Consequences of a Weak Vector Boson for the Decay $K \to \mu+\nu+\gamma$*

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(Received 24 January 1963)

Consequences of an intermediate vector boson, with arbitrary anomalous magnetic and quadrupole moments, have been investigated for the decay $K \to \mu+\nu+\gamma$. The results of a numerical calculation indicate that such a boson will have observable effects if its mass is not too great. Emission of photons at large backward angles with respect to energetic muons is significantly enhanced. Emission of high-energy photons at small forward angles is suppressed. The dependence of these effects on the mass and magnetic moment of the boson is discussed. Photon polarization phenomena are noted.

I. INTRODUCTION

RECENTLY, there has been renewed speculation that weak interactions are mediated by the exchange of a vector boson, called W.[1] Experimental evidence for the existence of two different types of neutrinos[2] has removed a serious obstacle to the intermediate vector boson theory, namely, the absence of the neutrinoless decay $\mu \to e+\gamma$.[3] The same high-energy neutrino experiments offer the hope of directly producing the W boson;[4] electromagnetic production of boson pairs has also been discussed.[5] These experiments are difficult, and it is natural to inquire what consequences the existence of W should have for other processes. Such indirect effects are generally small. In ordinary muon decay Lee and Yang[1] found that W-boson effects were of order $\lambda=(m_\mu/m_W)^2$, the square of the ratio of the muon to the boson mass. The absence of the fast decay $K \to W+\gamma$ implies $m_W \geq m_K$, so that $\lambda \leq 0.046$. In radiative muon decay ($\mu \to e+\nu+\bar{\nu}+\gamma$) Eckstein and Pratt[6] found that W-boson effects were again of order λ compared to the leading terms. The effects of W in $K \to e+\nu+\gamma$ were discussed by Kanazawa, Sugawara, and Tanaka,[7] and more recently Neville[8] has dealt with the radiative modes of kaon leptonic decays, including all possible structure effects in two form factors. However, it remained for Berman, Ghani, and Salmeron[9] to remark that the presence of a W boson could enhance the branching ratio $\Gamma(K \to e+\nu+\gamma)/\Gamma(K \to e+\nu)$ by a factor of a thousand over what it would be if there were only inner bremsstrahlung.[10] Similar, though smaller, effects occur in $\pi \to e+\nu+\gamma$, and a preliminary attempt to look for them appeared encouraging.[11] However, the study of these rare radiative electron modes is extremely difficult, and so it is interesting to ask what can be learned from a study of the muon modes.

It is the purpose of this note to point out some interesting features of the decay $K \to \mu+\nu+\gamma$ which may provide a test for the existence of the W boson, or some boson-like structure. The transition probability for this decay is given in Sec. II under the assumption that the decay is mediated by a W boson and that all other structure effects are approximated by a phenomenological boson-kaon coupling. The relative complexity of the analytic expression had deterred previous investigators and tends to obscure the interesting features. In the present work the differential decay rate, and also integrals corresponding to some possible experimental conditions, were evaluated on the IBM 7090 of the Stanford University Computation Center. The results of the machine calculation are described in Sec. III, and it is concluded that a W boson (of not too large a mass) can produce observable effects in $K \to \mu+\nu+\gamma$. The decay rate is significantly enhanced for the emission of photons at large backward angles with respect to energetic muons, while the emission of high-energy photons at small forward angles is suppressed. The dependence of these effects on mass m_W and magnetic moment μ_W of the W boson is also discussed in Sec. III.

II. CALCULATION OF DECAY RATES

The most general form of the differential decay rate for $K \to \mu+\nu+\gamma$ via a V-A interaction has been given by Neville.[8] The contribution from inner bremsstrahlung can be separated from the structure-dependent terms in a gauge-invariant manner. For a W boson with

* This work was supported in part by the National Science Foundation and by the United States Air Force Office of Scientific Research under Contract AF 49(638)-388.

