Calculation of Deuteron Stripping Amplitudes Using S-Matrix Reduction Techniques*

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(Received 14 January 1963; revised manuscript received 15 February 1963)

The transition matrix for deuteron stripping is calculated by use of the Lehmann, Symanzik, and Zimmerman technique for S-matrix reduction. The interaction is assumed to occur via two-body local potentials and all terms are retained in order to give insight into the difference between "direct" and "indirect" interactions. In the "direct" interaction approximation, it is shown that the stripping amplitude can be expressed in terms of a source term and the phase shifts for elastic scattering of deuterons and nucleons incident, respectively, on the initial and final nuclei. The source term includes the usual Born approximation as well as other terms arising from the identity of nucleons.

I. INTRODUCTION

RECENTLY, a very interesting paper[1] has appeared which applies the new methods[2] of S-matrix reduction to the problem of deuteron stripping: $d+B\rightarrow p+C$. Assuming a direct interaction, Amado is able to show that the stripping amplitude may be expressed by the Born approximation plus multiple-scattering corrections. The multiple scattering makes little change in the angular distribution but can produce considerable effect upon the yield; in particular, it correlates the energy variation of the stripping yield with the variations in the total $d+B$ and $n+B$ scattering cross sections. Since the correlation with $d+B$ scattering is experimentally observed, the result is very much like the predictions of distorted-wave Born calculations.[3]

However, the conditions under which the direct-interaction approximation should be valid are not obvious from Amado's derivation. And even if conditions guaranteeing the validity of the direct interaction were to be satisfied, it is still not clear when the Born approximation—distorted or otherwise—should be expected to give a good description of the stripping data. Furthermore, the derivation involves several assumptions whose implications are not clear and which may very well be self-contradictory. For example, the stripping transition matrix-element is expressed as the sum of an equal-time commutator and a time-dependent commutator.[1] Amado drops the former term and is able to evaluate the latter term under the assumption that the nucleon-B interaction is not a potential interaction but a Lee-model interaction in which nucleus B is treated as an elementary particle. This assumption is based upon the equivalence between the Lee model and a separable-potential interaction.[4] However, it is not obvious that such an assumption will be valid when several bound states are allowed, a situation which must be expected in a nuclear problem. Finally, the complexity of Amado's final formulas makes comparison with experiment very tedious. Perhaps this complexity is partly due to the introduction of additional degrees of freedom via the use of new "elementary particles" and the dropping of the simplifying concept of a universal nucleon-nucleon interaction.

This paper attempts to avoid such ambiguities in the calculation of the stripping amplitude by S-matrix reduction techniques. The number of degrees of freedom is kept to an absolute minimum. A local, two-body nucleon-nucleon potential is assumed. All interactions, such as absorption of nucleons by nuclei and nucleon-nucleus scattering, will then be the result of appropriate sums of such two-body potentials. All terms will be retained as long as possible and nothing will be dropped until its form has been made explicit. It is, thus, hoped that the meaning of all approximations will be clear and their realms of applicability made apparent. In an attempt to keep this heuristic calculation simple and transparent, all spin and isotopic spin-indexes will be dropped and no Coulomb effects will be allowed. The nucleons are, thus, treated as neutral, spinless, identical fermions.

II. STRIPPING FORMALISM

The formalism of second quantization[5] will be adopted here in order to manifest the identity and Fermi-Dirac character of the particles throughout the calculation. We introduce nucleon single-particle destruction and creation operators $\psi(\mathbf{x},t)$, $\psi^\dagger(\mathbf{x},t)$ which satisfy the equations of motion,

$$i\frac{\partial}{\partial t}\psi(\mathbf{x},t)=[\psi(\mathbf{x},t),H],\quad i\frac{\partial}{\partial t}\psi^\dagger(\mathbf{x},t)=[\psi^\dagger(\mathbf{x},t),H]. \quad (1)$$

The Hamiltonian H need not be specified as yet. For simplicity in the calculations we adopt units such that

* Work performed under the auspices of the U. S. Atomic Energy Commission.
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[1] R. D. Amado, Phys. Rev. **127**, 261 (1962).
[2] H. Lehmann, K. Symanzik, and W. Zimmerman, Nuovo Cimento **1**, 205 (1955); W. Zimmerman, *ibid.* **10**, 597 (1958).
[3] A review and list of references will be found in *Proceedings of the International Conference on Nuclear Structure* (University of Toronto Press, Toronto, 1960).
[4] M. T. Vaughn, R. Aaron, and R. D. Amado, Phys. Rev. **124**, 1258 (1961).
[5] See, for example, P. J. Redmond and J. L. Uretsky, Ann. Phys. (N. Y.) **9**, 106 (1960); S. Schweber and E. C. G. Sudarshan, *ibid.* **19**, 351 (1962).

