformed to the body-fixed coordinate system by using the well-known relation[14,14a]

$$\chi_{J\,M-\mu}(\mathbf{x})=\sum_\Lambda D_{M-\mu\,\Lambda}^J(\xi_i)\chi_{J\Lambda}(\mathbf{x}'), \quad (18)$$

where Λ is the eigenvalue of $\hat{J}_3'$ and $\mathbf{x}'$ represents the configuration in the body-fixed coordinate system. The orthonormality of the rotation matrices $D_{M-\mu\,\Lambda}^J(\xi_i)$ requires that $\chi_{J\Lambda}(\mathbf{x}')$ be also orthonormal in $\mathbf{x}'$ space.

More precisely, the Euler angles ξ_i in Eq. (18) have different meaning from those in Eq. (17). It is obvious that the normalization volume elements for both sides of Eq. (18) are in $\mathbf{x}$ space and $\mathbf{x}'$ space, respectively; therefore, the Euler angles are regarded as fixed. On the other hand, in Eq. (17), the three Euler angles, together with β and γ, are considered as dynamic variables for specifying the shape of the core and the directions of the body-fixed axes, so the normalization volume element for Eq. (17) does include the Euler angles. In order to remove this discrepancy, we shall renormalize the whole expression on the right-hand side of Eq. (18) by including the Euler angles in the normalization volume element. After doing this, we have

$$\chi_{J\,M-\mu}(\mathbf{x})=N^{1/2}\sum_\Lambda D_{M-\mu\,\Lambda}^J(\xi_i)\chi_{J\Lambda}(\mathbf{x}'), \quad (18')$$

where N is the normalization constant.

It is clear that the step from Eq. (18) to (18′) has been taken in order to identify the Euler angles which specify the transformation of the intrinsic wave function with the Euler angles which play the role of dynamic variables. This identification is always permissible if the residual deformation exists (because the body-fixed coordinate system can be defined). Furthermore, the Euler angles as the arguments of the rotation matrices in both Eqs. (18) and (18′) cannot be interpreted as additional degrees of freedom for the extra-core nucleons. Therefore, there is no question of redundancy of coordinates (as usually raised in arguing the theory of Bohr and Mottelson).

After imposing Bohr's symmetry requirements, the expansion coefficients $\varphi_{Qk}^\tau(\beta,\gamma)$ must satisfy the fol-

[14] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957).

[14a] It is interesting to note that this equation has exactly the same form as Eq. (16). However, conventionally, it should be written as $\chi_{JM-\mu}(\mathbf{x})=\Sigma_\Lambda D_{M-\mu\Lambda}^{J*}(\xi_i)\chi_{J\Lambda}(\mathbf{x}')$, where the star indicates complex conjugate. The reason for dropping the complex conjugate here is that we can replace $D_{\mu k}^Q(\xi_i)$ by $D_{\mu k}^{Q*}(\xi_i)$ in Eq. (16) without any loss of generality, and then all derivations following Eq. (18) would not be changed except for replacing all rotation matrices by their complex conjugates.

lowing conditions[2]:

$$\varphi_{Qk}{}^{\tau}(\beta,\gamma)=(-1)^{Q}\varphi_{Q-k}{}^{\tau}(\beta,\gamma), \quad (19)$$

$$=(-1)^{k/2}\varphi_{Qk}{}^{\tau}(\beta,-\gamma), \quad (20)$$

$$=\sum_{k'} D_{kk'}{}^{Q}(\pi/2,\pi/2,0)\varphi_{Qk'}{}^{\tau}(\beta,\gamma-\tfrac{2}{3}\pi). \quad (21)$$

(Note that k must be even.) By using Eq. (19) and some other formulas involving the Clebach-Gordan coefficients and the rotation matrices, we can easily show that Eqs. (17) and (18′) can be combined in a rather compact form. (See the derivation in the Appendix.)

$$\psi_{IM}(\tau,Q,J)=(N')^{1/2}\sum_{k\Lambda}\langle QJk\Lambda\,|\,Ik+\Lambda\rangle \times\varphi_{Qk}{}^{\tau}(\beta,\gamma)\{D_{Mk+\Lambda}{}^{I}(\xi_i)\chi_{J\Lambda}(\mathbf{x}') +(-1)^{I-J}D_{M-(k+\Lambda)}{}^{I}(\xi_i)\chi_{J-\Lambda}(\mathbf{x}')\}, \quad (22)$$

where N' is another normalization constant.

The wave function given by Eq. (22) is in general very complicated, because the expansion coefficients $\varphi_{Qk}{}^{\tau}(\beta,\gamma)$ should satisfy a very complicated set of coupled differential equations. However, there is one simple case (actually two simple cases as will be seen later) in which only one expansion coefficient is necessary. Suppose that, in the low-energy region, the excited states of the core can be entirely neglected. In other words, the core remains in the state of zero-point oscillation. Then, for the reason mentioned earlier, Λ can only take on the value J (or $-J$) in Eq. (22). Therefore, after setting $Q=k=0$ in Eq. (22), we have

$$\psi_{JM}=\left(\frac{2J+1}{16\pi^2}\right)^{1/2}\varphi_0(\beta,\gamma) \times\{D_{MJ}{}^{J}\chi_{JJ}(\mathbf{x}')+D_{M-J}{}^{J}\chi_{J-J}(\mathbf{x}')\}. \quad (23)$$

[The symbol τ in Eq. (22) represents additional quantum numbers necessary to specify the state of the core; so long as we restrict ourselves to zero-point oscillation, this symbol can be dropped without loss of generality.]

It is important to point out that the function given by Eq. (23) should not be misinterpreted as a strong-coupling wave function in the sense of Bohr and Mottelson. On the contrary, it represents the extreme opposite situation. The strong-coupling is a different representation in which $k+\Lambda$ and Λ are good quantum numbers. Therefore, so far as the angular part is concerned, the terms under the summation signs of Eq. (22) form the proper basis of this representation. It is noted that the expression in the curly bracket of Eq. (22) is identical to the angular part of the strong-coupling wave function given by Bohr and Mottelson if $k+\Lambda$ and Λ are good quantum numbers. [In this case too, there is only one expansion coefficient in each wave function; but these expansion coefficients are different from those in Eq. (22).] Furthermore, it is noted that the function ψ_{JM} is not identical to the shell-model wave function which is a complete analog of the atomic case. Therefore, this function may be called the semi-atomic-model wave function.

The differential equation for the expansion coefficient $\varphi_0(\beta,\gamma)$ has been solved by Bohr, and the solution is[2]

$$\varphi_0(\beta,\gamma)=\left[4\pi^{-1/2}\left(\frac{BC}{\hbar^2}\right)^{5/4}\right]^{1/2}\exp\left\{-\frac{1}{2\hbar}(BC)^{1/2}\beta^2\right\}. \quad (24)$$

The normalization volume element of $\varphi_0(\beta,\gamma)$ is $dv=\beta^4\sin3\gamma\,d\beta\,d\gamma$. In writing Eq. (24), a misprint which appeared in reference 2 has been corrected; and a factor of $(3/2)^{1/2}$, which results from the choice of the variation range of γ, has been introduced. We shall now discuss the question of the variation range of γ, which is an essential point in our theory.

In the semimolecular case, γ is usually treated as a parameter representing average value over one intrinsic state. However, strictly speaking, γ should be considered as a dynamical variable, especially in the semi-atomic case in which the parametric approach is totally unjustified. Therefore, the variation range of γ should be chosen in such a way that any γ-dependent operator, particularly contained in the total Hamiltonian of the nuclear system, is single-valued.[15] By using this condition of single-valuedness and the fact that the three body-fixed axes are physically equivalent, an appropriate definition of the range of γ can be obtained.

It is noted that in the theory of Bohr and Mottelson, the range of γ has not been clearly defined although it has been pointed out by Bohr[2] that Eqs. (20) and (21) "effectively" limit the range of γ to the interval $0\leq\gamma<\pi/3$. Furthermore, there is no theoretical basis for making the conventional assumption that the range of γ is the interval $0\leq\gamma<2\pi$. On the contrary, it is easy to see from the Hamiltonian of the core [Eq. (12)] that, in order to satisfy the condition of single-valuedness, this interval should be excluded in the first place. (The operator $\sin3\gamma$ is obviously not single-valued in the interval $0\leq\gamma<2\pi$.)