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[2] G. Danby, J.-M. Gaillard, K. Goulianos, L. M. Lederman, N. Mistry, M. Schwartz, and J. Steinberger, Phys. Rev. Letters **9**, 36 (1962).

[3] D. Bartlett, S. Devons, and A. Sachs, Phys. Rev. Letters **8**, 120 (1962); S. Frankel, J. Halpern, L. Holloway, W. Wales, M. Yerian, D. Chamberlain, A. Lemonick, and F. M. Pipin, *ibid.* **8**, 123 (1962).

[4] L. M. Lederman, Bull. Am. Phys. Soc. **7**, 430 (1962).

[5] S. M. Berman and Y. S. Tsai (private communication).

[6] S. G. Eckstein and R. H. Pratt, Ann. Phys. (N. Y.) **8**, 297 (1959).

[7] A. Kanazawa, M. Sugawara, and K. Tanaka, Phys. Rev. **122**, 341 (1961).

[8] D. E. Neville, Phys. Rev. **124**, 2037 (1961).

[9] S. M. Berman, A. Ghani, and R. A. Salmeron, Nuovo Cimento **25**, 685 (1962).

[10] By inner bremsstrahlung we mean the structure-independent part of the radiation (which persists in the limit that the boson mass m_W becomes infinite) arising from the principle of minimal electromagnetic interaction.

[11] P. Depommier, J. Heintze, A. Mukhin, C. Rubbia, V. Soergel, and K. Winter, in *Proceedings of the 1962 International Conference on High Energy Physics at CERN*, edited by J. Prentki (CERN, Geneva, 1962), p. 414.

arbitrary anomalous magnetic and quadrupole moments, $(e/2m_W)(\mu_W-1)$ and (eQ_W/m_W^2), respectively,[12] we may, using Neville's definition of the form factors, set

$$h_1=[2-\mu_W-p\cdot q(Q_W/m_W^2)]m_K^2[m_W^2-(p-q)^2]^{-1}, \quad (1)$$

$$h_2=0, \quad (2)$$

where p and q are the four momenta of the kaon and photon, respectively. This choice is equivalent (in lowest order perturbation theory) to the following simple model for the decay:

(a) The usual coupling of the W boson to the lepton current:

$$L_W=f[\bar{\psi}_\mu\gamma_\alpha\tfrac{1}{2}(1+i\gamma_5)\psi_\nu]W_\alpha^*+\text{c.c.}; \quad (3)$$

(b) A local phenomenological coupling of the W boson to the kaon:

$$L_{\text{eff}}=f'(\partial_\alpha K)W_\alpha^*+\text{c.c.}; \quad (4)$$

(c) Electromagnetic interactions given by the gauge-invariant substitution $\partial_\alpha\to\partial_\alpha-ieA_\alpha$, plus the contribution of anomalous moments:

$$L_{\text{A.M.}}=\tfrac{1}{2}ie(\mu_W-1)W_\alpha^*W_\beta F_{\alpha\beta}+\tfrac{1}{2}ie(Q_Wm_W^{-2})W_{\alpha\beta}^*W_\gamma(\partial_\gamma F_{\alpha\beta})+\text{c.c.}, \quad (5)$$

where $W_{\alpha\beta}=\partial_\alpha W_\beta-\partial_\beta W_\alpha$ and $F_{\alpha\beta}=\partial_\alpha A_\beta-\partial_\beta A_\alpha$.

The differential decay rate can be obtained by inserting Eq. (1) and (2) into Eq. (2.7) through (2.11) of Neville. It is more convenient, however, for the performance of the numerical integrations to rewrite the transition probability in terms of the *muon* energy and angle.[13] In the c.m. system (the rest system of the initial kaon) the differential decay rate is

$$\frac{d^2\Gamma}{d\epsilon dx}=\frac{\alpha}{4\pi^2}\left(\frac{ff'}{m_W^2}\right)^2\frac{m_K^3}{4}\times\left(\frac{\epsilon\beta}{2-\epsilon+\epsilon\beta x}\right)F\left(\epsilon,\frac{2(1+r-\epsilon)}{2-\epsilon+\epsilon\beta x},x\right), \quad (6)$$