$\hbar=1$, and drop all spin and isotopic spin indexes. The deuteron is, then, a bound state of two nucleons and is represented by the single-particle creation operator $\psi_d^\dagger(\mathbf{x},t)$ which, in turn, is just a bilinear combination of the nucleon creation operators:

$$\psi_d^\dagger(\mathbf{x},t)=(2!)^{-1/2}\int d\mathbf{y}d\mathbf{y}' \times h_d(\mathbf{y}+\mathbf{x},\,\mathbf{y}'+\mathbf{x},\,t)\delta(\mathbf{y}+\mathbf{y}')\psi^\dagger(\mathbf{y},t)\psi^\dagger(\mathbf{y}',t),$$

where h_d is the complete, exact, deuteron wave function. Similarly, the nuclei B and C are bound states of A and $A+1$ nucleons, respectively, represented by the single-particle creation operators $\psi_B^\dagger(\mathbf{x},t)$ and $\psi_C^\dagger(\mathbf{x},t)$. These operators are formed similarly from products of the nucleon creation operators. At time zero, the nucleon operators satisfy the anticommutation relations

$$\begin{aligned}&[\psi(\mathbf{x},0),\psi^\dagger(\mathbf{y},0)]_+=\delta(\mathbf{x}-\mathbf{y}),\\&[\psi(\mathbf{x},0),\psi(\mathbf{y},0)]_+=[\psi^\dagger(\mathbf{x},0),\psi^\dagger(\mathbf{y},0)]_+=0.\end{aligned}\tag{2}$$

Let $f_p(\mathbf{x},t)$ be a normalizable solution to the free-particle Schrödinger equation

$$i\frac{\partial}{\partial t}f_p(\mathbf{x},t)=\frac{1}{2m}\nabla^2 f_p(\mathbf{x},t)\tag{3}$$

for the outgoing proton, where m is the nucleon mass. In an analogous manner, single-particle wave functions $f_d(\mathbf{x},t)$, $f_B(\mathbf{x},t)$, and $f_C(\mathbf{x},t)$ are defined for the free deuteron, and for the two free nuclei. These wave functions are normalized wave packets so that when integrals over space are rearranged via Green's theorem, the surface terms at infinity vanish. Allowing them to become plane waves on completion of the calculation guarantees conservation of momentum. We can now define the bounded single-particle operators

$$\begin{aligned}&\psi_p(t)=\int d\mathbf{x}\,f_p^*(\mathbf{x},t)\psi(\mathbf{x},t),\quad \psi_d(t)=\int d\mathbf{x}\,f_d^*(\mathbf{x},t)\psi_d(\mathbf{x},t),\\&\psi_{B,C}(t)=\int d\mathbf{x}\,f_{B,C}^*(\mathbf{x},t)\psi_{B,C}(\mathbf{x},t),\end{aligned}\tag{4}$$

where $\psi_p(t)$ represents the destruction of a single nucleon in state f_p, etc. Finally, we define the interaction currents

$$J_X(t)=i\frac{\partial}{\partial t}\psi_X(t),\tag{5}$$

where $X=p$, d, B, or C. Following the method of Redmond and Uretsky,[5] $J_X(t)$ can be represented in terms of the internucleon potentials and the nuclear wave functions. Using the known asymptotic time dependence of the wave packets, f_X,[5] the limiting time dependence of the appropriate two-nucleus matrix elements of $J_X(t)$ can be found and it is, therefore, possible to define "in" and "out" operators corresponding to $\psi_X(t)$.

The S-matrix element to be calculated is

$$S(d+B\to p+C)\equiv\langle pC(-)|dB(+)\rangle,\tag{6}$$

where, for example, $|dB(+)\rangle=\psi_d^\dagger(\text{in})|B\rangle=\psi_B^\dagger(\text{in})|d\rangle$. Define the transition-matrix element via

$$S=1-2\pi i\delta(E_p+E_C-E_d-E_B)T(d+B\to p+C)\tag{7}$$

and contract[2,6] on each of the scattered particles in turn. This yields the T matrix in the equivalent forms (on the energy shell)

$$\begin{aligned}T(d+B\to p+C)&=\langle C|J_p(0)|dB(+)\rangle=\langle p|J_C(0)|dB(+)\rangle\\&=\langle pC(-)|J_d^\dagger(0)|B\rangle=\langle pC(-)|J_B^\dagger(0)|d\rangle.\end{aligned}\tag{8}$$

When contracting a second time, the usual methods[1,5,6] which introduce an anticommutator of two of the fields are avoided. This is due to a desire to introduce as few terms as possible, thus cutting down on the number of terms which must be dropped without ample justification. For example, Amado[1] is forced to drop $\frac{1}{2}$ of the anticommutator on the basis of "a large energy denominator" without any attempt to estimate the corresponding numerator. In this case the second contraction is performed using the identity $f(-\infty)=f(0)-\int_{-\infty}^0 dt \times f'(t)$. Choosing the first form and contracting on the incoming deuteron state yields

$$T=\langle C|J_p(0)\psi_d^\dagger(0)|B\rangle+i\int_{-\infty}^{\infty}dt\,\theta(-t)\langle C|J_p(0)J_d^\dagger(t)|B\rangle.\tag{9}$$

Introducing a complete set of intermediate eigenstates of the Hamiltonian, and using the known time dependence $J_d^\dagger(t)=\exp(iHt)J_d^\dagger(0)\exp(-iHt)$ allows the integration in Eq. (9) to be carried out. This gives

$$T(d+B\to p+C)=\langle C|J_p(0)\psi_d^\dagger(0)|B\rangle+\sum_N\frac{\langle C|J_p(0)|N\rangle\langle N|J_d^\dagger(0)|B\rangle}{E_N-E_B-E_d-i\epsilon},\tag{10}$$

which is a form of the Low equation.[6,7]

The states N include all $(A+2)$-nucleon eigenstates having the same symmetries as the $d+B$ and $p+C$ states. Among these are the bound states of the $(A+2)$-

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[6] M. L. Goldberger, in *Dispersion Relations and Elementary Particles*, edited by DeWitt and Omnes (Hermann & Cie., Paris, 1960).