In order to go further, we have to consider different specific cases. Let us first consider an even-even nucleus in the so-called deformed region and restrict ourselves to the consideration of the lowest rotational band. Since the total angular momentum of the extra-core nucleons vanishes in this case, the surface-particle interaction can be entirely ignored. Therefore, we are effectively considering the motion of the core only. In the Hamiltonian of the core [Eq. (12)], there are three more γ-dependent operators which need careful consideration, namely, the three principal moments of inertia of the core,

$$M_\lambda=8B\beta^2\sin^2(\gamma-\tfrac{2}{3}\lambda\pi), \quad \lambda=1,2,3. \quad (25)$$

A simple calculation will show that for fixed β all

[15] P. A. M. Dirac, *The Principle of Quantum Mechanics* (Oxford University Press, London, 1947), 3rd ed., p. 43.

possible values of M_λ can be exhausted by varying γ in any one of the intervals $n\pi/3 \leq \gamma < (n+1)\pi/3$ ($n = \pm$integer or zero) although relabeling the index λ is necessary when γ changes from one interval to another. Obviously, relabeling the index λ is equivalent to interchanging the body-fixed axes, which leaves Eq. (12) invariant. Therefore, all the intervals mentioned above are physically equivalent in the sense that all these intervals describe the same physical situation. However, in order to satisfy the condition of single-valuedness for the operators M_λ, the range of γ must be limited to one (but any one) of these intervals, say, the interval $0 \leq \gamma < \pi/3$. It is noted that this conclusion is consistent with the "nonsymmetric-top" theory of Davydov and Fillippov.[16]

For later arguments, let us consider the case discussed above a step further. It is easy to show that Eq. (12) is invariant under the transformations: $\gamma \rightarrow -\gamma$ and $\gamma \rightarrow \gamma + n\pi/3$ ($n = \pm$integer). Then, following the same procedure of proving the usual parity rule, we can show that $\varphi_{Qk}{}^{\tau}(\beta,\gamma)$ as a function of γ must satisfy the following conditions:

$$\varphi_{Qk}{}^{\tau}(\beta,\gamma) = \pm \varphi_{Qk}{}^{\tau}(\beta, -\gamma), \tag{26}$$

$$= \pm \varphi_{Qk}{}^{\tau}(\beta, \gamma + n\pi/3). \tag{27}$$

It is noted that Eq. (26) is identical to Eq. (20) if we recall that k must take on even integers.

From the above discussion, we can only conclude that the range of γ must be limited to one (but any one) of the intervals $n\pi/3 \leq \gamma < (n+1)\pi/3$ when the interaction Hamiltonian $\hat{H}_{\text{int}}$ can be entirely ignored. Now let us consider the case of odd-even nuclei and suppose that the surface-particle interaction is weak and $\hat{H}_{\text{int}}$ can be treated as perturbation. Such a case may be represented by the wave function given by Eq. (22). Then, by using Eq. (26) and Eq. (27), we can easily show that the expectation value of $\hat{H}_{\text{int}}$ has the following dependence on the range of γ:

$$\langle \hat{H}_{\text{int}} \rangle_{[0,\pi/3]} = I, \quad \langle \hat{H}_{\text{int}} \rangle_{[\pi/3,2\pi/3]} = 0, \quad \langle \hat{H}_{\text{int}} \rangle_{[2\pi/3,\pi]} = -I,$$

$$\langle \hat{H}_{\text{int}} \rangle_{[n\pi/3,(n+1)\pi/3]} = \langle \hat{H}_{\text{int}} \rangle_{[(n+3)\pi/3,(n+4)\pi/3]},$$

$$\langle \hat{H}_{\text{int}} \rangle_{[-(n+1)\pi/3,-n\pi/3]} = -\langle \hat{H}_{\text{int}} \rangle_{[n\pi/3,(n+1)\pi/3]},$$

where I represents the expectation value of $\hat{H}_{\text{int}}$ when the interval $0 \leq \gamma < \pi/3$ is chosen as the range of γ and n is a positive integer or zero. From these relations, we can easily see that the expectation value of $\hat{H}_{\text{int}}$ can be made arbitrary by changing the range of γ. For instance, $\langle \hat{H}_{\text{int}} \rangle$ vanishes identically when the interval $0 \leq \gamma < 2\pi$ is taken as the range of γ. This undesirable situation is obviously a serious contradiction to the assumption of surface-particle interaction. On the other hand, in order to avoid this contradiction, there are effectively only two choices for the range of γ, namely, $0 \leq \gamma < \pi/3$ and $2\pi/3 \leq \gamma < \pi$. As for the final decision between these two, we consider the extremely weak coupling case (or the semiatomic case) in which the expectation value of $\hat{H}_{\text{int}}$ can be calculated explicitly. (It is noted that the "zero-point" surface-particle interaction does not vanish when the range of γ is limited to either one of the intervals mentioned above.) Then, by comparing the result with Eq. (1), we found that it is proper to choose the interval $0 \leq \gamma < \pi/3$ as the range of γ.

Finally, we come to the case of odd-even nuclei with strong surface-particle interaction. In this case, we can no longer treat $\hat{H}_{\text{int}}$ as a perturbation. Furthermore, since the total Hamiltonian including the diagonal part of $\hat{H}_{\text{int}}$ is not invariant under the transformation $\gamma \rightarrow \gamma + n\pi/3$ ($n \neq 6m$ and $m = \pm$integer), similar condition to Eq. (27) is not valid. Therefore, there is some ambiguity about the range of γ in this case. However, with a consideration of the condition of single-valuedness and the general principle of physical continuation, we may assume that in this case the range of γ is also limited to the interval $0 \leq \gamma < \pi/3$.

In the semimolecular model of Bohr and Mottelson, the equilibrium shape of the core is supposed to be a prolate or oblate spheroid (or symmetric top) depending upon the intrinsic structure of the extra-core nucleons. For the case of oblate spheroid the average value of γ (denoted by $\bar{\gamma}$) is usually taken as π. It is noted that this assignment of π to $\bar{\gamma}$ is in some sense not a contradiction to the above discussion of restricting the range of γ to the interval $0 \leq \gamma < \pi/3$. However, this consistency is rather incidental. We know that the assignment of π to $\bar{\gamma}$ comes from the classical relation $\langle \beta \cos\gamma \rangle_{\text{av}} = -\bar{\beta}$ which actually tells nothing about the average value of γ. The only thing which we can derive from this relation is that we may set $\langle \cos\gamma \rangle_{\text{av}} = -1$. Therefore, in so far as this classical relation is concerned, we may effectively change $\langle \cos\gamma \rangle_{\text{av}}$ into $\cos\bar{\gamma}$ and consider π as the average value of γ. However, the actual average value of γ may be rather different from π. Furthermore, the average value of a function of β and γ should not just be a physically meaningless number such as zero or π but a function of the more fundamental parameters B and C which specify the dynamical properties of the core. However, this rigorous point of view is usually not taken (the average value of $\beta \cos\gamma$ has not been rigorously carried out except for the simple case considered in this paper), because the Schrödinger equation is, in general, too difficult to solve. As a final remark about the range of γ, we notice the fact that the average value of a function of β and γ is in general not equal to the function of the average values of β and γ and that we fail to recognize this fact leads to the doubtful belief that the equilibrium shape of the core should be limited to a prolate or oblate symmetric top.

By using the semiatomic-model wave function [Eq. (23)], the expectation value of the total Hamiltonian in-

[16] A. S. Davydov and G. F. Fillippov, Nucl. Phys. 8, 237 (1958).

cluding first-order perturbation can be easily calculated

$$E_J=E_0+5/2\hbar\omega_0+E_J'+\langle\hat{H}_{\rm int}\rangle_J. \quad (28)$$

The first two terms of Eq. (28) are "constants" which come from $\hat{H}_p{}^0$ and the zero-point energy, $\omega_0=(C/B)^{1/2}$. The contribution from the residual interaction, given by the third term, will be discussed for a specific case later. As for the last term, it can be written in two different forms, depending on whether Eq. (13a) or Eq. (13b) is used in the calculation. The derivations are given in the Appendix, and the final results are: For even values of ν,

$$\langle\hat{H}_{\rm int}\rangle_J=5^{1/2}\left(\frac{9K}{32\pi}\right)\left(\frac{\hbar^2}{BC}\right)^{1/4}(2J-1)/(J+1), \quad (J\neq 0),$$
$$=0, \quad (J=0); \quad (29a)$$

and for odd values of ν,

$$\langle\hat{H}_{\rm int}\rangle_J=-5^{1/2}\left(\frac{9K}{16\pi}\right)\left(\frac{\hbar^2}{BC}\right)^{1/4}\frac{1}{\langle r^2\rangle}\left\{(-1)^{J+j-1\frac{1}{2}}\langle r^2\rangle\left(\frac{2j+2-\nu}{2J+1}\right)\left[\frac{(2j-1)(2j+1)(2j+3)J(2J+1)(2J-1)}{j(j+1)(J+1)(2J+3)}\right]^{1/2}\right.$$
$$\left.\times\sum_{J_1}(2J_1+1)\,|\,(j^{2j+2-\nu}J_1[\![j^{2j+1-\nu}(J)jJ_1)\,|^2W(jjJJ;2J_1)\right\}. \quad (29b)$$

Here $(j^{2j+2-\nu}J_1[\![j^{2j+1-\nu}(J)jJ_1)$ is the parentage coefficient and $W(jjJJ;2J_1)$ is the Racah coefficient. It is noted that the expression in the curly bracket of Eq. (29b) is just the shell-model quadrupole moment $Q_J{}^s$ and, therefore, Eq. (29b) has the exact form of Eq. (1) after setting the parameter $\epsilon=5^{1/2}(27/16\pi)\times(\hbar^2/BC)^{1/4}$ in that equation. However, the intrinsic quadrupole moment in the semiatomic model is not identical to the shell-model quadrupole moment as will be seen later.