where

$$F(\epsilon,\omega,x)=A-zB+z^2C, \quad (7)$$

$$A=r\left[\frac{\beta^2(1-x^2)(1-r)}{\omega(1-\beta x)^2}+\frac{(1+r-\epsilon)}{\epsilon(1-\beta x)}\right], \quad (8)$$

$$B=\frac{1}{2}r\left[\frac{\beta^2\epsilon(1-x^2)}{(1-\beta x)}+\frac{2(1+r-\epsilon)}{\epsilon(1-\beta x)}\right], \quad (9)$$

$$C=\tfrac{1}{8}\omega[1-r-\omega-\tfrac{1}{2}\epsilon^2\beta^2(1-x^2)], \quad (10)$$

$$z=\left(2-\mu_W-\frac{\omega Q_W}{2K}\right)\left(\frac{\omega}{K-1+\omega}\right). \quad (11)$$

In these expressions $\frac{1}{2}m_K\epsilon$ is the total muon energy, $\frac{1}{2}m_K\omega$ is the photon energy, β is the speed of the muon, $r=(m_\mu/m_K)^2$, and $K=(m_W/m_K)^2$. The cosine of the angle between the muon and photon directions is denoted by x. For the relations between kinematic variables and the restrictions on their ranges, imposed by conservation of energy and momentum, the reader is referred to Neville.

It should be noted that the entire dependence on the W-boson mass is contained in the factor $\omega(K-1+\omega)^{-1}$ in Eq. (11). In the limit that $m_W\to\infty$, then $z\to 0$, hence $F\to A$. Equation (8) for A, therefore, represents the contribution of the ordinary inner bremsstrahlung. On the other hand, in the limit of vanishing lepton mass, i.e., $r\to 0$, C clearly dominates (except for $x\simeq 1$). Equation (10) for C, then, is the relevant term in the decay $K\to e+\nu+\gamma$. It is interesting to note that the anomalous moments appear only in the combination $(2-\mu_W-\omega Q_W/2K)$ of z and that Eq. (7) for F is simply a polynomial of second degree in z. It is curious that if $\mu_W=2$ and $Q_W=0$, then $F=A$ and the W boson has no effect whatever on the decay.

The differential decay rate for muons polarized perpendicular to the reaction plane can be obtained by substitution of Eq. (1) and (2) into Eq. (4.1) and (4.2) of Neville.[8] These effects will not be discussed further in the present note.

The differential decay rate for polarized photons has also been calculated. If $\hat{e}_x$, $\hat{e}_y$, and $\mathbf{q}$ form a right-handed orthogonal triad with $\hat{e}_x$ parallel to $\mathbf{p}\times\mathbf{q}$ ($\mathbf{p}$ and $\mathbf{q}$ are the momenta of the muon and photon in the c.m. system), then the polarization of the photon may be specified by the Stokes parameters[14]

$$\xi_i=(e^*,\sigma_ie),\quad i=1,2,3, \quad (12)$$

$$\xi_0=(e^*,e)^{1/2}=(\xi_1^2+\xi_2^2+\xi_3^2)^{1/2}\equiv 1, \quad (13)$$

where e is a two-component "spinor" with components along $\hat{e}_x$ and $\hat{e}_y$, and σ_i are the Pauli matrices. The differential decay rate for photons with polarization $\boldsymbol{\xi}$ is

$$\frac{d^2\Gamma(\boldsymbol{\xi})}{d\epsilon dx}=\frac{d^2\Gamma}{d\epsilon dx}\times\tfrac{1}{2}(1+P). \quad (6a)$$

The degree of polarization $P\equiv F'/F$, where F is given by Eq. (7) and F' is defined as follows:

$$F'(\epsilon,\omega,x)=A'-B'z+C'z^2, \quad (7a)$$

[12] In the ξ-limiting formalism of T. D. Lee and C. N. Yang [Phys. Rev. **128**, 885 (1962)], the interaction Lagrangian contains only a single time derivative of the electromagnetic field. Within this scheme one must set the intrinsic anomalous quadrupole moment equal to zero. Although the machine program for the present calculation allows for an arbitrary anomalous quadrupole moment, Q_W has been taken equal to zero in this paper. From Eq. (11) it can be seen that varying Q_W will have much the same effect as varying μ_W (and/or m_W).