[7] See, for example and references, S. Schweber, *Introduction to Relativistic Quantum Field Theory* (Row, Peterson & Co., Evanston, 1961), pp. 394–414.

nucleon system, the elastic and inelastic scattering states $|d'B'(-)\rangle$ and $|p'C'(-)\rangle$, and the scattering states in which three or more particles are present. In terms of the total incident energy defined as $E=E_d+E_B$, Eq. (10) implies that the stripping amplitude T will have poles along the negative real E axis corresponding to the bound states of the $(A+2)$-nucleon system, and that cuts corresponding to the various possible scattering states will extend to infinity along the positive real E axis.[7]

It is reasonable to assume that there are no other singularities, i.e., the stripping amplitude T is assumed to be an analytic function of the incident energy in the upper half of the E plane.[6] In the next section it is proved that the inhomogeneous term $\langle C|J_p(0)\psi_d^\dagger(0)|B\rangle$ is a real function for real energies. Thus, by using Cauchy's integral theorem and assuming the integrals to converge, the stripping matrix element T can be written as a function of energy in the form[7]

$$T_E(d+B\to p+C)=\langle C|J_p(0)\psi_d^\dagger(0)|B\rangle +\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{Im}T_{E'}(d+B\to p+C)}{E'-E-i\epsilon}dE'. \quad (11)$$

Write $\theta(t)=\frac{1}{2}[1+\epsilon(t)]$ in Eq. (9). Then it can be shown,[6] with the assumption of time reversal and parity invariance, that the imaginary part of T (absorptive part) is given by

$$\begin{aligned}\mathrm{Im}T_E&=\frac{1}{2}\int_{-\infty}^{\infty}dt\,\langle C|J_p(0)J_d^\dagger(t)|B\rangle\\&=\pi\textstyle\sum_N\delta(E_N-E)\langle C|J_p(0)|N\rangle\langle N|J_d^\dagger(0)|B\rangle,\end{aligned} \quad (12)$$

where the known time dependence of $J_d^\dagger(t)$ and the complete set of states N have again been introduced. Note that because of the appearance of the energy-conserving δ function, $\mathrm{Im}T_E$ is zero for negative E except for δ-function singularities at the energies of the bound states. [It should be pointed out that Amado[1] has omitted these pole terms from his Eqs. (3.9), (3.11), and (3.12), and hence, his conclusion, (3.19), must be suspect.] If the energy is less than the separation energy, three-particle scattering states are excluded, and for low enough energies only elastic-scattering states will be included.

By use of the same methods that led to Eq. (8), it is easy to show that

$$\begin{aligned}T_E(d+B\to d'+B')&=\langle d'B'(-)|J_d^\dagger(0)|B\rangle,\\T_E(p+C\to p'+C')&=\langle p'C'(-)|J_p^\dagger(0)|C\rangle,\end{aligned} \quad (13)$$

where B' and C' may represent excited states of the B and C nuclei so that Eq. (13) includes inelastic as well as elastic events. The imaginary part of the stripping matrix may now be written

$$\begin{aligned}\mathrm{Im}T_E(d+B\to p+C)=&\pi\sum_{d'B'}T_E(d'+B'\to p+C)T_E^*(d'+B'\to d+B)\delta(E-E_{d'}-E_{B'})\\&+\pi\sum_{p'C'}T_E(d+B\to p'+C')T_E^*(p+C\to p'+C')\delta(E-E_{p'}-E_{C'})\\&+\text{contributions from states with three or more particles.}\end{aligned} \quad (14)$$

The right-hand side of Eq. (14) is continuous only for physical E and has point singularities at the bound states so that the integral in Eq. (11) now is restricted to the positive real E axis but in addition there is now a sum over bound-state poles. All the scattering matrices on the right of Eq. (14) are at the same total incident energy; since the left-hand side of Eq. (14) must be real, Eq. (14) implies a relation between the $p+B$ and $d+A$ scattering amplitudes and the stripping amplitude at a fixed total positive energy.

The direct-interaction-model approximation is defined by the dropping of all terms in Eq. (14) except the ground states of B and C, i.e., the sum is restricted to elastic-scattering states.[1] This is analogous to the one-meson approximation in Chew-Low theory.[7] Essentially, this implies neglecting the nucleon-nucleon correlation in the incident and final nuclei.[8] For a very crude estimate of the validity of this approximation, note that the nucleus cannot be said to be in an excited state unless the energy difference between states is large compared to the uncertainty in energy implied by $\Delta E\Delta t\approx 1$, where Δt is the amount of time during which the nucleus can be excited. If $\epsilon_i-\epsilon_0$ is the energy difference between the ground state and the ith state of the nucleus in question, scattering to excited states and back down to the ground states will be important only if

$$|\epsilon_i-\epsilon_0|\gg\frac{1}{\Delta t}\approx\frac{1}{R}\left(\frac{E}{m_d}\right)^{1/2}, \quad (15)$$

where R is the nuclear radius. Thus, making the direct interaction approximation, the sums in Eq. (14) become integrations of elastic scattering amplitudes over scattering angles in the c.m. system. In terms of partial waves, the stripping amplitude may be expressed as

$$\begin{aligned}T_E&(d+B\to p+C)\\&=\textstyle\sum_L T_{E,L}(d+B\to p+C)Y_L(\cos\theta_p),\end{aligned} \quad (16)$$

where the Y_L are the normalized Legendre polynomials and θ_p is the angle between the proton and the deuteron in the c.m. system. By introducing the (complex) $d-B$