III. NUCLEAR PROPERTIES OF $_{25}Mn^{55}$

A. Energy Level Structure

According to the shell model, the last five protons are in $(1f_{7/2})$ shell and the last two neutrons are in $(2p_{3/2})$ shell. Since these two shells are well separated, we shall assume that the last two neutrons also couple to zero-angular momentum and consider only the last five protons as the extra-core nucleons. Then the configuration of this nucleus can be simply written as $(1f_{7/2})^{-3}$. The possible intrinsic states of this configuration are $J=3/2$, $5/2$, $7/2$, $9/2$, $11/2$, and $15/2$.[4] All these states are degenerate, if both $\hat{H}_p'$ and $\hat{H}_{\rm int}$ are neglected. In general, the residual interaction is difficult to obtain from the experimental energy levels, because the contribution from $\hat{H}_{\rm int}$ is also included in the experimental values. In addition, the configuration interaction may complicate the situation further. However, for the special case in which we are interested, more reliable estimation seems possible.

Assuming two-body force for the residual interaction and following the treatment of Goldstein and Talmi,[17] we can write the energy difference $E_{J_1}(f_{7/2}{}^{-3})-E_{J_2}(f_{7/2}{}^{-3})$ as a linear combination of the differences $E_{I_1}'(f_{7/2}{}^{-2})-E_{I_2}'(f_{7/2}{}^{-2})$ for appropriate values of I_1 and I_2. Here $E_I'(f_{7/2}{}^{-2})$ is the residual interaction for a nucleus with configuration $(f_{7/2})^{-2}$, the possible states of which are $I=0$, 2, 4, and 6.[4] After some rather straightforward manipulation of Racah coefficients and parentage coefficients, we have the results

$$E_{3/2}-E_{7/2}=(19/84)(E_2'-E_0')+(45/28)(E_4'-E_0')-(13/12)(E_6'-E_0'), \quad (30a)$$

$$E_{5/2}-E_{7/2}=(17/12)(E_2'-E_0')-(25/44)(E_4'-E_0')-(13/132)(E_6'-E_0'), \quad (30b)$$

$$E_{9/2}-E_{7/2}=-(3/28)(E_2'-E_0')+(369/308)(E_4'-E_0')-(45/132)(E_6'-E_0'), \quad (30c)$$

$$E_{11/2}-E_{7/2}=(5/12)(E_2'-E_0')-(7/44)(E_4'-E_0')+(65/132)(E_6'-E_0'), \quad (30d)$$

$$E_{15/2}-E_{7/2}=-(5/12)(E_2'-E_0')-(3/44)(E_4'-E_0')+(163/132)(E_6'-E_0'). \quad (30e)$$

For simplicity, the indication of configuration has been dropped in Eqs. (30a–e). If the experimental values of the excited states of $_{26}Fe^{54}$ (or $_{24}Cr^{52}$), which is supposed to have a core of double-closed shells and six extra-core protons in $(1f_{7/2})$-shell, are substituted in Eqs. (30a–e) the results are in agreement with the values calculated by Lawson and Uretsky.[18] However, it is important to observe that the energy differences on the right-hand side of Eqs. (30a–e) cannot be taken directly from the experimental values, because they represent the contributions from the residual interaction only. Furthermore, the structure of the core of $_{26}Fe^{54}$ is different from that of $_{25}Mn^{55}$. Therefore, in order to take into account these factors, the experimental values of the excited states of $_{26}Fe^{54}$ must be corrected for the contributions from $\hat{H}_{\rm int}'$ before they are substituted in Eqs. (30a–e). Here $\hat{H}_{\rm int}'$ means Eq. (13a) or Eq. (29a) except for replacing the parameter K, B, and C by K', B', and C', which corresponds to $_{26}Fe^{54}$.

Let us define $S=5^{1/2}(9K/32\pi)(\hbar^2/BC)^{1/4}$ and $S'=5^{1/2}(9K'/32\pi)(\hbar^2/B'C')^{1/4}$. Then, using Eqs. (28), (29)

[17] S. Goldstein and I. Talmi, Phys. Rev. **102**, 589 (1956).

[18] R. D. Lawson and J. L. Uretsky, Phys. Rev. **106**, 1369 (1957).

and (30), we have

$$E_{3/2}-E_{7/2}=0.987+0.774S'+1.244S, \quad (31a)$$

$$E_{5/2}-E_{7/2}=0.243+0.467S'+0.794S, \quad (31b)$$

$$E_{9/2}-E_{7/2}=1.818+1.034S'+1.110S, \quad (31c)$$

$$E_{11/2}-E_{7/2}=1.744+1.054S'+0.302S, \quad (31d)$$

$$E_{15/2}-E_{7/2}=3.141+1.212S'-0.552S. \quad (31e)$$

As pointed out before, the expression in the curly bracket of Eq. (29b) is just the shell-model quadrupole moment $Q_J{}^s$. For $j=7/2$ and $\nu=5$, the numerical values agree with the results given by Mayer and Jensen[4] except for a negative sign for $Q_{3/2}{}^s$. This correction has been included in Eq. (31a). (The values of $Q_J{}^s$ have been tabulated in the Appendix.) The ground state of $_{25}Mn^{55}$ is $J=5/2$ and the first excited state is $J=7/2$ (0.126 MeV). Assuming that the second excited state is $J=3/2$ (0.983 MeV), then, using Eqs. (31a, 31b), we can determine $S=0.147$ MeV and $S'=-0.487$ MeV. With these two parameters and Eqs. (31c–e), the other excited states of $_{25}Mn^{55}$ can be calculated. The results, presented in Fig. 1, are in reasonable agreement with the experimental values. Moreover, the experimental energy separation between the fourth and fifth levels and their symmetric position between the third and sixth levels are exactly produced in our calculation.

It is noted that Eqs. (31a–e) can also be used to calculate the level structure of $_{23}V^{51}$.[18] However, in this case, both S and S' should be vanishingly small. This is probably due to the fact that the core of $_{23}V^{51}$ has the same double-closed-shells structure as that of $_{26}Fe^{54}$ and, therefore, their intrinsic motions can be compared on equal footing.

From the above discussion, it is clear that each nucleus as an unique entity by its own, in general, shares no exact common ground with other nuclei. Only until this exact common ground is built up, it is impossible to predict the exact relation among nuclei just by looking at the shell-model level scheme (as looking at the periodic table in atomic physics). From this point of view, the unified model may be considered as a semiempirical method for building up such a common ground.

B. Magnetic Moment

In the semiatomic model, the core makes no contribution to the nuclear magnetic moment. By transforming the total angular momentum operator $\hat{J}$ of the extra-core nucleons to the body-fixed coordinate system, the magnetic moment operator can be written as follows:

$$\hat{\mu}=g_j'\hat{J}=g_j'\sum_{\lambda=-1}^{+1}\left\{D_{0\lambda}{}^1\hat{J}_\lambda'-\frac{(1-i)}{\sqrt{2}}D_{1\lambda}{}^1\hat{J}_\lambda'+\frac{(1+i)}{\sqrt{2}}D_{-1\lambda}{}^1\hat{J}_\lambda'\right\}, \quad (32)$$

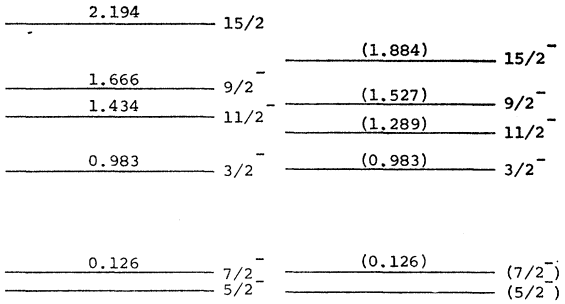

FIG. 1. Level structure of $_{25}Mn^{55}$. On the left are shown the predicted energy levels (in MeV), parity, and spins. The experimental values which are taken from reference 24 are listed in brackets.

where $\hat{J}_0'=\hat{J}_3'$,

$$\hat{J}_{+1}'=-(\hat{J}_1'-i\hat{J}_2')/\sqrt{2}, \text{ and } \hat{J}_{-1}'=(\hat{J}_1'+i\hat{J}_2')/\sqrt{2}.$$

(Note that $\hat{J}_{+1}'$ should not be mistaken for $\hat{J}_1'$.) The operators $\hat{J}_{\pm1}'$ are connected with $\hat{J}_{\pm1}$ (in the space-fixed coordinate system) by the relation $(\hat{J}_{\pm1}')^*=(\hat{J}_{\pm1})$,[19] where the star indicates complex conjugate. The gyromagnetic ratio of a single extra-core nucleon is represented by g_j'. Since the intrinsic gyromagnetic ratio of a bound nucleon may not be identical to that of a free nucleon, we have deliberately added a prime on the symbol g_j to indicate that the gyromagnetic ratio of a nucleon bound in the nucleus may be rather different from the Schmidt value.

In calculating the expectation value of the operator $\hat{\mu}$ with the semiatomic-model wave function, the last two terms of Eq. (32) drop out and, using the well-known integral formula for the product of three rotation matrices, we obtain the simple result

$$\mu_J\equiv\langle\hat{\mu}\rangle_J=[J^2/(J+1)]g_j'. \quad (33)$$

It should be noted that Eq. (33) is not valid for $J=\frac{1}{2}$, because in that case the residual deformation vanishes and the shell-model wave function should apply. This may partially explain the fact that small deviation from the Schmidt limits, in general, happens in such cases.[4] (A detailed discussion about this point will be presented in a separate paper.)