[13] In terms of the photon energy and angle, the muon energy is given by the roots of a quadratic equation, both of which are possible for high photon energies. See the Appendix of reference 8.

[14] J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1955), p. 42.

$$A'=r\left[\left(\frac{\beta^2(1-x^2)}{(1-\beta x)^2}+\frac{(1+r-\epsilon)}{\epsilon(1-\beta x)}\right)\xi_2-\frac{\beta^2(1-x^2)(1-r)}{\omega(1-\beta x)^2}\xi_3\right],\quad (8a)$$

$$B'=\frac{1}{2}r\left[\frac{2(1+r-\epsilon)}{\epsilon(1-\beta x)}\xi_2-\frac{\epsilon\beta^2(1-x^2)}{(1-\beta x)}\xi_3\right],\quad (9a)$$

$$C'=\tfrac{1}{16}\omega\{[(2-\omega)\epsilon\beta x+\omega\epsilon]\xi_2+\epsilon^2\beta^2(1-x^2)\xi_3\}.\quad (10a)$$

The quantity z is given by Eq. (11). The absence of ξ_1 in Eqs. (8a), (9a), and (10a) follows from the time-reversal invariance of the decay via a W boson.

It can be seen from Eqs. (6a) through (10a) that the sign of the photon polarization which the W-boson terms favor can be opposite to that which would come from inner bremsstrahlung alone. Hence, if there is a W boson, the sign of the predicted photon polarization will be reversed in regions where the boson terms dominate. Presumably, there will also be an effect on the orientation of Dalitz pairs from internal conversion of the photon.[15]

III. RESULTS

Equations (6)–(10) for the differential decay rate were evaluated by a machine calculation. The program also called for numerical integrations over an arbitrary region of energies and angles. Because the general kinematic treatment is complicated it is most convenient to integrate first over angles, x, and then over muon energy, ϵ, subject to a subsidiary condition on the photon energy, ω.

Although the total decay rate is only slightly affected by the presence of the boson (in contrast to the mode $K\to e+\nu+\gamma$), there are regions of energies and angles in which the effect of the W boson is quite marked.

(i) The emission of photons at large, backward angles with respect to the muon direction is significantly enhanced for energetic muons. The enhancement is greatest for an angle somewhat less than 180°. (The tendency is to evaluate the differential decay rate for $x=-1$, because the expressions simplify considerably, but then the maximum boson effects are missed. At exactly $x=-1$ the W boson actually causes the opposite effect, i.e., a suppression.[16])

(ii) Over most other regions of energies and angles the W boson tends to suppress the transition probability. The region where the suppression is most appreciable is at small forward angles and high photon energies. The magnitude of the decay rate in this region is comparable to that in region (i), since it is the emission of *low*-energy photons which is highly favored by the infrared divergent term in Eq. (8). The suppression of the rate in region (ii) is not as marked as the enhancement in region (i) and is probably more difficult to observe.

Representative of the effects of the W boson on the differential decay rate are the curves in Fig. 1, drawn for various muon energies. The ratio $R(\epsilon,x)$ of the differential transition probabilities with and without the W boson is plotted as a function of angle x. The case $m_W=1.1m_K$ ($\simeq$544 MeV), $\mu_W=Q_W=0$ was considered "typical." The effects described under (i) and (ii) are quite noticeable.