[8] A. M. Saperstein, Progr. Theoret. Phys. (Kyoto) **26**, 489 (1961).

and $p-C$ elastic-scattering phase shifts $\delta_{dB}{}^L$, $\delta_{pC}{}^L$, and carrying out the integration over intermediate scattering angles, Eq. (14) can be written as

$$\mathrm{Im}T_{E,L}(d+B\rightarrow p+C)\\ =-(2\pi)^3\Lambda_L e^{-i\chi_L}T_{E,L}(d+B\rightarrow p+C), \quad (17)$$

where the real quantities Λ_L and χ_L are defined by

$$\Lambda_L e^{-i\chi_L}=e^{-i\delta_{dB}{}^L}\sin\delta_{dB}{}^L+e^{-i\delta_{pC}{}^L}\sin\delta_{pC}{}^L. \quad (18)$$

Since the right-hand side of Eq. (17) must be real, it follows that the phase of $T_{E,L}$ must be $+\chi_L$. In the energy region for which Eq. (17) is exact, i.e., below the excited-state threshold, this result is just an extension of the final-state interaction theorem of Watson.[6] Let $S_{B,C}(E,L)$ be the Y_L projection of the source term $\langle C|J_p(0)\psi_d{}^\dagger(0)|B\rangle$ at energy E, and let $S'_{B,C}(E,L)\equiv S_{B,C}(E,L)+$(the pole terms from Eq. (17)); it is, thus, possible [using Eq. (16)] to put Eq. (11) into the form

$$T_{E,L}(d+B\rightarrow p+C)=S'_{B,C}(E,L)\\ +\frac{1}{\pi}\int_0^\infty \frac{T_{E',L}(d+B\rightarrow p+C)\sin\chi_L(E')e^{-i\chi_L(E')}}{E'-E-i\epsilon}dE'. \quad (19)$$

Equation (19) is a standard singular inhomogeneous integral equation[9] for $T_{E,L}$. Assuming that $\chi_L(E)\rightarrow 0$ as $E\rightarrow\infty$ (otherwise subtractions will be necessary) the solution of Eq. (19) which approaches the source term S' as the energy approaches infinity is[9]

$$T_{E,L}(d+B\rightarrow p+C)=S'_{B,C}(E,L)\\ +\frac{1}{\pi}e^{\rho_L(E)+i\chi_L(E)}\int_0^\infty \frac{S'_{B,C}(E',L)\sin\chi_L(E')e^{-\rho_L(E')}}{E'-E-i\epsilon}dE'. \quad (20)$$

Here

$$\rho_L(E)=\frac{1}{\pi}P\int_0^\infty \frac{\chi_L(E')}{E'-E}dE'. \quad (21)$$

Thus, the deuteron-stripping amplitude can be expressed in terms of a source term $S'_{B,C}$ and the phase shifts for elastic $d-B$ and $p-C$ scattering whenever the direct-interaction approximation is valid. Because the pole terms decrease rapidly with distance from the bound states, we can use S instead of S' in (20) whenever the bombarding energy is not too small. In Eq. (20), the source term may be interpreted as the direct-scattering term, whereas the second term on the right is the correction due to multiple-scattering events.

III. THE SOURCE TERM

The interaction between nucleons will now be specified to be a local, two-body force with potential $v(\mathbf{x}-\mathbf{y})$. The proton interaction current can then be written as[5]

$$J_p(t)=\int d\mathbf{x}d\mathbf{y}\, f_p{}^*(\mathbf{y},t)\psi^\dagger(\mathbf{x},t)v(\mathbf{x}-\mathbf{y})\psi(\mathbf{x},t)\psi(\mathbf{y},t). \quad (22)$$

The complete deuteron wave function for $t=0$—including its c.m. motion—will be written as $h_d(\mathbf{x},\mathbf{y})$. Similarly, the wave functions for the two nuclei are $h_B(\mathbf{x}_1,\cdots,\mathbf{x}_A)$ and $h_C(\mathbf{x}_1,\cdots,\mathbf{x}_{A+1})$. As a short-hand notation, write

$$\psi(\mathbf{x})\equiv\psi(\mathbf{x},0),\quad \psi^\dagger(\mathbf{x})\equiv\psi^\dagger(\mathbf{x},0). \quad (23)$$

