give

$$M'(\alpha)\sim -Kg^2G^2(1+\alpha)^{-1}\int_0^1\cdots\int_0^1 dx'dudydz$$
$$\times x'^{\alpha}(y+z)^{-2}\left[\left(x'+u+\frac{yz}{y+z}\right)^{-1}-(x'+u)^{-1}\right]. \quad \text{(C4)}$$

The integration over u gives simply

$$M'(\alpha)\sim Kg^2G^2(1+\alpha)^{-1}\int_0^1\cdots\int_0^1 \frac{dx'dydz\, x'^{\alpha}}{(y+z)^2}$$
$$\times\ln\left[1+\frac{yz}{x'(y+z)}\right]. \quad \text{(C5)}$$

Next the integration over x' yields

$$M'(\alpha)\sim Kg^2G^2(1+\alpha)^{-3}$$
$$\times\int_0^1\cdots\int_0^1 dydz\, y^{1+\alpha}z^{1+\alpha}(y+z)^{-3-\alpha}. \quad \text{(C6)}$$

The right-hand side of (C6) is easily evaluated to give finally

$$M'(\alpha)\sim Kg^2G^2(1+\alpha)^{-4}, \quad \text{(C7)}$$

for α near -1. Hence, as $s\to\infty$,

$$\mathfrak{M}'\sim\tfrac{1}{6}Kg^2G^2s^{-1}(\ln s)^3, \quad \text{(C8)}$$

which is negligible when compared with the corresponding quantity for the graph of Fig. 6(b).

PHYSICAL REVIEW VOLUME 130, NUMBER 6 15 JUNE 1963

Higher Random-Phase Approximations and the Theory of the Electron Gas*

GIU DO DANG AND ABRAHAM KLEIN†
University of Pennsylvania, Philadelphia, Pennsylvania
(Received 12 February 1963)

Using the example of the degenerate electron gas, it is shown how the operator equations of motion for a many-particle system may be exploited to generate systematically a sequence of nonperturbative approximations of which some version of the random-phase approximation is the first. Some questions left unsettled by previous attempts in this direction are resolved by close attention to the structure of the spectrum of the system. The solution of the equations is carried only so far as to make contact with previously substantiated results. Finally, a rigorous proof is given that the plasmon frequency approaches the classical plasmon frequency in the long-wavelength limit.

I. INTRODUCTION

THERE have been several attempts recently to go beyond the extreme high-density limit in the treatment of the problem of the degenerate electron gas in a uniform background of positive charge.[1] Using the diagram techniques developed for the electron gas by Hubbard[2] and DuBois,[3] Osaka[4] has shown how to sum an infinite but well-defined class of higher order diagrams not previously included in the calculation of the polarization propagator[2,3] or effective interaction and has applied the results to obtain corrected values of the screening constant, plasmon dispersion relation, and low-temperature specific heat. We shall not be concerned here with this kind of technique, though as we shall show in a sequel to this paper, Osaka's theory coincides with an accurate solution of a suitably defined extended random-phase approximation (RPA).

It is, in fact, well known that the results based on the lowest order polarization propagator are completely equivalent to those which can be obtained from the equation of motion method in the RPA without exchange. It is also well known that the next approximation in the diagram method is equivalent to the *partial inclusion* of the exchange interaction and exchange self-energy effects in the equations of motion i.e., in the use of an extended RPA method as the basis for the theory.[5] In consequence, several authors have felt that the equation-of-motion method warranted deeper investigation as the foundation for a nonperturbative approach to the many-body problem. Here we mention first the work of Suhl and Werthamer,[6] whose technique has been applied to nuclear physics by Sawicki.[7] Their method is based on the extended

* Supported in part by the U. S. Atomic Energy Commission.
† Alfred P. Sloan Foundation Fellow.
[1] For a general discussion, earlier references, and a summary of most of the known results, see D. Pines, *The Many Body Problem* (W. A. Benjamin, Inc., New York, 1961).
[2] J. Hubbard, Proc. Roy. Soc. (London) **A243**, 336 (1957).
[3] D. F. DuBois, Ann. Phys. (N. Y.) **7**, 174 (1959); **8**, 24 (1959).
[4] Y. Osaka, J. Phys. Soc. Japan **17**, 547 (1962).
[5] We shall, thus, distinguish between the simple and extended RPA, respectively, where the latter includes exchange and exchange self-energy corrections.
[6] H. Suhl and N. R. Werthamer, Phys. Rev. **122**, 359 (1961); N. R. Werthamer and H. Suhl, *ibid.* **125**, 1402 (1962). Reference should also be made to the work of K. Sawada, *ibid.* **119**, 2090 (1960).
[7] J. Sawicki, Phys. Rev. **126**, 2231 (1962); G. Fano and J. Sawicki, Nuovo Cimento **25**, 586 (1962).

RPA as first approximation. On the other hand, von Roos[8] has developed and applied a technique based on the simple RPA as the leading approximation, as appears particularly useful for the electron gas.

It is the view of the authors, however, that the physical and mathematical basis of such methods requires further clarification which we shall attempt to supply in this note. Though the reasoning will be carried out in detail for the electron gas problem, it will be evident that it applies with suitable modifications to any many-body system. It will be equally manifest, however, that the way in which the required modifications are to be made cannot be given once for all, but must be determined by the physics of each problem. It is for this reason that we work from the beginning with matrix elements of operators between different states of the system and that approximate evaluations of these matrix elements are always referred to supposed properties of the states.

What follows contains many points of contact with the work of Suhl and Werthamer[6] and of von Roos[8] as well as their predecessors. We shall confine our attention to a single problem, the description of the plasmon, though the initiated[9] will recognize that suitably supplemented, one finds here a complete method for treating the properties of the electron gas. The basic idea is the often used one that the states of the interacting electron gas can be put into a certain correspondence with those of the uncoupled system in a sense which we now describe.

Let $a(\mathbf{k},s)$, $a^\dagger(\mathbf{k},s)$ be destruction, creation operators for particles of momentum $\mathbf{k}$, spin s, with $A(\mathbf{k},\mathbf{q})=a^\dagger(\mathbf{k}+\mathbf{q},s)a(\mathbf{k},s)$. We establish a hierarchy of states of the N-particle system. We focus most of our attention on the so-called one pair states $|1,\mathbf{q}\rangle$ with the supposed property that the matrix element $\langle 1,\mathbf{q}|A(\mathbf{k},\mathbf{q})|0\rangle$ far exceeds in value[10] any matrix element obtained *either* by replacing $\langle 1,\mathbf{q}|$ by any other state of momentum $\mathbf{q}$ or by replacing the ground state $|0\rangle$ by any other state of momentum zero. The state $|1,\mathbf{q}\rangle$ is, thus, approximately a superposition of one pair configurations with respect to the ground state and is the more so the weaker the coupling. Our main object, though, will be to learn to generate systematically the corrections to this description.