Tables I–III give the results of the numerical integration of Eq. (6) over a kinematic region D defined by $\{E_\mu\geq E_\mu^{\text{exp}},\ E_\gamma\geq E_\gamma^{\text{exp}},\ \theta_1^{\text{exp}}\geq\theta\geq\theta_2^{\text{exp}}\}$, where E_μ is the

TABLE I. Ratio $R=\Gamma_D(m_W)/\Gamma_D(\infty)$ of the decay rate with a W boson to the decay rate for inner bremsstrahlung ($m_W=\infty$) in a kinematic region defined by E_μ^{exp}, $E_\gamma^{\text{exp}}=23.5$ MeV, $\theta_1^{\text{exp}}=180°$, $\theta_2^{\text{exp}}=154°$. m_W in units of m_K; μ_W in W-boson magnetons.

		E_μ^{exp} in MeV			
m_W	μ_W	137	122	107	92
1.0	−1	60.0	50.2	41.8	35.6
1.1	−1	28.7	25.2	21.3	18.2
1.2	−1	17.0	15.2	12.9	11.1
1.3	−1	11.1	10.0	8.60	7.41
1.5	−1	5.71	5.25	4.53	3.94
1.9	−1	2.46	2.29	2.02	1.80
1.0	0	26.9	22.5	18.7	15.9
1.1	0	13.1	11.5	9.67	8.29
1.2	0	7.92	7.08	6.03	5.20
1.3	0	5.34	4.83	4.16	3.61
1.5	0	3.00	2.76	2.41	2.13
1.9	0	1.60	1.49	1.36	1.25
1.0	+1	7.24	6.07	5.08	4.37
1.1	+1	3.86	3.39	2.91	2.55
1.2	+1	2.64	2.35	2.06	1.83
1.3	+1	1.99	1.85	1.65	1.49
1.5	+1	1.43	1.34	1.23	1.15
1.9	+1	1.11	1.06	1.02	0.984
$B=\Gamma_D(\infty)/\Gamma_0$		1.01×10^{-5}	1.47×10^{-5}	1.87×10^{-5}	2.25×10^{-5}

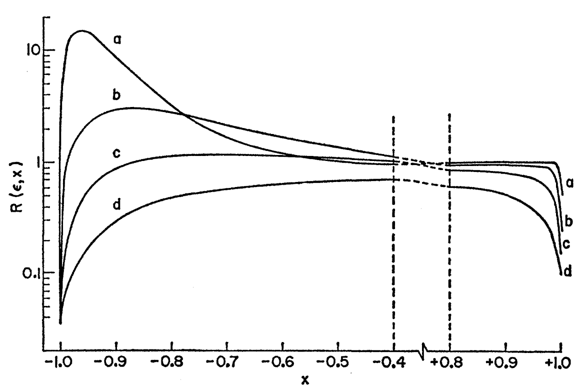

FIG. 1. The ratio, $R(\epsilon,x)$, of the differential decay rate for $m_W=1.1m_K$, $\mu_W=Q_W=0$, and the differential decay rate for $m_W=\infty$, as given by Eqs. (6)–(10), is plotted against x, the cosine of the angle between the muon and photon, for (a) $E_\mu=137$ MeV, (b) $E_\mu=107$ MeV, (c) $E_\mu=76$ MeV, (d) $E_\mu=46$ MeV.

[15] The authors are indebted to Dr. S. M. Berman for calling to their attention the possible importance of these polarization effects.

[16] See Fig. 1 for $x=-1$. The suppression at $x=-1$ is due to the fact that there is cancellation between A and B, while C is small since large ϵ means $\omega\simeq1-r$.

TABLE II. Ratio $R=\Gamma_D(m_W)/\Gamma_D(\infty)$ of the decay rate with a W boson to the decay rate for inner bremsstrahlung ($m_W=\infty$) in a kinematic region defined by E_γ^{exp}, $E_\mu^{\text{exp}}=137$ MeV, $\theta_1^{\text{exp}}=180°$, $\theta_2^{\text{exp}}=143°$. m_W in units of m_K, μ_W in boson magnetons.