By use of Eq. (22) and the anticommutation relations, Eq. (2), the source-term operator can be put into the normal form

$$J_p(0)\psi_d{}^\dagger(0)\\ =\int d\mathbf{x}d\mathbf{y}d\mathbf{z}d\mathbf{w}\, f_p{}^*(\mathbf{x})h_d(\mathbf{y},\mathbf{z})v(\mathbf{w}-\mathbf{x})\psi^\dagger(\mathbf{x})\\ \times\{\psi^\dagger(\mathbf{y})\psi^\dagger(\mathbf{z})\psi(\mathbf{w})\psi(\mathbf{x})\\ +\delta(\mathbf{x}-\mathbf{z})[\psi^\dagger(\mathbf{y})\psi(\mathbf{w})-\delta(\mathbf{w}-\mathbf{y})]\\ +\delta(\mathbf{x}-\mathbf{y})[\delta(\mathbf{w}-\mathbf{z})-\psi^\dagger(\mathbf{z})\psi(\mathbf{w})]\\ +\delta(\mathbf{y}-\mathbf{w})\psi^\dagger(\mathbf{z})\psi(\mathbf{x})-\delta(\mathbf{w}-\mathbf{z})\psi^\dagger(\mathbf{y})\psi(\mathbf{x})\}. \quad (24)$$

The time-independent state vectors for the nuclei can be written

$$|B\rangle=(A!)^{-1/2}\int d\mathbf{x}_1\cdots d\mathbf{x}_A\\ \times h_B(\mathbf{x}_1,\cdots,\mathbf{x}_A)\psi^\dagger(\mathbf{x}_A)\cdots\psi^\dagger(\mathbf{x}_1)|0\rangle,$$
$$\langle C|=[(A+1)!]^{-1/2}\int d\mathbf{x}_1\cdots d\mathbf{x}_{A+1}\\ \times h_C{}^*(\mathbf{x}_1,\cdots,\mathbf{x}_{A+1})\langle 0|\psi(\mathbf{x}_1)\cdots\psi(\mathbf{x}_{A+1}), \quad (25)$$

where $|0\rangle$ is the no-particle or vacuum state. With the aid of Eq. (2), the source term matrix elements of Eq. (24) may be written in terms of the following overlap integrals[10]:

$$\langle C|\psi^\dagger(\mathbf{y})|B\rangle=(-1)^A(A+1)^{1/2}\int d\mathbf{x}_1\cdots d\mathbf{x}_A\\ \times h_C{}^*(\mathbf{y},\mathbf{x}_1\cdots\mathbf{x}_A)h_B(\mathbf{x}_1,\cdots,\mathbf{x}_A), \quad (26)$$

$$\langle C|\psi^\dagger(\mathbf{y})\psi^\dagger(\mathbf{z})\psi(\mathbf{w})|B\rangle\\ =(-1)^{A-1}A(A+1)^{1/2}\int d\mathbf{x}_1\cdots d\mathbf{x}_{A-1}\\ \times h_C{}^*(\mathbf{z},\mathbf{y},\mathbf{x}_1,\cdots,\mathbf{x}_{A-1})h_B(\mathbf{w},\mathbf{x}_1,\cdots,\mathbf{x}_{A-1}), \quad (27)$$

[9] R. Omnes, Nuovo Cimento 8, 316 (1958).

[10] L. Cooper, lectures, Brown University, 1959 (unpublished).

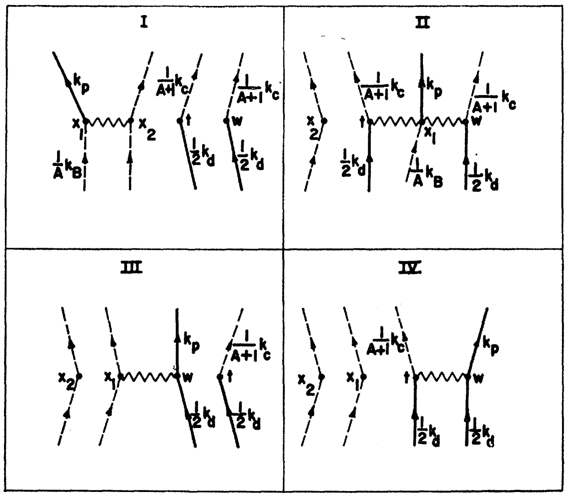

FIG. 1. Graphs describing the four parts of the source term.

$$\langle C|\psi^\dagger(\mathbf{u})\psi^\dagger(\mathbf{t})\psi^\dagger(\mathbf{w})\psi(\mathbf{z})\psi(\mathbf{y})|B\rangle$$
$$=(-1)^A A(A-1)(A+1)^{1/2}\int d\mathbf{x}_1\cdots d\mathbf{x}_{A-2}$$
$$\times h_C^*(\mathbf{u},\mathbf{t},\mathbf{w},\mathbf{x}_1,\cdots,\mathbf{x}_{A-2})h_B(\mathbf{y},\mathbf{z},\mathbf{x}_1,\cdots,\mathbf{x}_{A-2}). \quad (28)$$

The c.m. motions can now be separated out and will be assumed to be plane waves. Let

$$g_B(\xi_1,\cdots,\xi_A),\quad \text{with}\quad \sum \xi_i=0 \quad (29)$$

be the real wave function for the B nucleus in its center-of-mass system. Similarly, define $g_C(\xi_1,\cdots,\xi_{A+1})$ and $g_d(\xi_1-\xi_2)$ for the C nucleus and the deuteron. Let

$$F_{BC}(\mathbf{t},\mathbf{w},\mathbf{x}_1,\mathbf{x}_2)\equiv\int d\mathbf{x}_3\cdots d\mathbf{x}_A\,\delta(\sum_{i=1}^{A} x_i)$$
$$\times g_C(\mathbf{t},\mathbf{w},\mathbf{x}_2,\cdots,\mathbf{x}_A)g_B(\mathbf{x}_1,\mathbf{x}_2,\cdots,\mathbf{x}_A). \quad (30)$$