At this point, it should be noted that the "single-particle" Schmidt model does not require the assumption of j-j coupling and, furthermore, the "many-particle" Schmidt formula is actually identical to the "single-particle" formula when $J=j$. Only when $J\neq j$ does the assumption of j-j coupling make a difference. Therefore, in order to test the comparative validity of Eq. (33), in which j-j coupling is also assumed, and the Schmidt formula $\mu_J=Jg_j$, it is more proper to consider such cases in which $J\neq j$. In addition, such a comparison may provide a good test for the idea of residual deformation.

[19] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics*, English translation (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1958), p. 373.

TABLE I. Experimental verification of Eq. (23) in the region of $(1f_{7/2})$ shell. The experimental data are taken from the compilation of D. Strominger *et al.* [Rev. Mod. Phys. **30**, 585 (1958)] and the configuration assignments are given by Mayer and Jensen (reference 4). The last three columns are the experimental and theoretical magnetic moment ratios, with $R_{\text{theor}}{}^s$ representing the Schmidt values [Eq. (35)]. It is noted that according to Eq. (35) the magnetic moments of the first three rows should be the same. This prediction is obviously not fulfilled. It is remarkable that the average value of the second row and third row of R_{exp} differs from R_{theor} only by 3%.

Z	N	Element	A	J ($\hbar$)	$(nlj)^\nu$	μ_{exp} (nm)	R_{exp}	R_{theor}	$R_{\text{theor}}{}^s$
21		Sc	45	7/2	$(1f_{7/2})^1$	+4.749	1.024	1.000	1.000
27		Co	59	7/2	$(1f_{7/2})^{-1}$	+4.639			
23		V	51	7/2	$(1f_{7/2})^3$	+5.139	1.484	1.524	1.400
25		Mn	55	5/2	$(1f_{7/2})^{-3}$	+3.461			
	23	Ca	43	7/2	$(1f_{7/2})^3$	−1.315	1.671	1.524	1.400
	25	Ti	47	5/2	$(1f_{7/2})^{-3}$	−0.787			

However, we cannot compare Eq. (33) and the Schmidt formula directly, because g_j' is still unknown and g_j is not correct. Therefore, we shall consider the magnetic moment ratio of a pair of nuclei (the configurations of which are conjugate to each other). The advantage of approaching the problem this way is that the formula of the magnetic moment ratio does not depend upon the gyromagnetic ratio g_j or g_j' and, therefore, a real test for the assumption of j-j coupling can be made. Furthermore, we can also have some idea about the degree to which the configuration mixing is important in the determination of the nuclear magnetic moments.

To start with, we have to make some conjecture about how g_j' depends on the configuration. It seems reasonable to assume that g_j' is a function not only of the total angular momentum j and the orbital angular momemtum l (as in the Schmidt model), but also on the total quantum number n and the number of nucleons (or holes) ν.[20] Then, according to Eq. (33), the magnetic moments of two odd-A nuclei with configurations $(nlj)^\nu$ and $(nlj)^{\nu'}$ and with total angular momenta J and J', respectively, must be connected by the following relation:

$$\mu_J[(nlj)^\nu]/\mu_{J'}[(nlj)^{\nu'}]=[(J'+1)/(J+1)](J/J')^2, \quad |\nu|=|\nu'|, \quad (34)$$

where ν and ν' can be positive or negative (odd) integers; when they are positive they represent the number of nucleons and when negative the number of holes. The counterpart of Eq. (34) in the Schmidt model is given by

$$\mu_J[(nlj)^\nu]/\mu_{J'}[(n'lj)^{\nu'}]=J/J', \quad \text{no restriction on } n, n', \nu \text{ and } \nu'. \quad (35)$$

Here "no restriction" means that n and n' can be different, and the same for ν and ν'.

A comparison of Eq. (34) and Eq. (35) with the experimental data in the region of the $(1f_{7/2})$ shell has been made in Table I. The results show that the relation given by Eq. (34) is better. From these results, we can also draw the following conclusions: (a) The assumption of residual deformation is valid; (b) j-j coupling is a good approximation; (c) The contribution of configuration mixing is small; (d) The Schmidt formulas are also approximately valid (provided the assumption made for g_j' also holds for g_j); this implies that the experimental magnetic moment deviation from the theoretical predictions are mainly due to the wrong value assigned to the single-particle gyromagnetic ratios.

For $J=J'$, Eq. (23) and Eq. (24) are identical except for the assumption made about g_j'. Therefore, a consideration of such cases will form a test for this assumption. However, this will be discussed in detail in a separate paper along with some systematic features of the magnetic moment distribution of odd-A nuclei.

Now let us calculate the magnetic moment of $_{25}\text{Mn}^{55}$, using different relations. Needless to say, a comparison of these results with the experimental value will also provide a test of the validity of the above conclusions. Taking the experimental value of $\mu_{7/2}(_{23}\text{V}^{51})=5.139$ nm, we have, from Eq. (34), $\mu_{5/2}(_{25}\text{Mn}^{55})=3.372$ nm which is in good agreement with the experimental value 3.461 nm. However, if we take the Schmidt value 1.655 for g_j', then from Eq. (33), $\mu_{5/2}(_{25}\text{Mn}^{55})=2.955$ nm. When compared with the experimental value, the error is approximately the same as the value 4.13 nm from the Schmidt formula (in one case the prediction is lower and in the other it is higher). From Eq. (35), using the experimental value of $_{23}\text{V}^{51}$, we have $\mu_{5/2}(_{25}\text{Mn}^{55})=3.671$ nm, which is 0.210 nm higher than the experimental value. However, the prediction of Eq. (34) is only 0.089 nm lower than the experimental value.

From the above discussion, it is clear that in the usual Schmidt plots all nuclei with same j and l are in general in different categories and, therefore, cannot be compared on equal footing. Although the small magnetic moment variation in each of these categories may be explained by the local irregularities such as core excitation, configuration mixing, etc., it is very unlikely that the clear distinction among these categories themselves as shown in Table I can be solely due to these local irregularities. Therefore, there must exist some dominating factor which persists over large region of the nuclear chart. This central idea derived from the present investigation will be discussed in detail in a separate paper.

[20] Here we shall follow the notations of reference 4. Strictly speaking, the number written before the orbital quantum number there is not the total quantum number, because it does not satisfy the relation which gives the number of radial nodes (the number of radial nodes is equal to the total quantum number minus the orbital quantum number and minus one). However, this is not our main concern. For a different system of terminology, one is referred to C. D. Coryell, Ann. Rev. Nucl. Sci. **2**, 305 (1953).

C. Electric Quadrupole Moment

Written in the body-fixed coordinate system, the quadrupole moment operator is given by[4,3,21]

$$\hat{Q}=\hat{Q}_p+\hat{Q}_c=\left(\frac{16\pi}{5}\right)^{1/2}\sum_{\mu i} r_i^2 D_{0\mu}^2 Y_{2\mu}(\Omega_i') + \frac{3Z'R_0'^2}{(5\pi)^{1/2}}\left[\frac{1}{\sqrt{2}}\beta\sin\gamma(D_{02}^2+D_{0\,-2}^2)+\beta\cos\gamma\, D_{00}^2\right]. \tag{36}$$

Here Z' is the total charge of the core divided by the proton charge, $R_0'=1.5A'^{1/3}\times10^{-13}$ cm and A' is the total number of nucleons in the core. By a simple, standard calculation, the expectation value of $\hat{Q}$ is given by (see the derivation in the Appendix)

$$Q_J\equiv\langle\hat{Q}\rangle_J=|\langle J2J0|JJ\rangle|^2 Q_J^s + \frac{27Z'R_0'^2}{4\pi(5)^{1/2}}\left(\frac{\hbar^2}{BC}\right)^{1/4}\frac{J(2J-1)}{(J+1)(2J+3)}, \tag{37}$$

where Q_J^s is the shell-model quadrupole moment given by the expression in the curly bracket of Eq. (29b). Obviously, the well-known conclusion that the quadrupole moment vanishes for total angular momentum $J=0$ and $J=\frac{1}{2}$ still holds.

It is interesting to observe that the magnitude of the intrinsic quadrupole moment in the semiatomic model is reduced by a factor of $|\langle J2J0|JJ\rangle|^2$ from the shell-model quadrupole moment Q_J^s and this reduction is "compensated" by the contribution from the core. Since the contribution of the core is always positive in this case, the compensation actually becomes suppression for $Q_J^s<0$. It should be pointed out that, even in the strong-coupling theory of Bohr and Mottelson, the intrinsic quadrupole moment should also be approximately the product of the shell-model value and the square of some Clebsch-Gordan coefficient. (The total angular momentum of the extra-core nucleons in this case can be considered only approximately as a good quantum number.) Therefore, the effect of suppression of negative quadrupole moments should also occur. However, in this case, the contribution from the core may be negative, because the average value of $\beta\cos\gamma$ can be negative. Therefore, the suppression of the negative quadrupole moments is suppressed. Finally, we come to such cases which may be represented by the type of wave function given by Eq. (22). Then, the intrinsic quadrupole moment becomes a linear combination of the shell-model values for states with definite J and different projections, and the collective contribution of the core is also a linear combination of the quadrupole moments for states with definite angular momentum of the core and different projections. This is a sort of smearing out of both negative and positive quadrupole moments.

[21] M. Moshinsky, lecture notes (unpublished).