The next most complicated type of state is the two pair state $|2,\mathbf{q}\rangle$ where it is essential to distinguish two species, namely, those of the form[11]

$$|2,\mathbf{q}\rangle\cong|1,\mathbf{q}_1\rangle\times|1,\mathbf{q}-\mathbf{q}_1\rangle \tag{1}$$

for some $\mathbf{q}_1$, which may be termed two pair incoherent states and which have the property[11]

$$\langle 2,\mathbf{q}|A(\mathbf{k},\mathbf{q}-\mathbf{q}_1)|1,\mathbf{q}_1\rangle\cong\langle 1,\mathbf{q}-\mathbf{q}_1|A(\mathbf{k}_1,\mathbf{q}-\mathbf{q}_1)|0\rangle \tag{2}$$

and those coherent states in which the decomposition (1) is impossible with the consequence[10]

$$\langle 2,\mathbf{q}|A(\mathbf{k},\mathbf{q}')|1,\mathbf{q}-\mathbf{q}'\rangle\ll\langle 1,\mathbf{q}'|A(\mathbf{k},\mathbf{q}')|0\rangle. \tag{3}$$

It is a consequence of the inequality (3) and similar inequalities that in studying the equation of motion for the matrix element $\langle 1,\mathbf{q}|A(\mathbf{k},\mathbf{q})|0\rangle$ by the means to be developed in the succeeding sections, we need never consider any matrix elements involving two or more pair *coherent* states. This is fortunate since the properties and/or existence of such states has received scant attention. The methods of this paper could, however, be applied (separately) to this problem.

By continuation of the above arguments, we can classify all the states of the system and decide which are the relevant matrix elements of the operators $A(\mathbf{k},\mathbf{q})$ between these states. These arguments will enter because the equation of motion for $\langle 1,\mathbf{q}|A(\mathbf{k},\mathbf{q}|0\rangle$ will bring in such products as

$$\langle 1,\mathbf{q}|A(\mathbf{k}_1,\mathbf{q}_1)A(\mathbf{k}_2,\mathbf{q}-\mathbf{q}_1)|0\rangle$$
$$=\sum_\alpha\langle 1,\mathbf{q}|A(\mathbf{k}_1,\mathbf{q}_1)|\alpha\rangle\langle\alpha|A(\mathbf{k}_2,\mathbf{q}-\mathbf{q}_1)|0\rangle. \tag{4}$$

A given stage of approximation consists in choosing a suitable subset of states $|\alpha\rangle$ and of obtaining subsequently equations of motion for the new amplitudes thus included.[12] A closed set of equations is finally generated.

Further details are best relegated to the text since as we show successively in Secs. II and III, alternate modes of development are still possible. Thus, in Sec. II we develop a method, allied to that of Suhl and Werthamer,[6] though distinct from it in detail, in which the first stage of approximation is the extended RPA. In practice, this method is most useful for the case of a short-range interaction where exchange interactions are of equal importance with direct ones. In Sec. III, an alternative method is developed, allied to that of von Roos[8] in which the initial stage of approximation is the simple RPA.

Aside from achieving the realization that we have in each instance a systematic approach to the problem at hand, we only carry the calculations far enough to obtain the first exchange corrections[13] to the plasmon

---

[8] O. von Roos, Phys. Rev. **128**, 911 (1962). The method of this paper is closely related to that of Sec. III of this paper. The results of von Roos must, however, be considered questionable since they contradict the theorem of Sec. IV.

[9] Thus, the function whose vanishing determines the plasmon dispersion relation is effectively the dielectric function which also determines the ground-state properties of the system, as shown in reference 1. The determination of single-particle properties requires additional considerations which are mentioned in the text, but not carried out in detail.

[10] By a factor of order Ω at least, where Ω is the volume of the system.

[11] The approximate equality becomes exact asymptotically (as $\Omega\to\infty$).

[12] This simple idea has also been used by von Roos, reference 8.

[13] For the most recent summary of this question with references, see O. von Roos and J. S. Smuidzinas, Phys. Rev. **121**, 941 (1961).

dispersion relations. A more ambitious attempt to solve these equations is under way.

It is generally believed that the classical plasma frequency constitutes the exact long-wavelength limit for the eigenfrequencies of density fluctuations of the system. To our knowledge, however, a rigorous proof of this fact is lacking. A simple proof is provided in Sec. IV, and it is seen to hold also for the charged boson gas[14] as well as for systems with *either* statistics which have additional short-range interactions of either sign.

II. THEORY BASED ON THE EXTENDED RPA AS FIRST APPROXIMATION

We work with the usual Hamiltonian

$$H=\sum_{\mathbf{k}s}\epsilon(\mathbf{k})a^\dagger(\mathbf{k}s)a(\mathbf{k}s)+\tfrac{1}{2}\sum_{\mathbf{q}}V(q)\sum_{\mathbf{k}_1s_1\mathbf{k}_2s_2}a^\dagger(\mathbf{k}_1+\mathbf{q},s_1)\times a^\dagger(\mathbf{k}_2-\mathbf{q},s_2)a(\mathbf{k}_2s_2)a(\mathbf{k}_1s_1), \quad (5)$$

$$V(q)=\frac{4\pi e^2}{q^2\Omega}, \quad q\neq 0, \quad V(0)=0, \quad (6)$$

$$\epsilon(\mathbf{k})=\hbar^2k^2/2m. \quad (7)$$

We suppress spin indices, understanding that with the momentum $\mathbf{k}_i$ or $\mathbf{k}_i+\mathbf{Q}$, where $\mathbf{Q}$ is arbitrary, there always occurs the spin variable s_i and that summation over $\mathbf{k}_i$ includes also the spin summation. We, furthermore, introduce the notation

$$\epsilon(a,b,c\cdots;x,y,z\cdots)=\epsilon(a)+\epsilon(b)+\epsilon(c)+\cdots-\epsilon(x)-\epsilon(y)-\epsilon(z)-\cdots. \quad (8)$$

Finally, when convenient we employ the variables

$$A(\mathbf{k},\mathbf{q})=a^\dagger(\mathbf{k}+\mathbf{q})a(\mathbf{k}),\qquad \rho(\mathbf{q})=\sum_{\mathbf{k}}A(\mathbf{k},\mathbf{q}), \quad (9)$$

which create a state of momentum $\mathbf{q}$ or destroy a state of momentum $-\mathbf{q}$.