Note that $F_{BC}(\mathbf{t},\mathbf{w},\mathbf{x}_1,\mathbf{x}_2)=F_{BC}(-\mathbf{t},-\mathbf{w},-\mathbf{x}_1,-\mathbf{x}_2)$ because of the parity invariance of the c.m. wave functions. By use of Eqs. (24)–(30) the source term for the stripping reaction may be written as

$$\langle C|J_p(0)\psi_d^\dagger(0)|B\rangle=\frac{(A+1)^{1/2}}{(2\pi)^3}\delta(\mathbf{k}_d+\mathbf{k}_B-\mathbf{k}_p-\mathbf{k}_C)$$
$$\times[S_{\rm I}+S_{\rm II}+S_{\rm III}+S_{\rm IV}], \quad (31)$$

where the $\mathbf{k}_X$ are the momenta of the particles partaking in the reaction and where

$$S_{\rm I}\equiv(-1)^{A-1}A(A-1)(A-2)^3\int d\mathbf{x}_1d\mathbf{x}_2d\mathbf{t}d\mathbf{w}\;g_d(\mathbf{t}-\mathbf{w})v(\mathbf{x}_1-\mathbf{x}_2)F_{BC}(\mathbf{t},\mathbf{w},\mathbf{x}_1,\mathbf{x}_2)\exp\left[i(\mathbf{t}+\mathbf{w})\cdot\left(\frac{\mathbf{k}_d}{2}-\frac{\mathbf{k}_C}{A+1}\right)\right]$$
$$\times\exp\left\{i\mathbf{x}_1\cdot\left[\frac{A}{A+1}\left(\frac{\mathbf{k}_B}{A}-\mathbf{k}_p\right)+\frac{2}{A+1}\left(\frac{\mathbf{k}_d}{2}-\mathbf{k}_p\right)\right]\right\}, \quad (32)$$

$$S_{\rm II}\equiv(-1)^{A-1}A(A-1)^3\int d\mathbf{x}_1d\mathbf{x}_2d\mathbf{t}d\mathbf{w}\;g_d(\mathbf{t}-\mathbf{w})[v(\mathbf{w}-\mathbf{x}_1)+v(\mathbf{t}-\mathbf{x}_1)]F_{BC}(\mathbf{t},\mathbf{w},\mathbf{x}_1,\mathbf{x}_2)\exp\left[i(\mathbf{t}+\mathbf{w})\cdot\left(\frac{\mathbf{k}_d}{2}-\frac{\mathbf{k}_C}{A+1}\right)\right]$$
$$\times\exp\left\{i\mathbf{x}_1\cdot\left[\frac{A}{A+1}\left(\frac{\mathbf{k}_B}{A}-\mathbf{k}_p\right)+\frac{2}{A+1}\left(\frac{\mathbf{k}_d}{2}-\mathbf{k}_p\right)\right]\right\}, \quad (33)$$

$$S_{\rm III}\equiv(-1)^A 2A(A-1)^3\int d\mathbf{x}_1d\mathbf{x}_2d\mathbf{t}d\mathbf{w}\;g_d(\mathbf{t}-\mathbf{w})v(\mathbf{x}_1-\mathbf{w})$$
$$\times F_{BC}(\mathbf{t},\mathbf{x}_1,\mathbf{x}_1,\mathbf{x}_2)\exp\left\{i\mathbf{t}\cdot\left(\frac{\mathbf{k}_d}{2}-\frac{\mathbf{k}_C}{A+1}\right)\right\}\exp\left\{i\mathbf{w}\cdot\left(\frac{\mathbf{k}_d}{2}-\mathbf{k}_p\right)\right\}, \quad (34)$$

$$S_{\rm IV}\equiv 2A^3\int d\mathbf{x}_1d\mathbf{x}_2d\mathbf{t}d\mathbf{w}\;g_d(\mathbf{t}-\mathbf{w})v(\mathbf{t}-\mathbf{w})F_{BC}(\mathbf{t},\mathbf{x}_1,\mathbf{x}_1,\mathbf{x}_2)\exp\left\{i\mathbf{t}\cdot\left(\frac{\mathbf{k}_d}{2}-\frac{\mathbf{k}_C}{A+1}\right)\right\}\exp\left\{i\mathbf{w}\cdot\left(\frac{\mathbf{k}_d}{2}-\mathbf{k}_p\right)\right\}. \quad (35)$$

With the aid of Eq. (30) it is obvious, by inspection, that the source term is a real quantity; it is assumed that g_d and v are parity invariant.