In the above discussion, we have presumed the general validity of our theory. This may not be legitimate at the present stage. However, the positive-negative asymmetry of the quadrupole moment distribution[22] for odd-even nuclei corroborates our picture very well. It seems that this asymmetry has been so far overlooked without explanation. Furthermore, in our theory, the signs of the quadrupole moments can always be explained. For example, according to the shell model, the quadrupole moment of $_{23}V^{51}$ should be negative, but the experimental value is $\sim+0.3$ b. However, in our theory, it can be positive.[23]

The experimental value of ground-state quadrupole moment of $_{25}Mn^{55}$ is 0.55 barns.[24] From this experimental value and Eq. (37) in which $\langle r^2\rangle$ is taken approximately as $(3/5)R_0^2$ ($R_0=1.5A^{1/3}\times10^{-13}$ cm and A is the atomic number),[25] the dimensionless parameter $(\hbar^2/BC)^{1/4}$ is found to be 0.218. By using this paremeter, the quadrupole moments for the excited states can also be calculated easily from Eq. (37). These values are not very interesting, at least for the time being, because their experimental determination is still very difficult. The present purpose of this numerical calculation is to get some general idea about the order of magnitudes of the parameter K and the surfon energy $\hbar\omega_0$; and to see if the assumption of weak coupling is valid.

From $S=5^{1/2}(9K/32\pi)(\hbar^2/BC)^{1/4}=0.187$ MeV and $(\hbar^2/BC)^{1/4}=0.218$, we have $K=4.284$ MeV. Since the parameters B and C cannot be empirically determined separately, we shall take the hydrodynamical estimation for B, namely, $B\approx3M'A'R_0'^2/8=1.902\times10^{-42}$ MeV-sec^2 (M' is the total mass of the core).[2,21] Then we have $C=19.172$ MeV and $\hbar\omega_0=2.089$ MeV. According to the criterion given by Bohr and Mottelson,[3] the "strength" of the coupling (not K alone) is determined by the magnitude of the dimensionless parameter $X=[5K^2/16\pi j\hbar\omega_0 C]^{1/2}$. Substituting the numerical values, we have $X=0.124$. Therefore, the assumption of weak coupling is valid.

Looking at the numerical calculations presented above, one may wonder why the surfon excitation cannot occur after the intrinsic excitation approaches the same order of magnitude of $\hbar\omega_0$. This misunderstanding originates from preoccupation with the semimolecular model. Up to now, the relative magnitudes of $\hat{H}_{\text{vib}}$ and $\hat{H}_{\text{rot}}$ have not been specified. In the semimolecular case, $\hat{H}_{\text{rot}}$ is supposed to be smaller than $\hat{H}_{\text{vib}}$. However, in the semiatomic case, the opposite situation may occur. If this is the case, then we may

[22] D. L. Hill and J. A. Wheeler, Phys. Rev. **89**, 1102 (1953).

[23] In Table X.1, p. 168 of reference 4, the shell-model quadrupole moment divided by $\langle r^2\rangle$ for $_{23}V^{51}$ is tabulated as $+0.66$, but it should be -0.222 [see (A22) and subsequent remark].

[24] *Nuclear Data Sheets*, compiled by K. Way *et. al.* (National Research Council: National Academy of Sciences—National Research Council: Washington, 25, D. C., 1959).

[25] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952).

solve $\hat{H}_{\text{vib}}$ for fixed $\hat{H}_{\text{rot}}$. Actually this case has already been considered by Bohr,[2] although on a different physical basis. The conclusion is that the first vibrational excited state is $3\hbar\omega_0$. Therefore, in calculating the parameter X, we should have used $3\hbar\omega_0$ instead of $\hbar\omega_0$. Then the value of X would be reduced by a factor of $(1/3)^{1/2}$ and the coupling would be very weak indeed.

We mention, in passing, a very interesting statement made by Ford and Levison.[26] After a general survey of the experimental situation, they conclude: "The surface coupling is self-reinforcing, being strong when strong and almost negligible when weak." The implication of this conclusion depends upon the approach adopted in the original investigation. However, if our theory is correct, this statement seems to indicate that, in general, the nuclear system can be approximated in the low-energy region either by the semi-molecular model or by the semiatomic model.

D. Transition Probability

According to our assumption, an $M1$ transition is forbidden. This is due to the fact that the off-diagonal matrix elements of the magnetic dipole operator vanish between states due to the recoupling of nucleons of the same type.[27] However, Coulomb excitation and lifetime measurements show that the transition $J=7/2\rightarrow J_0=5/2$ in ${}_{25}\text{Mn}^{55}$ is the $E2$-$M1$ mixture type.[28,29] Therefore, configuration mixing is possibly not negligible and/or the specification of the core is not appropriate.

It should be noted, however, that the statement that configuration mixing is not negligible is not a contradiction to the conclusion drawn from the magnetic moment investigation (Sec. B). In the case of the magnetic moment, only those configurations which can be mixed with the ground state contribute. However, in the case of a transition, both initial and final states can have configuration mixing. Furthermore, the magnetic moment is given by the diagonal matrix elements of the magnetic dipole operator in the space of total angular momentum, while the $M1$ transition probability is determined by the off-diagonal elements of the same operator [except for the factor $(e/4\pi)^{1/2}$] in the same space. Therefore, configuration mixing can be important for $M1$ transition probability and at the same time irrelevant for static magnetic dipole moment. It is also clear that comparison between the conclusions from static quadrupole moment and $E2$ transition probability investigations is also in general misleading.

The calculation of configuration mixing requires detailed information about the nuclear interaction; and, furthermore, our lack of the exact knowledge of the gyromagnetic ratio of nucleons in nuclei introduces further uncertainty in this type of calculation. In order to keep our main idea simple and clear, we shall confine ourselves to the consideration of $E2$ transition probabilities only. Another reason for doing this is that a nearly pure $E2$ transition can be experimentally produced by Coulomb excitation and, therefore, theory and experiment can be compared with less ambiguity.[30]

In the body-fixed coordinate system, the electric multipole operator can be written[3,21,25]

$$\mathfrak{M}(Elm)=\sum_{\mu i}\left(e_i-\frac{Ze}{A}\delta_{l1}\right)r_i^2 D_{m\mu}{}^{l}Y_{l\mu}(\Omega_i'); \qquad (38)$$

and the transition probability is then given by

$$T(El)=\frac{8\pi(l+1)}{l[(2l+1)!!]^2}\frac{1}{\hbar}\left(\frac{\omega}{c}\right)^{2l+1}\sum_{mM_f}|\langle i|\mathfrak{M}(Elm)|f\rangle|^2$$

$$\equiv\frac{8\pi(l+1)}{l[(2l+1)!!]^2}\frac{1}{\hbar}\left(\frac{\omega}{c}\right)^{2l+1}B(El), \qquad (39)$$

where $B(El)$ is usually called the reduced transition probability. In writing down Eq. (38), the surfon excitation has been neglected entirely.

Now our problem is to calculate $B(E2)$. In Secs. A, B, and C, the static nuclear properties are discussed without considering the surfon excitation, so the projection of the total angular momentum $\hat{J}$ on the body-fixed 3-axis must take on the value J (or $-J$). However, in the transition problem, we have to consider the finite probability of the surfon excitation. The "direct" contribution of the surfon excitation to the total transition probability may be negligible, but its effect on the intrinsic transition probability may not be ignored. In order to take this effect into account, we shall assume that the projection of $\hat{J}$ on the body-fixed 3-axis will be no longer restricted to the value J (and $-J$). Then, the proper wave function for this problem would be

$$\psi_{JM}=\left(\frac{2J+1}{16\pi^2}\right)^{1/2}\varphi_0(\beta,\gamma)$$

$$\times\sum_{\Lambda}\{D_{M\Lambda}{}^{J}\chi_{J\Lambda}(\mathbf{x}')+D_{M-\Lambda}{}^{J}\chi_{J-\Lambda}(\mathbf{x}')\}. \qquad (40)$$

It is noted that the wave function given by Eq. (40) makes no contribution to the diagonal matrix elements (in the space of the total angular momentum) of the operators $\hat{H}_{\text{int}}$, $\hat{Q}$, and $\hat{\mu}$ (discussed in Secs. A, B, and C). This is consistent with the assumption that the surfon excitation is so high that only the residual deformation is significant. In other words, the wave function given by Eq. (23), acting as a projection opera-

[26] K. W. Ford and C. A. Levinson, Phys. Rev. **100**, 1 (1955).
[27] S. A. Moszkowski, Phys. Rev. **89**, 474 (1953).
[28] G. M. Temmer, Phys. Rev. **104**, 967 (1956).
[29] R. E. Holland and F. J. Lynch, Phys. Rev. **121**, 1464 (1961).
[30] K. Alder, A. Bohr, T. Huus, B. Mottelson, and A. Winther, Rev. Mod. Phys. **28**, 432 (1956).

TABLE II. $E2$ reduced transition probabilities in the region of $(1f_{7/2})$ shell. The experimental data are taken from reference 28. The second and third columns are the spins of the ground state and the first excited state, respectively. In order to avoid ambiguity, only the experimental data from Coulomb excitation measurements are taken. In these measurements, nearly pure $E2$ excitation is assumed. The approximation for $\langle r^2\rangle$ in the calculation of $B(E2)$ for $_{23}V^{51}$ is taken as $\frac{3}{5}(1.5\times10^{-13}A^{1/3})^2$ cm^2 as usual [however, see V. F. Weisskopf, Phys. Rev. **83**, 1073 (1951), in which r_0 is taken as 1.2×10^{-13} cm].