By commuting $A(\mathbf{k},\mathbf{q})$ with H, we obtain the following equation of motion for the energy difference $\omega(q)$,

$$[\omega(\mathbf{q})-\epsilon(\mathbf{k}+\mathbf{q};\mathbf{k})]\langle 1,\mathbf{q}|A(\mathbf{k},\mathbf{q})|0\rangle=\sum_{\mathbf{q}_1}V(\mathbf{q}_1)\sum_{\mathbf{k}_1}\langle 1,\mathbf{q}|\{a^\dagger(\mathbf{k}+\mathbf{q}-\mathbf{q}_1)A(\mathbf{k}_1,\mathbf{q}_1)\times a(\mathbf{k})-a^\dagger(\mathbf{k}+\mathbf{q})A(\mathbf{k}_1\mathbf{q}_1)a(\mathbf{k}+\mathbf{q}_1)\}|0\rangle. \quad (10)$$

In evaluating the right-hand side of (10), leaving out single contributions to any sum over momenta is, in general, permissible giving an error which vanishes as $\Omega\to\infty$ unless the term omitted happens to be larger than the average by a factor of order Ω. Such terms exist and are easy to find. They are for the first and second terms, respectively, on the right-hand side of (10)

(a) $\mathbf{q}_1=\mathbf{q}$ and $\mathbf{q}_1=\mathbf{q}$;

(b) $\mathbf{k}_1=\mathbf{k}-\mathbf{q}_1$ and $\mathbf{k}_1=\mathbf{k}+\mathbf{q}$;

(c) $\mathbf{k}_1=\mathbf{k}+\mathbf{q}-\mathbf{q}_1$ and $\mathbf{k}_1=\mathbf{k}$.

Terms of type (a) contribute, for example, after a permissible commutation,

$$V(\mathbf{q})\langle 1\mathbf{q}|A(\mathbf{k},0)\rho(\mathbf{q})-A(\mathbf{k}+\mathbf{q},0)\rho(\mathbf{q})|0\rangle\cong V(\mathbf{q})[n(\mathbf{k})-n(\mathbf{k}+\mathbf{q})]\langle 1\mathbf{q}|\rho(\mathbf{q})|0\rangle, \quad (11)$$

where

$$n(\mathbf{k})\cong\langle 1,\mathbf{q}|A(\mathbf{k},0)|1,\mathbf{q}\rangle=\langle 1,\mathbf{q}|a^\dagger(\mathbf{k})a(\mathbf{k})|1,\mathbf{q}\rangle\cong\langle 0|a^\dagger(\mathbf{k})a(\mathbf{k})|0\rangle. \quad (12)$$

Though we have here indicated approximate equality in (11) and (12), it is essential to realize that (11) and (12) should be *asymptotically exact*, the difference vanishing as some inverse power of Ω. This follows, for example, by evaluating

$$\langle 1,\mathbf{q}|A(\mathbf{k},0)\rho(\mathbf{q})|0\rangle=\langle 1,\mathbf{q}|A(\mathbf{k},0)|1,\mathbf{q}\rangle\langle 1,\mathbf{q}|\rho(\mathbf{q})|0\rangle+\textstyle\sum_\alpha\langle 1,\mathbf{q}|A(\mathbf{k},0)|\alpha\neq 1,\mathbf{q}\rangle\langle\alpha\neq 1,\mathbf{q}|\rho(\mathbf{q})|0\rangle. \quad (13)$$

Now $n(\mathbf{k})\sim 1$ and as can be easily shown $\langle 1\mathbf{q}|\rho(\mathbf{q})|0\rangle\sim\Omega^{1/2}$. On the other hand, the order of magnitude of the remaining sum is, by the type of analysis indicated in the introduction, at most of order unity.[15] The term (11) is the direct-interaction term.

Similar arguments can be made for the remaining terms (b) and (c). For (b) we obtain

$$-\sum_{\mathbf{q}_1}V(\mathbf{q}_1)[n(\mathbf{k})-n(\mathbf{k}+\mathbf{q})]\langle 1,\mathbf{q}|A(\mathbf{k}+\mathbf{q}_1,\mathbf{q})|0\rangle, \quad (14)$$

the exchange-interaction term, whereas (c) becomes

$$+\sum_{\mathbf{q}_1}V(\mathbf{q}_1)[n(\mathbf{k}+\mathbf{q}_1)-n(\mathbf{k}+\mathbf{q}+\mathbf{q}_1)]\times\langle 1,\mathbf{q}|A(\mathbf{k},\mathbf{q})|0\rangle \quad (15)$$

and is identified as the exchange self-energy term. Equation (10) may now be rewritten

$$[\omega(\mathbf{q})-\epsilon(\mathbf{k}+\mathbf{q};\mathbf{k})]\langle 1,\mathbf{q}|A(\mathbf{k},\mathbf{q})|0\rangle=(11)+(14)+(15)+\sum_{\mathbf{q}_1}V(\mathbf{q}_1)\langle 1,\mathbf{q}|\{A(\mathbf{k},\mathbf{q}-\mathbf{q}_1)-A(\mathbf{k}+\mathbf{q}_1,\mathbf{q}-\mathbf{q}_1)\}\rho(\mathbf{q}_1)|0\rangle \quad (16)$$

where it is understood that in the last term we are to omit the contributions from those momentum values corresponding to conditions (a), (b), and (c) and indeed this has been used in rearranging the operators so as to

[14] L. L. Foldy, Phys. Rev. **124**, 649 (1961).

[15] The sum is of order Ω, the first factor of order Ω^{-1}, the second at most of order unity.

obtain the form given.[16] We now turn our attention to the problem of evaluating the contribution of the last, nonlinear term in (16).

To carry out this program, we introduce a sum over states as in Eq. (4). For example, we write ($\mathbf{q}_1 \neq \mathbf{q}$)

$$\begin{aligned}\langle 1,\mathbf{q}|A(\mathbf{k},\mathbf{q}-\mathbf{q}_1)\rho(\mathbf{q}_1)|0\rangle &= \sum\langle 1,\mathbf{q}|A(\mathbf{k},\mathbf{q}-\mathbf{q}_1)|1,\mathbf{q}_1\rangle\langle 1,\mathbf{q}_1|\rho(\mathbf{q}_1)|0\rangle \\ &\quad +\sum\langle 1,\mathbf{q}|A(\mathbf{k},\mathbf{q}-\mathbf{q}_1)|2,\mathbf{q}_1\rangle\langle(2,\mathbf{q}_1)|\rho(\mathbf{q}_1)|0\rangle \\ &\quad +\text{contributions from 3 or more pair intermediate states.} \quad (17)\end{aligned}$$

For purposes of illustration we shall omit the contributions from three or more pair states. Furthermore, making use of Eq. (2) we can simplify the second term since the only contributions that matter are those for which

$$|2,\mathbf{q}_1\rangle = |1,\mathbf{q}\rangle \times |1,\mathbf{q}_1-\mathbf{q}\rangle \equiv |1,\mathbf{q};1,\mathbf{q}_1-\mathbf{q}\rangle. \quad (18)$$

Thus, we shall illustrate an approximation in which we write

$$\begin{aligned}\langle 1,\mathbf{q}|A(\mathbf{k},\mathbf{q}-\mathbf{q}_1)\rho(\mathbf{q}_1)|0\rangle \cong &\langle 1\mathbf{q}|A(\mathbf{k},\mathbf{q}-\mathbf{q}_1)|1\mathbf{q}_1\rangle \\ &\times\langle 1\mathbf{q}_1|\rho(\mathbf{q}_1)|0\rangle + \langle 0|A(\mathbf{k}_1\mathbf{q}-\mathbf{q}_1)|1,\mathbf{q}_1-\mathbf{q}\rangle \\ &\times\langle 1,\mathbf{q};1,\mathbf{q}_1-\mathbf{q}|\rho(\mathbf{q}_1)|0\rangle. \quad (19)\end{aligned}$$

This approximation is based on the idea that amplitudes like $\langle 1,\mathbf{q}|A(\mathbf{k},\mathbf{q}-\mathbf{q}_1)|1,\mathbf{q}_1\rangle$ and $\langle 1,\mathbf{q};1\mathbf{q}_1-\mathbf{q}|A(\mathbf{k},\mathbf{q}_1)|0\rangle$ are, next to $\langle 1,\mathbf{q}|A(\mathbf{k},\mathbf{q})|0\rangle$ the most important ones characterizing the so-called one-pair states. They take cognizance for the first time that, in fact, such a state also contains two-pair configurations, but it is expected, or rather the treatment is valid, for the case that these do not dominate.