The four parts of the source term may be interpreted in terms of the diagrams in Fig. 1. Solid lines represent nucleons in the entrance or exit channels, dashed lines are nucleons bound in a nucleus, and a wiggly line is the potential interaction. The nucleon lines are labeled with their position and their momentum. In diagram I, each of the two incident nucleons $\mathbf{t}$ and $\mathbf{w}$ changes its momentum from the value $\mathbf{k}_d/2$ characteristic of nucleons in the incident deuteron to the value $k_C/(A+1)$ characteristic of a nucleon bound in the final nucleus. Meanwhile, two nucleons $\mathbf{x}_1$ and $\mathbf{x}_2$ in the initial nucleus interact, changing the initial momentum $\mathbf{k}_B/A$ of the former (with an addition from the absorption of $\mathbf{t}$ and $\mathbf{w}$) into the momentum $\mathbf{k}_p$ of the final free proton. Term $S_{\rm II}$ corresponding to diagram II is the contribution when both absorbed nucleons interact with a single nucleon in the nucleus, so that the momentum of the latter is changed to the free-particle value $\mathbf{k}_p$. Term S_{III} arises when one of the incident nucleons $\mathbf{t}$ is absorbed while the other one $\mathbf{w}$ interacts with a bound nucleon, then continues on its way as a free particle. In diagram IV, again only one incident nucleon $\mathbf{t}$ is absorbed, but this time the interaction is between the two deuteron particles, releasing $\mathbf{w}$ as the free proton.

Contributions S_{I} and S_{II}, being the results of a two-step process of deuteron absorption and subsequent proton emission, would not normally be included under the heading of "direct interaction." These indirect terms may be referred to as "heavy particle stripping" contributions. Notice that they appear with factors of A^5 and A^4, whereas the "direct interaction" terms S_{III} and S_{IV} go as A^4 and A^3. The justification for dropping the indirect terms is thus not immediately obvious. The magnitude of $(\mathbf{k}_d/2)-\mathbf{k}_p$ increases with increasing proton scattering angle θ_p. Thus, when integrated, the factor $\exp\{i\mathbf{w}[(\mathbf{k}_d/2)-\mathbf{k}_p]\}$ would produce a normal diffraction pattern—decreasing in amplitude with increasing θ_p. The factor $\exp\{i\mathbf{v}[(\mathbf{k}_d/2)-\mathbf{k}_C]/(A+1)\}$ produces just the opposite effect—its diffraction pattern amplitude increases with increasing θ_p. The resultant of these two factors in Eqs. (34) and (35) should produce a pattern whose amplitude does not change markedly with angle. The two indirect terms, Eqs. (32) and (33), do not contain factors producing normal diffraction patterns; instead, they each contain two factors tending to produce backward diffraction patterns. Hence, the indirect terms are expected to be very small—compared with the direct terms—at small scattering angles. Furthermore, the integrands of the indirect terms contain a third exponential

$$\exp\left(\frac{i\mathbf{x}_1}{A+1}\cdot\left\{A\left[\frac{\mathbf{k}_B}{A}-\mathbf{k}_p\right]+2\left[\frac{\mathbf{k}_d}{2}-\mathbf{k}_p\right]\right\}\right)=\exp\left[i\mathbf{x}_1\cdot\left(\frac{\mathbf{k}_C}{A+1}-\mathbf{k}_p\right)\right].$$

This factor will decrease the integrals in Eqs. (32) and (33) independent of the c.m. angle, the amount of damping increasing with increasing incident energy. Thus, when the incident energy is sufficiently large and the angle of scattering not too great, it should be reasonable to keep only the direct-interaction parts, S_{III} and S_{IV}, of the source term. Since the indirect terms arise from the identity of nucleons, this is equivalent to saying that when the incident and scattered particles have energy sufficiently high to distinguish them from the remaining particles, antisymmetrization is no longer necessary.[11]

[11] G. Takeda and K. M. Watson, Phys. Rev. **97**, 1339 (1955); F. Coester and H. Kummel, Nucl. Phys. **9**, 225 (1958–9).

The direct term S_{IV} can be written as

$$S_{\mathrm{IV}}=2A^3\left[\int d\mathbf{u}\, g_d(\mathbf{u})v(\mathbf{u})\exp\{i\mathbf{u}\cdot(\mathbf{k}_d/2-\mathbf{k}_p)\}\right]\times\left[\int d\mathbf{t}\exp\{i\mathbf{t}\cdot[\mathbf{k}_d-\mathbf{k}_p-\mathbf{k}_C/(A+1)]\}\times\int d\mathbf{x}_1 d\mathbf{x}_2\, F_{BC}(\mathbf{t},\mathbf{x}_1,\mathbf{x}_1,\mathbf{x}_2)\right]. \quad (36)$$

The second factor is just the Fourier transform of the "single-particle state" in nucleus C into which the neutron is captured. If the first factor is dropped, i.e., if the deuteron structure is neglected, the remainder is the ordinary Born approximation for stripping,[1] although here, the momentum transfer also includes the contribution from the final bound nucleon. The integrand of S_{III} is nonzero only when $\mathbf{x}_1\approx\mathbf{w}$ since the internucleon potential has a very short range compared with nuclear radii. If the structure of the deuteron is neglected so that $g_d(\mathbf{t}-\mathbf{x}_1)$ is approximately constant whenever F_{BC} is nonvanishing, S_{III} can be approximated by

$$S_{\mathrm{III}}\approx(-1)^A 2A(A-1)^3\times\int d\mathbf{u}\, g_d(\mathbf{u})v(\mathbf{u})\exp[i\mathbf{u}\cdot(\mathbf{k}_p-\tfrac{1}{2}\mathbf{k}_d)]\times\int d\mathbf{t}\exp\left[i\mathbf{t}\cdot\left(\frac{\mathbf{k}_d}{2}-\frac{\mathbf{k}_C}{A+1}\right)\right]\int d\mathbf{x}_1 d\mathbf{x}_2\times\exp[i\mathbf{x}_1\cdot(\tfrac{1}{2}\mathbf{k}_d-\mathbf{k}_p)]F_{BC}(\mathbf{t},\mathbf{x}_1,\mathbf{x}_1,\mathbf{x}_2). \quad (37)$$

Equation (37) has one more exponential in the integrand than does Eq. (36). Thus, under the conditions for which it is valid to drop the nondirect terms, one would expect $S_{\mathrm{III}}\ll S_{\mathrm{IV}}$. Under these conditions, then, the source term $\langle C|J_p(0)\psi_d^\dagger(0)|B\rangle$ can be approximated by the direct Born term, Eq. (36).