			$B(E2; J_0\to J)(e^2\times10^{-48}$ cm$^4)$		
				Theoretical prediction	
Element	J_0 ($\hbar$)	J ($\hbar$)	Coulomb excitation	Shell model	Semiatomic model
$_{23}V^{51}$	7/2	5/2	0.0055	0.0075	0.0058
$_{25}Mn^{55}$	5/2	7/2	0.075	···	···
$_{22}Ti^{47}$	5/2	7/2	0.040	···	···

tor, sorts out those parts of $\hat{H}_{\text{int}}$, $\hat{Q}$, and $\hat{\mu}$, which are related to the residual deformation.

Following a similar procedure in the derivation of Eq. (36), it can be easily shown that the $E2$ reduced transition probability can be written as follows (see the derivation in the Appendix)[31]

$$B(E2; J_i\to J_f)=\frac{2J_f+1}{2J_i+1}B^s(E2; J_i\to J_f), \quad (41)$$

where $B^s(E2; J_i\to J_f)$ is the $E2$ reduced transition probability in the usual shell theory.

Before presenting the numerical calculation, some remark about the ordering of the initial state and final state in calculating the matrix elements is in order. It is a general rule that in the body-fixed coordinate system, all expressions should be taken the complex conjugate with respect to the corresponding expressions in space-fixed coordinate system.[19] Therefore, in calculating the matrix elements of an operator written in the body-fixed coordinate system, we have to take the complex conjugate of the initial state instead of taking the complex conjugate of the final state.

The calculated value of $B(E2)$ for $_{23}V^{51}$ in the transition $J_0=7/2\to J=5/2$ is presented in Table II. The agreement with the experimental value is very good. However, the calculated value of $B(E2)$ for $_{25}Mn^{55}$ in the transition $J_0=5/2\to J=7/2$, which is approximately the same as that for $_{23}V^{51}$, is about ten times smaller than the experimental value.

The configurations of $_{25}Mn^{55}$ and $_{23}V^{51}$ are $(1f_{7/2})^{-3}$ and $(1f_{7/2})^3$, respectively, so their reduced transition probabilities should be approximately the same if their core difference can be ignored. However, in view of the big difference between their $E2$ reduced transition probabilities, the effect of the two $(2p_{3/2})$ neutrons in the core of $_{25}Mn^{55}$ cannot be neglected. This point of view is supported by considering the $E2$ reduced transition probability of $_{22}Ti^{47}$. In this case, if the two $(1f_{7/2})$ protons can be included in the core and the semiatomic model is valid, then the $E2$ transition probability would be vanishingly small. However, experiments show that $_{22}Ti^{47}$ and $_{25}Mn^{55}$ have approximately the same $E2$ reduced transition probability in the transition $J_0=5/2\to J=7/2$. Therefore, in both cases, the two $(1f_{7/2})$ protons and the two $(2p_{3/2})$ neutrons, whose effect on static nuclear properties may be approximately included in the core, become rather important in affecting the $E2$ transition.

From the arguments given above, we can also conclude that $_{20}Ca^{43}$ would show very small probability for Coulomb excitation. This conclusion is actually consistent with the experimental fact that no γ ray ($E_\gamma\lesssim0.5$ MeV) has been observed when $_{20}Ca^{43}$ is bombarded with α particles ($E_\alpha\sim3$ MeV).[32]

Detailed investigation about the $E2$ transition probabilities of $_{22}Ti^{47}$ and $_{25}Mn^{55}$ would require some modification of the intrinsic part of the wave function given by Eq. (40). However, this will be considered in a later publication. For the purpose of comparison, the experimental values of the $E2$ reduced transition probabilities of $_{25}Mn^{55}$ and $_{22}Ti^{47}$ are also included in Table II.

IV. GENERAL DISCUSSION

Historically, the unified model of Bohr and Mottelson is supposed to be an extension of the shell model by including the motion of the core, so the shell model must be logically a limiting case of the unified model. However, from a general consideration of the adiabatic principle, which is the physical basis of the semimolecular limit, it becomes clear that the other limiting case of the unified model is possibly the semiatomic model, but not the shell model. Fortunately, in many respects, these two models, namely, semiatomic model and shell model, are approximately equivalent except for the residual deformation. This idea needs further explanation.

In the calculation of the unified model, the constitution of the core should be specified beforehand (usually

[31] It is noted that in the calculation of static quadrupole moments, the parentage coefficients appear in the formula in their absolute values, so their phases are irrelevant. However, in the calculation of transition probability [Eq. (41)], the relative phases of the parentage coefficients become important. Unfortunately, there are some errors in the phase values in the classical tables given by A. R. Edmonds and B. H. Flowers [Proc. Roy. Soc. (London) **A214**, 515 (1952)]. The misprints, which we found in the tables we needed, are as follows: $((\frac{7}{2})^4J[(\frac{7}{2})^3(J')\frac{7}{2}J)$ must change sign for $J=2, 4, 6$ in the partition [1111](1100) and $J'=3/2, 5/2, 9/2, 11/2, 15/2$; and $((\frac{7}{2})^3J[(\frac{7}{2})^2(J')\frac{7}{2}J)$ must also change sign for $J=15/2$ and $J'=6$. A simple formula for checking the relative phase of two parentage coefficients can be found in reference 14, p. 207; but a general formula is given by P. J. Redmond [Proc. Roy. Soc. (London) **A222**, 84 (1954)]. (The author is indebted to Dr. R. D. Lawson for pointing out this formula.) For a practical way to check possible mistakes in sign of the parentage coefficients, see Eq. (A19).

[32] G. M. Temmer and N. P. Heydenburg, Phys. Rev. **93**, 351 (1954).

on the basis of the shell model), but it can be arbitrary with "some degree of freedom." For instance, we could also choose the double-closed shells of $_{25}Mn^{55}$ as its core, and consider the two $(2p_{3/2})$ neutrons and the five $(1f_{7/2})$ protons as the extra-core nucleons. Under this specification, the structure of the intrinsic part of the wave function would change (for instance, isotopic spin formulation becomes necessary) and the parameters B and C, which specify the dynamical properties of the core, should also be modified. Then we have a problem which is formally different from that discussed in the present paper. However, in principle, the results of these two problems should approximately agree, possibly with some improvement in the calculation of the transition probability. The main point at which we are driving here is that the effect of the residual deformation should be included in both cases. In general, when the nuclear system is divided (say, according to the shell model) into two parts, namely, the core and the extra-core nucleons, an interaction between these two parts always exists (no matter how strong or how weak it is) and may not be easily explained by the shell-model calculation alone; only when the extra-core nucleons are in a state with total angular momentum $J=0$ and $1/2$, will this interaction disappear. This is the central theme of the present paper, which has been constantly emphasized.

APPENDIX

Some of the derivations presented here are not entirely new. They are included only for the purpose of completeness and orientation.

A. Derivation of Eq. (22)

Combining Eqs. (17) and (18′), we have

$$\psi_{IM}(\tau,Q,J)=N^{1/2}\sum_{\mu k\Lambda}\langle QJ\mu\, M-\mu\,|IM\rangle \times\varphi_{Qk}{}^{\tau}(\beta,\gamma)D_{\mu k}{}^{Q}D_{M-\mu\Lambda}{}^{J}\chi_{J\Lambda}(\mathbf{x}'). \quad (A1)$$

By the well-known relation[14]

$$D_{\mu_1 m_1}{}^{J_1}D_{\mu_2 m_2}{}^{J_2}=\sum_{J'}\langle J_1J_2\mu_1\mu_2\,|J'\mu_1+\mu_2\rangle \times\langle J_1J_2m_1m_2\,|J'm_1+m_2\rangle D_{\mu_1+\mu_2 m_1+m_2}{}^{J'}, \quad (A2)$$

Eq. (A1) can then be written

$$\psi_{IM}(\tau,Q,J)=(N')^{1/2}\sum_{k\Lambda}\{\langle QJk\Lambda\,|Ik+\Lambda\rangle\varphi_{Qk}{}^{\tau}(\beta,\gamma)D_{Mk+\Lambda}{}^{I}\chi_{J\Lambda}(\mathbf{x}') +\langle QJ-k-\Lambda\,|I-(k+\Lambda)\rangle\varphi_{Q-k}{}^{\tau}(\beta,\gamma) \times D_{M-(k+\Lambda)}{}^{I}\chi_{J-\Lambda}(\mathbf{x}')\}. \quad (A3)$$

Finally, by using Eq. (19) and the symmetry relation of the Clebsch-Gordan coefficients,[14]

$$\langle QJ-k-\Lambda\,|I-(k+\Lambda)\rangle=(-1)^{Q+J-I}\langle QJk\Lambda\,|Ik+\Lambda\rangle, \quad (A4)$$

we obtain the final result.