For the new amplitudes that enter (19), we obtain by reasoning analogous to that which led to Eq. (16), the equations

$$\begin{aligned}&[\omega(\mathbf{q})-\omega(\mathbf{q}_1)-\epsilon(\mathbf{k}+\mathbf{q}-\mathbf{q}_1;\mathbf{k})]\langle 1,\mathbf{q}|A(\mathbf{k},\mathbf{q}-\mathbf{q}_1)|1,\mathbf{q}_1\rangle \\ &\quad = V(\mathbf{q}-\mathbf{q}_1)[n(\mathbf{k})-n(\mathbf{k}+\mathbf{q}-\mathbf{q}_1)]\langle 1,\mathbf{q}|\rho(\mathbf{q}-\mathbf{q}_1)|1,\mathbf{q}_1\rangle \\ &\qquad -\sum_{\mathbf{q}_2} V(\mathbf{q}_2)[n(\mathbf{k})-n(\mathbf{k}+\mathbf{q}-\mathbf{q}_1)]\langle 1,\mathbf{q}|A(\mathbf{k}+\mathbf{q}_2;\mathbf{q}-\mathbf{q}_1)|1,\mathbf{q}_1\rangle \\ &\qquad +\sum_{\mathbf{q}_2} V(\mathbf{q}_2)[n(\mathbf{k}+\mathbf{q}_2)-n(\mathbf{k}+\mathbf{q}-\mathbf{q}_1+\mathbf{q}_2)]\langle 1,\mathbf{q}|A(\mathbf{k},\mathbf{q}-\mathbf{q}_1)|1,\mathbf{q}_1\rangle \\ &\qquad +\sum_{\mathbf{q}_2}{}' V(\mathbf{q}_2)\langle 1\mathbf{q}|[A(\mathbf{k},\mathbf{q}-\mathbf{q}_1-\mathbf{q}_2-A(\mathbf{k}+\mathbf{q}_2,\mathbf{q}-\mathbf{q}_1-\mathbf{q}_2)]\rho(\mathbf{q}_2)|1,\mathbf{q}_1\rangle, \quad (20)\end{aligned}$$

and

$$\begin{aligned}&[\omega(\mathbf{q})+\omega(\mathbf{q}_1-\mathbf{q})-\epsilon(\mathbf{k}_1+\mathbf{q}_1;\mathbf{k}_1)]\langle 1,\mathbf{q};1,\mathbf{q}_1-\mathbf{q}|A(\mathbf{k}_1,\mathbf{q}_1)|0\rangle \\ &\quad = V(\mathbf{q}_1)[n(\mathbf{k}_1)-n(\mathbf{k}_1+\mathbf{q}_1)]\langle 1,\mathbf{q};1,\mathbf{q}_1-\mathbf{q}|\rho(\mathbf{q}_1)|0\rangle \\ &\qquad -\sum_{\mathbf{q}_2} V(\mathbf{q}_2)[n(\mathbf{k}_1)-n(\mathbf{k}_1+\mathbf{q}_1)]\langle 1,\mathbf{q};1,\mathbf{q}_1-\mathbf{q}|A(\mathbf{k}_1+\mathbf{q}_2,\mathbf{q}_1)|0\rangle \\ &\qquad +\sum_{\mathbf{q}_2} V(\mathbf{q}_2)[n(\mathbf{k}_1+\mathbf{q}_2)-n(\mathbf{k}_1+\mathbf{q}_1+\mathbf{q}_2)]\langle 1,\mathbf{q};1,\mathbf{q}_1-\mathbf{q}|A(\mathbf{k}_1,\mathbf{q}_1)|0\rangle \\ &\qquad +\sum_{\mathbf{q}_2}{}' V(\mathbf{q}_2)\langle 1,\mathbf{q};1,\mathbf{q}_1-\mathbf{q}|[A(\mathbf{k}_1,\mathbf{q}_1-\mathbf{q}_2)-A(\mathbf{k}_1+\mathbf{q}_2,\mathbf{q}_1-\mathbf{q}_2)]\rho(\mathbf{q}_2)|0\rangle. \quad (21)\end{aligned}$$

In order to understand the structure of these equations we must again examine the structure of the nonlinear terms. From Eq. (20), for instance, we must evaluate

$$\begin{aligned}&\sum_{\mathbf{q}_2}{}' V(\mathbf{q}_2)\langle 1,\mathbf{q}|A(\mathbf{k},\mathbf{q}-\mathbf{q}_1-\mathbf{q}_2)\rho(\mathbf{q}_2)|1,\mathbf{q}_1\rangle \\ &\quad = V(\mathbf{q}_1)\langle 1,\mathbf{q}|A(\mathbf{k},\mathbf{q})|0\rangle\langle 0|\rho(-\mathbf{q}_1)|1,\mathbf{q}_1\rangle \\ &\qquad +V(\mathbf{q})\langle 0|A(\mathbf{k}_1,-\mathbf{q}_1)|1,\mathbf{q}_1\rangle\langle 1,\mathbf{q}|\rho(\mathbf{q})|0\rangle+\cdots. \quad (22)\end{aligned}$$

Here the second term in (22) is the already simplified contribution from the two-pair incoherent state $|2,\mathbf{q}+\mathbf{q}_2\rangle = \delta(\mathbf{q}_2-\mathbf{q})|1,\mathbf{q};1,\mathbf{q}_1\rangle$. The terms shown are the only contributions quadratic in the large amplitudes like $\langle 1\mathbf{q}|A(\mathbf{k},\mathbf{q}|0\rangle$. We are ignoring terms quadratic in the correction amplitudes under consideration or involving yet higher amplitudes. In this approximation, Eqs. (16), (20), and (21) yield a closed set of coupled integral equations involving amplitudes schematically of types $\langle 1|A|0\rangle$, $\langle 1|A|1\rangle$, and $\langle 2|A|0\rangle$.