IV. CONCLUSION

There seems to be evidence[3] that the multiple-scattering contribution to the stripping amplitude decreases with decreasing incident kinetic energy and decreasing Q value for the reaction. This observation may be understood in terms of the solution for the stripping amplitude, Eq. (20). Noting that the numerator of the integrand in the multiple-scattering term of Eq. (20) vanishes as E' vanishes, it is obvious that the contribution of this term will be minimized when E is confined to the vicinity of zero. This implies small deuteron kinetic energy and a small Q value for the stripping reaction.

Further insight into the implications of the multiple-scattering term can be obtained in the following crude manner. Multiplying Eq. (20) by $Y_L(\cos\theta_p)$ and summing over L gives the complete stripping amplitude in the form

$$T_E(d+B \to p+C) = \langle C|J_p(0)\psi_d^\dagger(0)|B\rangle + \frac{1}{\pi}\int_0^\infty \frac{dE'}{E'-E-i\epsilon} \times\sum_L [\sin\chi_L(E')e^{i\chi_L(E)}e^{\rho_L(E)-\rho_L(E')}S_{B,C}(E',L) \times Y_L(\cos\theta_p)]. \quad (38)$$

If an average phase shift is defined by a relation of the form $\sum a_L b_L \equiv \langle a_L\rangle_{\text{av}} \sum b_L$, then Eq. (38) can be written as

$$T_E(d+B \to p+C) = \langle C|J_p(0)\psi_d^\dagger(0)|B\rangle[1+i\langle e^{i\chi_L(E)}\sin\chi_L(E)\rangle_{\text{av}}] + \frac{1}{\pi}P\int_0^\infty \frac{\langle C|J_p(0)\psi_d^\dagger(0)|B\rangle}{E'-E} \times\langle e^{i\chi_L(E)}e^{\rho_L(E)-\rho_L(E')}\sin\chi_L(E')\rangle_{\text{av}}dE'. \quad (39)$$

The average values will ordinarily depend upon the scattering angle θ_p. However, for not too large angles (such that the significant Y_L are all of the same sign) the average values should be roughly independent of angle. Then, if the principal-value term is dropped because of the energy denominator, Eq. (39) implies that, for small angles, the stripping amplitude should have roughly the same angular variation as the source term. This proportionality between the Born term and the stripping amplitude is well known experimentally, as is its breakdown at large angles.[3] It is reasonable to expect that a more careful evaluation of the multiple-scattering term[1,12] will also show the changes in angular dependence usually interpreted as a change of effective radius in a Born calculation.[3]

It should also be noted from Eq. (39) that the magnitude of the proportionality constant varies with energy, the variation being given by the energy dependence of the $d-B$ and $p-C$ elastic-scattering cross sections. This result differs from that of Amado,[1] who found that the energy dependence is given by the $d-B$ and $n-B$ elastic cross sections. It is, however, very close in spirit to the usual distorted-wave Born approximation,[3] in which the incoming plane wave is distorted by the $d-B$ elastic-scattering optical potential, whereas the final plane wave is distorted by the $p-C$ optical potential. The significant difference is that the present calculation depends only upon the elastic-scattering phase shifts, whereas the distorted-wave calculations depend upon the details of the respective optical potentials.

When comparison with Amado's results is to be made, the source term in Eq. (38) must be restricted to the direct Born part S_{IV}. In this case the stripping amplitude given in Eq. (38) has a much simpler form than the corresponding one found in Amado.[1] This was expected in view of the unifying use of the two-nucleon potential and the single-nucleon field. However, the results of Sec. III indicate that the simple Born term, even with multiple scattering corrections, may not give a good representation of the stripping amplitude at very large nucleon angles. The remaining parts S_{I}, S_{II}, and S_{III} of the source term may be expected to make their presence felt by a failure of the distorted-wave Born approximation to fit the stripping differential cross section at large angles. Such a failure has been observed[13] in the $\text{Ca}^{40}(d,p)\text{Ca}^{41}$ reaction at energies between 7 and 12 MeV. Careful calculations might make possible the experimental determination of the nondirect terms and thus throw additional light on the nuclear wave functions. There still remains, however, the problem of rigorously justifying the direct-interaction approximation itself and clearly delineating its realm of applicability.

ACKNOWLEDGMENTS

The author is indebted to Dr. Paul Benioff, Dr. Fritz Coester, and Dr. Jack Uretsky for many helpful conversations.

[12] A. M. Saperstein and D. Feldman, Nuovo Cimento **14**, 457 (1959).

[13] J. P. Schiffer (private communication).