B. Derivation of Eqs. (29a) and (29b)

The result of Eq. (29a) is trivial, and the only calculation we have to carry out is the following integral

$$\langle\beta\cos\gamma\rangle=\left[4\pi^{-1/2}\left(\frac{BC}{\hbar^2}\right)^{5/4}\right] \times\int_0^{\infty}\beta^5\exp\left\{-\frac{1}{\hbar}(BC)^{1/2}\beta^2\right\}d\beta\int_0^{\pi/3}\cos\gamma\sin3\gamma d\gamma =\frac{9}{4(\pi)^{1/2}}\left(\frac{\hbar^2}{BC}\right)^{1/4}.$$

The derivation of Eq. (29b) is a little bit complicated, but there is a standard procedure to follow. Expanding the intrinsic part of the semiatomic-model wave function by using the parentage coefficients,[26]

$$\chi_{JJ}(\mathbf{x}')\equiv|j^{\nu}JJ\rangle=\sum_{J_1}(j^{\nu}J[[j^{\nu-1}(J_1)jJ)|j^{\nu-1}(J_1)jJJ\rangle,$$

where symbols for additional quantum numbers such as seniority are dropped for simplicity. Then we have

$$\langle\hat{H}_{\rm int}\rangle_J=-\frac{9K}{4(\pi)^{1/2}}\left(\frac{\hbar^2}{BC}\right)^{1/4}\sum_{J_1}|(j^{\nu}J[[j^{\nu-1}(J_1)jJ)|^2 \times\langle j^{\nu-1}(J_1)jJJ|\sum_{i=1}^{\nu}Y_{20}(\Omega_i')|j^{\nu-1}(J_1)jJJ\rangle. \quad (A5)$$

The matrix element in the summation is well known[14,26] [also see G. Racah, Phys. Rev. **62**, 438 (1942)]:

$$\langle j^{\nu-1}(J_1)jJJ|\sum_{i=1}^{\nu}Y_{20}(\Omega_i')|j^{\nu-1}(J_1)jJJ\rangle =\nu(2J+1)(-1)^{J_1-j+J}\langle j\|Y_2\|j\rangle W(jjJJ;2J_1) \times V(JJ2;-JJ0), \quad (A6)$$

where

$$\langle j\|Y_2\|j\rangle=-\left(\frac{5}{64\pi}\right)^{1/2}\frac{(2j-1)(2j+1)(2j+3)}{j(j+1)}, \quad (A7)$$

and

$$V(JJ2;-JJ0)=(-1)^{2J}(2J+1)^{-1/2}\langle J2J0\,|JJ\rangle. \quad (A8)$$

In order to use the tabulated values of the parentage coefficients, we need the following relation [G. Racah,

Phys. Rev. **63**, 367 (1943)]:

$$(j^\nu J[[j^{\nu-1}(J')jJ)$$
$$=(-1)^{j-J-J'}\left[\frac{(2j+2-\nu)(2J'+1)}{(2J+1)}\right]^{1/2}$$
$$\times(j^{2j+2-\nu}J'[[j^{2j+1-\nu}(J)jJ'). \quad \text{(A9)}$$

Combining all equations from (A5) to (A9) and multiplying and dividing the final result by $\langle r^2\rangle$ at same time, we obtain Eq. (29b).

C. Derivation of Eq. (37)

The result of the second term of Eq. (37) is easily obtained by using the well-known formula[14]

$$\int D_{m_3\mu_3}{}^{J_3*}D_{m_2\mu_2}{}^{J_2}D_{m_1\mu_1}{}^{J_1}d\Omega$$
$$=\left(\frac{8\pi^2}{2J_3+1}\right)\delta_{\mu_1+\mu_2\mu_3}\delta_{m_1+m_2m_3}\langle J_1J_2\mu_1\mu_2|J_3\mu_3\rangle$$
$$\times\langle J_1J_2m_1m_2|J_3m_3\rangle. \quad \text{(A10)}$$

Similarly, the first term is given by

$$Q_J(\text{first term})$$
$$=\sum_\mu\langle J2J0|JJ\rangle\langle J2J\mu|JJ\rangle$$
$$\times\delta_{\mu0}\left\langle j^\nu JJ\left|\left(\frac{16\pi}{5}\right)^{1/2}\sum_{i=1}^{\nu}r_i^2Y_{2\mu}(\Omega_i')\right|j^\nu JJ\right\rangle$$
$$=|\langle J2J0|JJ\rangle|^2\left\langle j^\nu JJ\left|\left(\frac{16\pi}{5}\right)^{1/2}\sum_{i=1}^{\nu}r_i^2Y_{20}(\Omega_i')\right|j^\nu JJ\right\rangle, \quad \text{(A11)}$$

where the expression in the curly bracket is just $Q_J{}^s$. For numerical calculation of Eq. (A11), all equations from (A6) to (A9) should be used.

D. Derivation of Eq. (41)

By using Eq. (A10), it can be easily shown that

$$\langle i|\mathfrak{M}(E2m)|f\rangle=\left(\frac{2J_f+1}{2J_i+1}\right)^{1/2}\sum_\mu\{\langle J_f2M_fm|J_iM_i\rangle\langle J_f2\Lambda_f\mu|J_i\Lambda_i\rangle\langle\chi_{J_i\Lambda_i}(\mathbf{x}')|\sum_{i=1}^{\nu}er_i^2Y_{2\mu}(\Omega_i')|\chi_{J_f\Lambda_f}(\mathbf{x}')\rangle\}. \quad \text{(A12)}$$

Therefore,

$$B(E2;J_i\to J_f)=\sum_{mM_f}|\langle J_f2M_fm|J_iM_i\rangle|^2\left(\frac{2J_f+1}{2J_i+1}\right)|\sum_{\mu\Lambda}\langle J_f2\Lambda_f\mu|J_i\Lambda_i\rangle\langle\chi_{J_i\Lambda_i}(\mathbf{x}')|\sum_{i=1}^{\nu}er_i^2Y_{2\mu}(\Omega_i')|\chi_{J_f\Lambda_f}(\mathbf{x}')\rangle|^2. \quad \text{(A13)}$$

But

$$\langle\chi_{J_i\Lambda_i}(\mathbf{x}')|\sum_{i=1}^{\nu}er_i^2Y_{2\mu}(\Omega_i')|\chi_{J_f\Lambda_f}(\mathbf{x}')\rangle=\frac{1}{(2J_i+1)}\langle J_f2\Lambda_f\mu|J_i\Lambda_i\rangle\{\nu e\langle r^2\rangle[(2J_f+1)(2J_i+1)]^{1/2}\langle j\|Y_2\|j\rangle$$
$$\times\sum_{J_1}(-1)^{J_1-j-J_i}(j^\nu J_i[[j^{\nu-1}(J_1)jJ_i)(j^{\nu-1}(J_1)jJ_f]]j^\nu J_f)W(jjJ_fJ_i;2J_1)\}, \quad \text{(A14)}$$

where the expression in the curly bracket is independent of μ. Combining Eqs. (A13) and (A14), we have the final result.

E. Numerical Calculation

The calculation of $Q_J{}^s$ and $B^s(E2)$ can be made simpler by considering first the reduced matrix elements defined as

$$F_{J_1J_2}\equiv\langle j^\nu J_1\|\sum_i Y_2(\Omega_i)\|j^\nu J_2\rangle/$$
$$[(2J_1+1)(2J_2+1)]^{1/2}. \quad \text{(A15)}$$

For ν extra-core nucleons of the same type, the factor r_i^2 and e_i can be considered separately.

A nucleus with ν extra-core nucleons in the shell j can also be considered as $(2j+1-\nu)$ extra-core holes in the same shell. However, the reduced matrix elements should not depend on which point of view is taken. Let us define $F_{J_1J_2}{}^p$ as the value of $F_{J_1J_2}$ calculated from the former point of view and $F_{J_1J_2}{}^h$ that from the latter. Then we should have

$$F_{J_1J_2}{}^p=F_{J_1J_2}{}^h. \quad \text{(A16)}$$

It is noted that both $F_{J_1J_2}{}^p$ and $F_{J_1J_2}{}^h$ are not symmetric in sign in exchanging J_1 and J_2. In order to remove this "undesirable" feature, we define

$$F_{J_1J_2}{}^p(s)\equiv(-1)^{J_2-j-1}(2j+2-\nu)$$
$$\times[(2J_1+1)(2J_2+1)]^{1/2}F_{J_1J_2}{}^p, \quad \text{(A17)}$$

and

$$F_{J_1J_2}{}^h(s)\equiv(-1)^{j-J_1}(2j+1-\nu)^{-1}F_{J_1J_2}{}^h, \quad \text{(A18)}$$

where s means "symmetric in exchanging J_1 and J_2." These definitions are suggested by Eq. (A9). (Also note that J and J' in that equation are half-integer and integer, respectively—or vice versa.) Then, from Eqs. (A16), (A17), and (A18), we have

$$F_{J_1J_2}{}^p(s)=(-1)^{J_1+J_2}\left(\frac{2j+1-\nu}{2j+2-\nu}\right)$$
$$\times[(2J_1+1)(2J_2+1)]^{1/2}F_{J_1J_2}{}^h(s). \quad \text{(A19)}$$