A proviso must be added to the assertion that the equations are closed. This is the case if the number densities $n(\mathbf{k})$ are assumed to be known. Within the framework of the time-independent equation of motion method espoused here, they can be obtained only by studying the equations of motion for the matrix elements of single operators $a(\mathbf{k})$ or $a^\dagger(\mathbf{k})$ and by use of the formula

$$n(\mathbf{k}) = \sum_{\alpha}\langle 0|a^\dagger(\mathbf{k})|\alpha\rangle\langle\alpha|a(\mathbf{k})|0\rangle. \quad (23)$$

[16] For a superfluid system, we would have additional large individual matrix elements besides (11), (14), and (15), for example, those of the pairing interaction in the case of superconductivity.

This aspect of the problem will not be considered in this paper, but it can be handled by methods analogous to those being discussed.

For the remainder of this section, we indicate only the crudest possible treatment of our coupled equations. If we think of the large amplitudes $\langle 1|A|0\rangle$ as given, then Eqs. (20) and (21) constitute inhomogeneous integral equations in which the small amplitudes are driven by the term quadratic in the large amplitudes. We retain for purposes of illustration only the inhomogeneous terms arising from the decomposition (22) and related equations, and insert the corresponding solutions of (20) and (21) into (16). The general form of the resulting equation is

$$[\omega(\mathbf{q})-\epsilon(\mathbf{k}+\mathbf{q};\mathbf{k})]\langle 1,\mathbf{q}|A(\mathbf{k},\mathbf{q})|0\rangle = V(\mathbf{q})L(\mathbf{q},\mathbf{k})\langle 1,\mathbf{q}|\rho(\mathbf{q})|0\rangle + \sum_{\mathbf{p}} M(\mathbf{q},\mathbf{k},\mathbf{p})\langle 1\mathbf{q}|A(\mathbf{k}+\mathbf{p},\mathbf{q})|0\rangle. \quad (24)$$

We address some general remarks to this form before recording in detail its relevant pieces L and M. It is not difficult to see, if one stares at the coupled Eqs. (16), (20), and (21) long enough, that any reasonable method of elimination of the higher amplitudes, and not only the simple approximation entertained here, will lead to an equation with the structure of (24). One may easily imagine that the same will hold for a larger set of equations embracing additional amplitudes. It is to be supposed that the elimination is carried out so that Eq. (24) contains amplitudes of the form $\langle 1|A|0\rangle$ or $\langle 0|A|1\rangle$ only. On the other hand, as we shall see in detail shortly, the kernels L and M are, in general, *nonlinear functionals of these amplitudes*.

The only practical way we see of attacking the resulting nonlinear integral Eq. (24), at least initially, is by the following processes of linearization. We first write

$$L=\sum_{0}^{\infty} L_n, \quad M=\sum_{1}^{\infty} M_n, \quad (25)$$

where the order n of a given term is determined by some reasonable criterion such as the power of V which occurs. There may, however, depending on the mode of elimination of higher amplitudes, i.e., whether all potentials are screened, single-particle energies renormalized, etc., be other nonpower dependence on V; and therefore, the best definition of order will only emerge from experience. Starting with [cf. Eqs. (11) and (16)]

$$L_0(\mathbf{q},\mathbf{k})=n(\mathbf{k})-n(\mathbf{k}+\mathbf{q})\cong\theta(k-k_F) - \theta(|\mathbf{k}+\mathbf{q}|-k_F), \quad (26)$$

where we have replaced the momentum distribution functions by their appropriate free-gas values, we know from the work of Sawada[17] and Brout[18] that the resulting simple RPA equation has a complete set of solutions consisting of the scattering states and (below the decay threshold), the plasmon.

In the next approximation we add the terms L_1 and M_1 and consider the approximate iterated equation

$$[\omega(\mathbf{q})-\epsilon(\mathbf{k}+\mathbf{q};\mathbf{k})]\langle 1\mathbf{q}|A(\mathbf{k},\mathbf{q})|0\rangle = V(q)\{L_0(\mathbf{q},\mathbf{k})+L_1(\mathbf{q},\mathbf{k})+\sum_{p} M_1(\mathbf{q},\mathbf{k},p)[\omega(\mathbf{q})-\epsilon(\mathbf{k}+\mathbf{q}+\mathbf{p},\mathbf{k}+\mathbf{p})]^{-1} \times L_0(\mathbf{q},\mathbf{k}+\mathbf{p})\}\langle 1\mathbf{q}|\rho(\mathbf{q})|0\rangle. \quad (27)$$

Here

$$M_1(\mathbf{q},\mathbf{k},\mathbf{p})=-V(\mathbf{p})[n(\mathbf{k})-n(\mathbf{k}+\mathbf{q})]+\delta(\mathbf{p})\sum_{\mathbf{q}'} V(\mathbf{q}')[n(\mathbf{k}+\mathbf{q}')-n(\mathbf{k}+\mathbf{q}+\mathbf{q}')] \quad (28)$$

are the exchange and exchange self-energy terms and $L_1(\mathbf{q},\mathbf{k})$ will be discussed below. As an example of the type of improved result obtainable from (27) we derive in the standard way the following improved dispersion relation for the plasmon

$$1=V(q)\sum_{\mathbf{k}}[\omega(\mathbf{q})-\epsilon(\mathbf{k}+\mathbf{q};\mathbf{k})]^{-1}\{L_0(\mathbf{q},\mathbf{k})+\sum_{\mathbf{p}} M_1(\mathbf{q},\mathbf{k},\mathbf{p})[\omega(\mathbf{q})-\epsilon(\mathbf{k}+\mathbf{q}+\mathbf{p};\mathbf{k}+\mathbf{p})]^{-1}L_0(\mathbf{q},\mathbf{k}+\mathbf{p})+L_1(\mathbf{q},\mathbf{k})\} \quad (29)$$

It is simple to show that the second term gives the well-known exchange correction to the plasmon dispersion relation. The role of the term $L_1(\mathbf{q},\mathbf{k})$ requires some explication to which we now turn.

The term in question is the first one which arises from the second stage of approximation considered in Eqs. (20) and (21). From the latter we obtain, respectively, the contributions.

$$L_1(\mathbf{q},\mathbf{k})=L_1[(20)]+L_1[(21)], \quad (30)$$

where

$$L_1[(20)]=\sum_{\mathbf{q}_1} V(\mathbf{q}_1)\sum_{\mathbf{k}_1}\{[\omega(\mathbf{q})-\omega(\mathbf{q}_1)-\epsilon(\mathbf{k}+\mathbf{q}-\mathbf{q}_1;\mathbf{k})]^{-1}\langle 0|[A(\mathbf{k},-\mathbf{q}_1)-A(\mathbf{k}+\mathbf{q},-\mathbf{q}_1)]|1,\mathbf{q}_1\rangle \times\langle 1,\mathbf{q}_1|A(\mathbf{k}_1,\mathbf{q}_1)|0\rangle-[\omega(\mathbf{q})-\omega(\mathbf{q}_1)-\epsilon(\mathbf{k}+\mathbf{q};\mathbf{k}+\mathbf{q}_1)]^{-1}\langle 0|[A(\mathbf{k}+\mathbf{q}_1,-\mathbf{q}_1)-A(\mathbf{k}+\mathbf{q}+\mathbf{q}_1-\mathbf{q}_1)|1,\mathbf{q}_1\rangle \times\langle 1,\mathbf{q}_1|A(\mathbf{k}_1,\mathbf{q}_1)|0\rangle\}, \quad (31)$$