		E_γ^{exp} in MeV				
m_W	μ_W	24	47	71	94	118
1.0	−1	30.4	41.4	52.0	66.2	88.0
1.1	−1	13.9	19.5	26.0	35.0	49.3
1.2	−1	8.15	11.4	15.3	21.1	30.6
1.3	−1	5.42	7.46	10.0	13.8	20.3
1.5	−1	3.01	3.95	5.17	7.07	10.4
1.9	−1	1.60	1.88	2.27	2.89	4.02
1.0	0	13.8	18.7	23.3	29.5	39.2
1.1	0	6.57	9.04	11.9	15.8	22.1
1.2	0	4.07	5.48	7.19	9.70	13.8
1.3	0	2.88	3.76	4.86	6.52	9.34
1.5	0	1.84	2.23	2.75	3.57	5.00
1.9	0	1.23	1.35	1.51	1.77	2.24
1.0	+1	4.04	5.19	6.56	7.82	10.2
1.1	+1	2.28	2.86	3.53	4.47	5.99
1.2	+1	1.69	2.01	2.41	3.00	3.99
1.3	+1	1.41	1.60	1.85	2.24	2.91
1.5	+1	1.17	1.25	1.36	1.55	1.87
1.9	+1	1.04	1.05	1.08	1.07	1.24
$B=\Gamma_D(\infty)/\Gamma_0$		2.85×10^{-5}	1.93×10^{-5}	1.33×10^{-5}	8.25×10^{-6}	4.37×10^{-6}

muon kinetic energy, E_γ is the photon energy, and θ is the angle between the muon and photon (all in the c.m. system). The entries in the tables are again ratios of the rate including W-boson effects to the rate for inner bremsstrahlung alone, i.e., $R=\Gamma_D(m_W)/\Gamma_D(\infty)$. As a basis for reference the branching ratio $B=\Gamma_D(\infty)/\Gamma_0$ is given in the last line of each table, where Γ_0 is the total rate for the nonradiative decay $K\to\mu+\nu$. For example, to find the branching ratio for a particular choice of D, m_W, and μ_W, multiply the corresponding entry for R by B.

Table I shows that the W-boson effects gradually decrease as a result of relaxing the requirement of high muon energy. This, however, is accompanied by an increase in the transition probability. It should be emphasized that the requirement of high muon energy is just what is needed to distinguish the radiative mode

$$K\to\mu+\nu+\gamma \tag{14}$$

from the major background mode

$$K\to\mu+\nu+\pi^0 \quad\quad \llcorner\!\!\to\gamma+\gamma \tag{15}$$

In the mode (15) the maximum c.m. muon kinetic energy is only 135 MeV, while in mode (14) it is 152 MeV. By requiring $E_\mu^{\text{exp}}>135$ MeV the background may be greatly reduced.

In Table II, therefore, E_μ^{exp} has been set equal to 137 MeV (90% of $E_{\mu\max}$) and E_γ^{exp} has been varied. Simultaneously, the angular region has been doubled. As E_γ^{exp} is raised the ratio R increases; however, the decay rate falls. Enlarging the angular region has just the opposite effect, as can be seen by comparing the first columns of Tables I and II. W-boson effects are beginning to be washed out (although they are still significant) while the rate is nearly tripled. The final choice of the region D will have to be made by the experimenter.

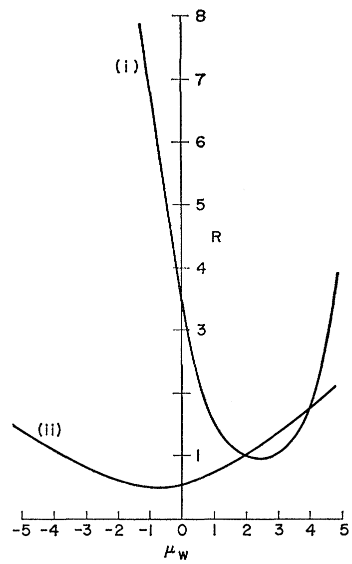

FIG. 2. Qualitative dependence of $R=\Gamma_D(m_W)/\Gamma_D(\infty)$ on the W-boson magnetic moment μ_W (in boson magnetons), for ϵ and x in the kinematic regions (i) and (ii) discussed in the text.