Equation (A19) is a practically useful relation for checking the numerical calculations. By using the tables of parentage coefficients [see footnotes to Eq. (41)] and Racah coefficients (K. M. Howell, Research Report 59–1, University of Southampton, England), both $F_{J_1J_2}{}^p(s)$ and $F_{J_1J_2}{}^h(s)$ have been calculated for the case in which $j=7/2$ and $\nu=5$. The results, when written in the matrix form, are

$$F_{J_1J_2}{}^p(s)=4\left(\frac{150}{63\pi}\right)^{1/2}\begin{Vmatrix} -\frac{7}{20}\left(\frac{3}{14}\right)^{1/2} & +\frac{1}{10}\left(\frac{11}{7}\right)^{1/2} & -\frac{1}{2}\left(\frac{3}{35}\right)^{1/2} & 0 & 0 & 0 \\ +\frac{1}{10}\left(\frac{11}{7}\right)^{1/2} & -\frac{13}{40}\frac{1}{\sqrt{2}} & -\frac{1}{2}\left(\frac{11}{30}\right)^{1/2} & +\frac{1}{8}\left(\frac{65}{42}\right)^{1/2} & 0 & 0 \\ -\frac{1}{2}\left(\frac{3}{35}\right)^{1/2} & -\frac{1}{2}\left(\frac{11}{30}\right)^{1/2} & +\frac{1}{12} & -\frac{1}{6}\left(\frac{13}{14}\right)^{1/2} & +\frac{1}{2}\frac{1}{\sqrt{3}} & 0 \\ 0 & +\frac{1}{8}\left(\frac{65}{42}\right)^{1/2} & -\frac{1}{6}\left(\frac{13}{14}\right)^{1/2} & +\frac{5}{24}\left(\frac{7}{22}\right)^{1/2} & +\frac{1}{6}\left(\frac{21}{11}\right)^{1/2} & 0 \\ 0 & 0 & +\frac{1}{2}\frac{1}{\sqrt{3}} & +\frac{1}{6}\left(\frac{21}{11}\right)^{1/2} & +\frac{1}{28}\left(\frac{13}{22}\right)^{1/2} & -\frac{2}{7} \\ 0 & 0 & 0 & 0 & -\frac{2}{7} & +\frac{1}{28}(34)^{1/2} \end{Vmatrix}, \tag{A20}$$

and

$$F_{J_1J_2}{}^h(s)=3\left(\frac{150}{63\pi}\right)^{1/2}\begin{Vmatrix} +\frac{1}{20}\left(\frac{7}{6}\right)^{1/2} & +\frac{1}{15}\left(\frac{11}{42}\right)^{1/2} & +\frac{1}{12}\left(\frac{6}{35}\right)^{1/2} & 0 & 0 & 0 \\ +\frac{1}{15}\left(\frac{11}{42}\right)^{1/2} & +\frac{13}{180}\frac{1}{\sqrt{2}} & -\frac{1}{18}\left(\frac{11}{10}\right)^{1/2} & -\frac{1}{36}\left(\frac{13}{14}\right)^{1/2} & 0 & 0 \\ +\frac{1}{12}\left(\frac{6}{35}\right)^{1/2} & -\frac{1}{18}\left(\frac{11}{10}\right)^{1/2} & -\frac{1}{72} & -\frac{1}{18}\left(\frac{13}{70}\right)^{1/2} & -\frac{1}{18}\frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{36}\left(\frac{13}{14}\right)^{1/2} & -\frac{1}{18}\left(\frac{13}{70}\right)^{1/2} & -\frac{1}{36}\left(\frac{7}{22}\right)^{1/2} & +\frac{1}{9}\left(\frac{7}{110}\right)^{1/2} & 0 \\ 0 & 0 & -\frac{1}{18}\frac{1}{\sqrt{2}} & +\frac{1}{9}\left(\frac{7}{110}\right)^{1/2} & -\frac{1}{252}\left(\frac{13}{22}\right)^{1/2} & +\frac{1}{21}\frac{1}{\sqrt{3}} \\ 0 & 0 & 0 & 0 & +\frac{1}{21}\frac{1}{\sqrt{3}} & -\frac{17}{168}\left(\frac{1}{34}\right)^{1/2} \end{Vmatrix}, \tag{A21}$$

where the rows and columns are labeled by $J_1(J_2)=3/2$, $5/2$, $7/2$, $9/2$, $11/2$, and $15/2$. It is a simple matter to check that Eqs. (A20) and (A21) satisfy the relation given by Eq. (A19). In making numerical calculations of $Q_J{}^s$ and $B^s(E2)$, one of these two matrices is essentially the only thing we need.

The numerical values of $Q_J{}^s$ for the configurations $(1f_{7/2})^{\pm 3}$ can be calculated by using the general expression in the curly bracket of Eq. (29b); for the configuration $(1f_{7/2})^{-3}$, we have

$$\begin{Vmatrix} J(\hbar) & 3/2 & 5/2 & 7/2 & 9/2 & 11/2 & 15/2 \\ Q_J{}^s/\langle r^2\rangle & -0.400 & +0.619 & +0.222 & -0.333 & +0.071 & +0.498 \end{Vmatrix} \tag{A22}$$

and for the configuration $(1f_{7/2})^3$, the magnitudes of $Q_J{}^s/\langle r^2\rangle$ in (A22) remain the same but all signs must be changed.

As mentioned previously, the sign of $Q_{3/2}{}^s$ for the configuration $(1f_{7/2})^{-3}$ in Table VIII.3 of reference 4 is incorrect. This can be alternatively checked by using the shell-model wave function expressed in terms of Slater determinants. For three particles (not for three holes), this wave function can be easily constructed:

$$\Psi_{J=3/2}{}^{M=3/2}=(3/14)^{1/2}|\psi_{7/2}{}^{5/2}\psi_{7/2}{}^{3/2}\psi_{7/2}{}^{-5/2}|+(3/14)^{1/2}|\psi_{7/2}{}^{3/2}\psi_{7/2}{}^{1/2}\psi_{7/2}{}^{-1/2}|+(3/10)^{1/2}|\psi_{7/2}{}^{7/2}\psi_{7/2}{}^{-1/2}\psi_{7/2}{}^{-3/2}|.$$
$$-(1/10)^{1/2}|\psi_{7/2}{}^{7/2}\psi_{7/2}{}^{1/2}\psi_{7/2}{}^{-5/2}|-(6/35)^{1/2}|\psi_{7/2}{}^{5/2}\psi_{7/2}{}^{1/2}\psi_{7/2}{}^{-3/2}|, \quad \text{(A23)}$$

where $|\psi_j{}^{m_j}\psi_j{}^{m_j'}\psi_j{}^{m_j''}|$ is the totally antisymmetric normalized Slater determinant.

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Some Regularities of the Magnetic Moment Distribution of Odd-A Nuclei

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Some systematic features of the magnetic moment distribution of odd-A nuclei are established and theoretical explanation for them is given. The explanation suggested here is based on the idea of quenching of the intrinsic magnetic moments of nucleons in nuclei. This idea was created independently by several investigators many years ago and has been re-examined in detail recently. Essentially all these investigations predict that the magnetic moment of an odd-A nucleus should be somewhere between the so-called Schmidt and Dirac limits. However, if the exclusion principle is the only reason for the quenching, the magnitude of the quenching turns out to be too small to explain the large magnetic moment deviations from the Schmidt limit. Therefore, the question is whether this idea is valid or whether, even if it is valid, other factors such as configuration mixing, core excitation, etc., are more important. In this paper, these questions are attacked empirically. Furthermore, it is shown that the parity rule may also be an important factor in quenching the intrinsic magnetic moments of nucleons in nuclei.

I. INTRODUCTION

It has been shown by Bow that the magnetic moment ratio of two odd-Z–even-N (or odd-N–even-Z) nuclei with the configurations $(nlj)^\nu$ and $(nlj)^{\nu'}$ for the incomplete-shell nucleons of the odd parts, respectively, is given by the following relation (assuming j-j coupling and pure configuration)[1]:

$$\mu_J[(nlj)^\nu]/\mu_{J'}[(nlj)^{\nu'}]=(J'+1)J^2/(J+1)J'^2, \quad |\nu|=|\nu'|, \quad (1)$$

where ν and ν' are positive or negative odd integers (absolute value $\leqslant j+\frac{1}{2}$). When they are positive they represent the number of protons (or neutrons), and when negative the number of holes. The total angular momentum of the ground states of these two nuclei are J and J', respectively.

The result of Eq. (1) is derived on the basis of the semiatomic model which is an extremely weak-coupling case of the unified model given by Bohr and Mottelson. The counterpart of Eq. (1) in the shell model (or Schmidt model) is as follows (also assuming j-j coupling and pure configuration)[2]:

$$\mu_J[(nlj)^\nu]/\mu_{J'}[(nlj)^{\nu'}]=J/J', \quad \text{no restriction on } \nu,\ \nu',\ n, \text{ and } n'. \quad (2)$$

Here "no restriction" means that n and n' can be either different or equal for arbitrary ν and ν' (absolute value $\leqslant j+\frac{1}{2}$).

A test for the comparative validity of Eqs. (1) and (2) has been made in the region of $(1f_{7/2})$ shell.[1] The results show that Eq. (1) is better. It is the purpose of this paper to extend this investigation to the whole range of the nuclear chart and, furthermore, to see if there is any regularity (in addition to the Schmidt and Dirac limits) in the magnetic moment distribution of odd-A nuclei. Theoretical explanation for the systematic features found in this investigation is also attempted.

When $J=J'$, Eqs. (1) and (2) become identical except for the more strict restrictions on ν and n in

[1] Y. F. Bow, preceding paper, Phys. Rev. **130**, 1931 (1963).

[2] M. G. Mayer and J. H. D. Jensen, *Elementary Theory of Nuclear Shell Structure* (John Wiley & Sons, Inc., New York, 1955).