[17] K. Sawada, Phys. Rev. **106**, 372 (1957).
[18] R. Brout, Phys. Rev. **108**, 515 (1957).

and

$$L_1[(21)]=\sum_{\mathbf{q}_1} V(\mathbf{q}_1) \sum_{\mathbf{k}_1} [\omega(\mathbf{q})+\omega(\mathbf{q}_1-\mathbf{q})-\epsilon(\mathbf{k}_1+\mathbf{q}_1;\mathbf{k}_1)]^{-1}\langle 0|[A(\mathbf{k},\mathbf{q}-\mathbf{q}_1)-A(\mathbf{k}+\mathbf{q}_1,\mathbf{q}-\mathbf{q}_1)]|1,\mathbf{q}_1-\mathbf{q}\rangle$$
$$\times\langle 1,\mathbf{q}_1-\mathbf{q}|[A(\mathbf{k}_1,\mathbf{q}_1-\mathbf{q})-A(\mathbf{k}_1+\mathbf{q},\mathbf{q}_1-\mathbf{q})]|0\rangle. \quad (32)$$

According to the scheme outlined above (31) and (32) are to be evaluated by inserting for the matrix elements the solutions of the simple RPA for the plasmon and for the scattering states, summing over all possibilities. Here, however, (and this will be in contrast to the method of the next section) a caution must be observed. This we learn by studying a simple approximation to these equations obtained by trying the free-particle values of these matrix elements. Consider, for example,

$$\langle 0|A(\mathbf{k},-\mathbf{q}_1)|1,\mathbf{q}_1\rangle\langle 1,\mathbf{q}_1|A(\mathbf{k},\mathbf{q}_1)|0\rangle$$
$$=\langle 0|a^\dagger(\mathbf{k}-\mathbf{q}_1)a(\mathbf{k})|1,\mathbf{q}_1\rangle$$
$$\times\langle 1,\mathbf{q}_1|a^\dagger(\mathbf{k}_1+\mathbf{q}_1)a(\mathbf{k}_1)|0\rangle$$
$$\cong\delta(\mathbf{k}_1-\mathbf{k}+\mathbf{q}_1)n(\mathbf{k}-\mathbf{q}_1)[1-n(\mathbf{k})]. \quad (33)$$

This kind of evaluation, inserted in (31) and (32), would bring them to a form quite comparable to the exchange correction of (29) and there would be a contribution of the same order.

Further reflection shows, however, that the contribution (33) is to be excluded because it corresponds precisely to the conditions (b) and (c) which determine the extra terms included already in the extended RPA. If it is simultaneously realized that the terms like (33) would arise from the free-particle parts of the pair-scattering states, we are led to the altered prescription that we are to insert in (31) the plasmon "wave function" and the nonfree parts of the pair-scattering wave functions. This leads in each case to an additional factor, at least, of V. We, thus, reach the conclusion that the contribution from $L_1(\mathbf{q},\mathbf{k})$ may safely be left for the next order.

We leave further development of this method to a future publication. Despite the wealth of detail already given, there are several points which require elaboration in order to carry the method forward: (a) the addition of a systematic method for generating corrections to the distribution functions $n(\mathbf{k})$; (b) a systematic method for generating corrections to the Sawada-Brout amplitudes. Assuming that these matters can be disposed of satisfactorily, at least within the iterative framework espoused above (and they can), we have gone far enough to show that we have here what is in principle a complete, nonperturbative approach to the electron gas problem in terms initially of a set of coupled nonlinear integral equations. It is true that the provisional method of solving these equations outlined above comes uncomfortably close to the perturbation method. Our aim here, however, was to make contact with previous results. It remains to be seen how far one can escape from these limitations. For this purpose, the method of the next section may have some advantages.

III. THEORY BASED ON THE SIMPLE RPA AS FIRST APPROXIMATION

The method of the previous section should prove most useful, if at all so, for the case of a short range interaction where direct and exchange interactions tend to be of comparable magnitude. For the electron gas, where the direct interaction dominates, the method, though completely viable, as we have tried to indicate, is also somewhat clumsy, and an alternative method based more directly on the simple RPA may be superior in practice. As the basis for this method, we notice that Eq. (10) may, without approximation, be rewritten as

$$[\omega(\mathbf{q})-\epsilon(\mathbf{k}+\mathbf{q};\mathbf{k})]\langle 1,\mathbf{q}|A(\mathbf{k},\mathbf{q})|0\rangle=\sum_{\mathbf{q}_1}\langle 1,\mathbf{q}|\{A(\mathbf{k},\mathbf{q}-\mathbf{q}_1)\rho(\mathbf{q}_1)-\rho(\mathbf{q}_1)A(\mathbf{k}+\mathbf{q}_1,\mathbf{q}-\mathbf{q}_1)\}|0\rangle$$
$$=V(\mathbf{q})[n(\mathbf{k})-n(\mathbf{k}+\mathbf{q})]\langle 1,\mathbf{q}|\rho(\mathbf{q})|0\rangle+\sum_{\mathbf{q}_1\neq\mathbf{q}} V(\mathbf{q}_1)\langle 1,\mathbf{q}|\{A(\mathbf{k},\mathbf{q}-\mathbf{q}_1)\rho(\mathbf{q}_1)-\rho(\mathbf{q}_1)A(\mathbf{k}+\mathbf{q}_1,\mathbf{q}-\mathbf{q}_1)\}|0\rangle, \quad (34)$$

where we have thus chosen only to linearize the direct interaction term. We now treat the entire remainder by the method of spectral decomposition described after Eq. (16) of the previous section. In the same approximation as that corresponding to the inclusion of Eqs. (20) and (21), we here consider the equation

$$[\omega(\mathbf{q})-\omega(\mathbf{q}_1)-\epsilon(\mathbf{k}+\mathbf{q}-\mathbf{q}_1;\mathbf{k})]\langle 1,\mathbf{q}|A(\mathbf{k},\mathbf{q}-\mathbf{q}_1)|1,\mathbf{q}_1\rangle=V(\mathbf{q}-\mathbf{q}_1)[n(\mathbf{k})-n(\mathbf{k}+\mathbf{q}-\mathbf{q}_1)]\langle 1,\mathbf{q}|\rho(\mathbf{q}-\mathbf{q}_1)|1,\mathbf{q}_1\rangle$$
$$+\sum_{\mathbf{q}_2\neq\mathbf{q}-\mathbf{q}_1} V(\mathbf{q}_2)\langle 1,\mathbf{q}|\{A(\mathbf{k},\mathbf{q}-\mathbf{q}_1-\mathbf{q}_2)\rho(\mathbf{q}_2)-\rho(\mathbf{q}_2)A(\mathbf{k}+\mathbf{q}_2,\mathbf{q}-\mathbf{q}_1-\mathbf{q}_2)\}|1\mathbf{q}_1\rangle \quad (35)$$

and several similar equations which we do not record. When we treat these in a manner similar to Eq. (22) and substitute the result in Eq. (34), we obtain an equation of the form (24) with $L(\mathbf{q},\mathbf{k})$ replaced by a new kernel $\bar{L}(\mathbf{q},\mathbf{k})$ and $M(\mathbf{q},\mathbf{k},\mathbf{p})$ replaced by $\bar{M}(\mathbf{q},\mathbf{k},\mathbf{p})$. We now write

$$\bar{L}=\sum_0^\infty \bar{L}_n, \quad \bar{M}=\sum_2^\infty \bar{M}_n. \quad (36)$$