Table III provides an example of the suppression of the transition probability in region (ii). A particular choice of D has been made, but the variation of the ratio R as a function of the W-boson mass and magnetic moment is explicit. It can be seen that R is generally much closer to unity in region (ii) than in region (i), although the branching ratio B is greater in region (ii). In addition, muon energies will be low so that the background from (15) will be greatly increased in region (ii).

The dependence of the decay rate on the mass m_W and magnetic moment μ_W should be noted. As stated earlier, the W-boson effects approach zero as $m_W\to\infty$. Just how fast they approach zero can be seen from the previous tables. For m_W bigger than the nucleon mass ($m_W>1.9m_K$) it will probably be difficult to observe the W-boson effects.

Regarding the magnetic moment μ_W, if we set the anomalous quadrupole moment equal to zero (as has been done in all our discussion), then for fixed m_W, the ratio $R=\Gamma_D(m_W)/\Gamma_D(\infty)$ behaves as

$$R=1-(2-\mu_W)a+(2-\mu_W)^2b. \tag{16}$$

This represents a parabola, opening upwards, with a minimum at $(2-\mu_W)=\frac{1}{2}ab^{-1}$. The orientation and shape

TABLE III. Ratio $R=\Gamma_D(m_W)/\Gamma_D(\infty)$ of the decay rate with a W boson to the decay rate for inner bremsstrahlung ($m_W=\infty$), in a kinetic region defined by $E_\mu^{\text{exp}}=23$ MeV, $E_\gamma^{\text{exp}}=94$ MeV, $\theta_1^{\text{exp}}=37°$, $\theta_2^{\text{exp}}=0°$. m_W in units of m_K, μ_W in W-boson magnetons.

m_W	μ_W	R	m_W	μ_W	R	m_W	μ_W	R
1.0	−1	0.766	1.0	0	0.518	1.0	+1	0.597
1.1	−1	0.526	1.1	0	0.527	1.1	+1	0.686
1.2	−1	0.513	1.2	0	0.587	1.2	+1	0.750
1.3	−1	0.550	1.3	0	0.646	1.3	+1	0.796
1.5	−1	0.642	1.5	0	0.737	1.5	+1	0.856
1.9	−1	0.772	1.9	0	0.840	1.9	+1	0.916

$B=\Gamma_D(\infty)/\Gamma_0=0.810\times10^{-4}$

of the parabola depend on the values of a and b, which in turn depend on m_W, ϵ, and x. A qualitative idea of the dependence of R on μ_W can be obtained from Fig. 2. The first curve represents a choice of ϵ and x in region (i) and the second in region (ii).

A similar argument can explain the variation of R with m_W shown in Table III. For fixed μ_W, ϵ, and x, the ratio of the differential decay rates, $R(\epsilon,x)=d^2\Gamma(m_W)/d^2\Gamma(\infty)$, is quadratic in the variable $z'=\omega(K-1+\omega)^{-1}$:

$$R(\epsilon,x)=1-(2-\mu_W)a'z'+(2-\mu_W)^2b'z'^2. \qquad (17)$$

$R(\epsilon,x)$ has a minimum at $z'_{\min}=a'[2(2-\mu_W)b']^{-1}$. Depending on the value of μ_W, $z'_{\min}$ can lie inside or outside the allowed range for z', i.e., $0\leq z'\leq 1$. As μ_W becomes increasingly negative, $z'_{\min}$ moves into the interval $[0,1]$ from the right, so that when $\mu_W=-1$ the ratio R has a minimum near $m_W=1.2m_K$.

ACKNOWLEDGMENTS

The authors are indebted to Dr. S. M. Berman for many useful comments and suggestions, and to Dr. G. H. Trilling for a helpful conversation.