In analogy with the work of the previous section we consider briefly the approximation in which we drop $\bar{M}$ and write

$$L=\bar{L}_0+\bar{L}_1. \tag{37}$$

Here obviously $\bar{L}_0=L_0$ and for $\bar{L}_1(\mathbf{q},\mathbf{k})$ we now find

$$\begin{aligned}\bar{L}_1(\mathbf{q},\mathbf{k})=\sum_{\mathbf{q}_1} V(\mathbf{q}) \sum_{\mathbf{k}_1} \{&[\omega(\mathbf{q})-\omega(\mathbf{q}_1)-\epsilon(\mathbf{k}+\mathbf{q}-\mathbf{q}_1;\mathbf{k})]^{-1}\langle 0|[A(\mathbf{k},-\mathbf{q}_1)-A(\mathbf{k}+\mathbf{q},-\mathbf{q}_1)]|1,\mathbf{q}_1\rangle\langle 1,\mathbf{q}_1|A(\mathbf{k}_1,\mathbf{q}_1|0\rangle\\
&-[\omega(\mathbf{q})-\omega(\mathbf{q}-\mathbf{q}_1)-\epsilon(\mathbf{k}_1+\mathbf{q}_1;\mathbf{k}_1)]^{-1}\langle 0|[A(\mathbf{k}_1,\mathbf{q}_1-\mathbf{q})-A(\mathbf{k}_1+\mathbf{q},\mathbf{q}_1-\mathbf{q})]|1,\mathbf{q}-\mathbf{q}_1\rangle\\
&\times\langle 1,\mathbf{q}-\mathbf{q}_1|A(\mathbf{k}+\mathbf{q}_1,\mathbf{q}-\mathbf{q}_1)|0\rangle+[\omega(\mathbf{q})+\omega(\mathbf{q}_1-\mathbf{q})-\epsilon(\mathbf{k}_1;\mathbf{k}_1-\mathbf{q}_1)]^{-1}\\
&\times\langle 0|A(\mathbf{k},\mathbf{q}-\mathbf{q}_1)|1,\mathbf{q}_1-\mathbf{q}\rangle\langle 1,\mathbf{q}_1-\mathbf{q}|[A(\mathbf{k}_1-\mathbf{q}_1,\mathbf{q}_1-\mathbf{q})-A(\mathbf{k}_1+\mathbf{q}-\mathbf{q}_1,\mathbf{q}_1-\mathbf{q})]|0\rangle\\
&-[\omega(\mathbf{q})+\omega(-\mathbf{q}_1)-\epsilon(\mathbf{k}+\mathbf{q},\mathbf{k}+\mathbf{q}_1)]^{-1}\langle 0|A(\mathbf{k}_1,\mathbf{q}_1)|1,-\mathbf{q}_1\rangle\\
&\times\langle 1,-q_1|[A(\mathbf{k}+\mathbf{q}_1,-\mathbf{q}_1)-A(\mathbf{k}+\mathbf{q}+\mathbf{q}_1,-\mathbf{q}_1)]|0\rangle\}.\end{aligned} \tag{38}$$

If, for purposes of orientation, we make the type of approximation indicated in (33) we obtain as the contribution of (38) to the plasmon-dispersion relation

$$\begin{aligned}\sum_{\mathbf{k}}[\omega(\mathbf{q})-\epsilon(\mathbf{k}+\mathbf{q};\mathbf{k})]^{-1}\bar{L}_1(\mathbf{q},\mathbf{k})=\sum_{\mathbf{q}_1} V(\mathbf{q}_1)\sum_{\mathbf{k}}&[n(\mathbf{k})-n(\mathbf{k}+\mathbf{q})][n(\mathbf{k}+\mathbf{q}_1)-n(\mathbf{k}+\mathbf{q}+\mathbf{q}_1)]\\
&\times[\omega(\mathbf{q})-\epsilon(\mathbf{k}+\mathbf{q};\mathbf{k})]^{-1}\{[\omega(\mathbf{q})-\epsilon(\mathbf{k}+\mathbf{q};\mathbf{k})]^{-1}-[\omega(\mathbf{q})-\epsilon(\mathbf{k}+\mathbf{q}+\mathbf{q}_1;\mathbf{k}+\mathbf{q}_1)]^{-1}\}.\end{aligned} \tag{39}$$

This is identical with the second term on the right-hand side of Eq. (27) and, therefore, with the first exchange correction.

IV. LONG-WAVELENGTH LIMIT OF THE PLASMON ENERGY

It is generally believed that as $q\to 0$, $\omega(\mathbf{q})$ must be given by the classical plasmon frequency

$$\omega^2(\mathbf{q})=\omega_{\mathrm{cl}}^2=4\pi ne^2/m, \tag{40}$$

where n is the particle density. Moreover, the literature abounds with near proofs.[1] We consider here what is essentially a simple and rigorous proof, based on Eq. (34), though we could equally well base it on Eq. (16).

We first note that if we sum Eq. (34) with respect to k, we should and do obtain the equation of continuity

$$\omega(\mathbf{q})\langle\mathbf{q}|\rho(\mathbf{q})|0\rangle=\langle\mathbf{q}|\mathbf{q}\cdot\mathbf{j}(\mathbf{q})|0\rangle \tag{41}$$

where the current

$$\mathbf{j}(\mathbf{q})=\sum_{\mathbf{k}}(2m)^{-1}(2\mathbf{k}+\mathbf{q})A(\mathbf{k},\mathbf{q}). \tag{42}$$

This is so because, first

$$\sum_{\mathbf{k}}[n(\mathbf{k})-n(\mathbf{k}+\mathbf{q})]=0, \tag{43}$$

and second because we encounter, in the second term, the commutator

$$[\rho(\mathbf{q}-\mathbf{q}_1),\rho(\mathbf{q}_1)]=0. \tag{44}$$

We next derive from (34) the equation

$$\begin{aligned}\langle 1,\mathbf{q}|\rho(\mathbf{q})|0\rangle=V(\mathbf{q})\sum_{\mathbf{k}}\frac{[n(\mathbf{k})-n(\mathbf{k}+\mathbf{q})]}{[\omega(\mathbf{q})-\epsilon(\mathbf{k}+\mathbf{q};\mathbf{k})]}\langle 1,\mathbf{q}|\rho(\mathbf{q})|0\rangle+\sum_{\mathbf{q}_1\neq\mathbf{q}} V(\mathbf{q}_1)\sum_{\mathbf{k}\mathbf{k}_1}&[\omega(\mathbf{q})-\epsilon(\mathbf{k}+\mathbf{q};\mathbf{k})]^{-1}\\
&\times\langle 1,\mathbf{q}|\{A(\mathbf{k},\mathbf{q}-\mathbf{q}_1)\rho(\mathbf{q}_1)-\rho(\mathbf{q}_1)A(\mathbf{k}+\mathbf{q}_1,\mathbf{q}-\mathbf{q}_1)\}|0\rangle.\end{aligned} \tag{45}$$

We look for a solution of this equation with the properties

$$\lim_{q\to 0}\langle 1\mathbf{q}|\rho(\mathbf{q})|0\rangle\neq 0,\quad \lim_{q\to 0}\omega(\mathbf{q})=\omega_0\neq 0. \tag{46}$$

With the momentary supposition that the last term of (45) vanishes as $q\to 0$, we obtain the standard condition

$$1=\lim_{q\to 0} V(\mathbf{q})\sum_{\mathbf{k}}[n(\mathbf{k})-n(\mathbf{k}+\mathbf{q})]\times[\omega(\mathbf{q})-\epsilon(\mathbf{k}+\mathbf{q};\mathbf{k})]^{-1}, \tag{47}$$

which indeed gives the solution (40).[19] It, therefore,

[19] This is true even though we find in (47) the exact distribution functions $n(\mathbf{k})$.

remains to consider this last term. As $q \to 0$ we may ignore $\epsilon(\mathbf{k}+\mathbf{q};\mathbf{k})$ relative to $\omega(\mathbf{q})$ since the former vanishes linearly with q. What remains vanishes for arbitrary q in virtue of Eq. (44) [thus showing the connection with the equation of continuity (41)]. This completes the proof.

We show below that the first nonvanishing contribution of the last term of (45) behaves like q^2. First, however, we address ourselves to the possible sources of doubt concerning this apparently trivial proof. There is, in fact, only one such source. We have assumed that the system is such that only for the particular value of $\mathbf{q}_1=\mathbf{q}$ do we get an abnormally large matrix element and a singular factor $V(\mathbf{q})$ simultaneously. It is, therefore, supposed that for any $\mathbf{q}_1=c\mathbf{q}$, with c a constant not equal to unity for sufficiently small q, which would also give a singular factor of $V(\mathbf{q})$ as $q \to 0$, the matrix element multiplying this singular factor is $0(\Omega^{-1})$ compared to the first term of (45). The proof, moreover, is equally applicable to the case of a charged boson gas and to a gas with either statistics with additional local short-range interaction.

We complete our considerations by noting explicitly that the leading nonvanishing contribution of the last term of (45) must go like q^2, as must indeed follow from inversion symmetry. Here it is only necessary to show that the term linear in q which comes from expanding the denominator must vanish, since the other linear term from expansion of the matrix element has already been indicated to vanish. We must, therefore, show that

$$\begin{aligned}0=&\sum_{\mathbf{k}}\sum_{\mathbf{q}_1} V(\mathbf{q}_1)(\mathbf{k}\cdot\mathbf{q})[A(\mathbf{k},-\mathbf{q}_1)\rho(\mathbf{q}_1)\\ &\qquad -\rho(\mathbf{q}_1)A(\mathbf{k}+\mathbf{q}_1,-\mathbf{q}_1)]\\ =&\sum_{\mathbf{k}}\sum_{\mathbf{q}_1} V(\mathbf{q}_1)(\mathbf{k}\cdot\mathbf{q})[A(\mathbf{k},-\mathbf{q}_1),\rho(\mathbf{q}_1)]\\ &\qquad -\sum_{\mathbf{q}_1} V(\mathbf{q}_1)(\mathbf{q}_1\cdot\mathbf{q})\rho(-\mathbf{q}_1)\rho(\mathbf{q}_1), \quad (48)\end{aligned}$$

where the substitution $\mathbf{k} \to \mathbf{k}-\mathbf{q}_1$ has been used in the second term of the first form of (48). We have, further,

$$\begin{aligned}&\sum_{\mathbf{k}}\sum_{\mathbf{q}_1} V(\mathbf{q}_1)(\mathbf{k}\cdot\mathbf{q})[A(\mathbf{k}_1-\mathbf{q}_1),\rho(\mathbf{q}_1)]\\ &\quad=\sum_{\mathbf{k}}\sum_{\mathbf{q}_1} V(\mathbf{q}_1)(\mathbf{k}\cdot\mathbf{q})[A(\mathbf{k}-\mathbf{q}_1,0)-A(\mathbf{k}_1,0)]\\ &\qquad\qquad =\sum_{\mathbf{k}}\sum_{\mathbf{q}_1} V(\mathbf{q}_1)(\mathbf{q}_1\cdot\mathbf{q})A(\mathbf{k}), \quad (49)\end{aligned}$$

which vanishes upon averaging over $\mathbf{q}_1$ and $-\mathbf{q}_1$, as does also the last term of (48).

V. SUMMARY AND CONCLUSIONS

The purpose of this paper has been to indicate, by methods which are more widely applicable, but which have been carried out specifically for the electron gas, how the equation-of-motion method may be carried beyond the first RPA, be it the simple or the extended version, to give an, in principle, complete and nonperturbative method for studying a many-body system. The reasoning is based primarily upon (a) definite assertions about the order of magnitude of various kinds of matrix elements relevant to various classes of states of the system, and (b) the use of a spectral decomposition of products of more than two operators which defines in conjunction with (a) the manner in which one proceeds to higher order. We have indicated briefly one way in which the new equations may be solved which will make use of the known solutions to the simple RPA case, and we have made brief contact with previous results.

It remains to be seen, however, how far our methods can be pushed to obtain new results, and for this reason we have not undertaken any discussion of several elements of the method which will be of importance in this attempt, namely, the correction to the occupation numbers $n(\mathbf{k})$ that must come in higher order and the systematic correction of the RPA wave functions. The former is the more fundamental question, and as indicated in the text, will require the study of the equation of motion of single operators, a question not considered here, but amenable to the same methods. Though no single element of our method is really new, we believe we have achieved a new synthesis and helped clarify some basic questions left obscure in previous attempts of the same kind.

It should be clear that the same type of reasoning can be applied to the method of Green functions and in some contexts especially at finite temperatures it may be preferable to apply this latter method. For the sake of initial exposition, we have chosen to avoid this possibility in order to minimize the number of indices at which the reader need stare.

It should also be clear how the methods, especially that of Sec. II, can be extended to superfluid systems. One has merely to include in the lowest approximation the additional "abnormally large" correlations characteristic of these cases.

Finally, we have given a rigorous proof that the plasmon energy, in the long-wavelength limit, goes over into the classical plasmon energy.