PHYSICAL REVIEW VOLUME 130, NUMBER 5 1 JUNE 1963

Threshold Motion of Regge Poles*

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The motion of Regge poles as $E\to 0$ is examined in detail. It follows from the well-known threshold law of the S matrix that infinitely many poles approach $l=-\frac{1}{2}$, from the first and the third quadrants as $E\to 0+$, and from the second and third as $E\to 0-$. A possible way of including these poles in a representation of S is indicated.

It is well known that for potentials that fall off sufficiently rapidly at infinity the S matrix for an angular momentum l has the behavior[1]

$$S=1+O(k^{2l+1}) \quad \text{as} \quad k\to 0.$$

If the S matrix is expressed in terms of a single Regge pole,[2] or a finite sum of Regge poles the threshold dependence is clearly not satisfied.[3] We want to point out an intimate connection between this threshold behavior and the infinitely many Regge trajectories that arrive at $l=-\frac{1}{2}$ as $E\to 0$. As $E\to 0+$, there are infinitely many poles which approach $l=-\frac{1}{2}$ from the upper right-half and the lower left-half of the l plane, the approach being essentially independent of the potential. As $E\to 0-$, the poles approach $l=-\frac{1}{2}$ in complex conjugate pairs from the left-half l plane. The present authors had indicated recently that $l=-\frac{1}{2}$ is the low-energy end point of infinitely many Regge trajectories.[4,5] In this article, we give further details about the threshold motion of the poles. We shall also indicate the possible way in which an S matrix can be expressed so as to correctly take into account its threshold properties.[6] We consider only nonrelativistic potential scattering but we believe these results, coming as they do from the threshold dependence, should also hold in the relativistic case. We also find another class of infinitely many Regge poles in the left-half l plane whose energy dependence we have derived. The behavior of these poles is found to be analogous to the right-hand Regge poles associated with bound states and resonances.

An S matrix unitary for real λ $(=l+\frac{1}{2})$ can be written near $E=0$ as[4,7]

$$S(\lambda,k)=\frac{1-k^{2\lambda}e^{i\pi\lambda}C(\lambda)}{1-k^{2\lambda}e^{-i\pi\lambda}C(\lambda)}, \qquad (1)$$

where $C(\lambda)$ is a meromorphic function of $\lambda^{2,7}$ and is real for real λ.[8,9]

* Supported in part by the National Science Foundation.

† National Science Foundation Senior Postdoctoral Fellow, on leave of absence from Indiana University, Bloomington, Indiana.

[1] Here k is the momentum, $E=k^2$ is the energy.

[2] T. Regge, Nuovo Cimento **14**, 951 (1959).

[3] For instance, the threshold behavior of a single right-hand Regge pole is $S-1=O(k^{2\alpha(0)+1})$, where $\alpha(0)$ is the $k=0$ position of the pole and $-\frac{1}{2}<\alpha(0)<\frac{1}{2}$.

[4] B. R. Desai and R. G. Newton, Phys. Rev. **129**, 1445 (1963).

[5] V. N. Gribov and I. Ya. Pomeranchuk, Phys. Rev. Letters **9**, 238 (1962), had indicated that there are conjugate poles approaching $l=-\frac{1}{2}$ as $E\to 0-$ in the relativistic case.

[6] Note that a Regge-pole representation of $(S-1)/k^{2l+1}$ satisfies threshold dependence of S but clearly does not get rid of the infinite number of poles at $l=-\frac{1}{2}$.

[7] R. G. Newton, J. Math. Phys. **3**, 867 (1962).

[8] This follows from the relation given in reference 7,

$$S(\lambda,k)e^{-2\pi i\lambda}+S^{-1}(\lambda,ke^{-i\pi})=1+e^{-2\pi i\lambda}.$$

[9] We have only written the first two dominant terms in k^2 for $\lambda<1$. In general, one has, both in the numerator and the denominator, terms of the form $a_1(\lambda)k^2+a_2(\lambda)k^4+\cdots+b_1(\lambda)k^{2\lambda+2}+\cdots$. For $\lambda>1$, the $k^{2\lambda}$ term should be replaced by the k^2 term.